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# Escape components of McMullen maps

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*Abstract.* We consider the McMullen maps $f_\lambda(z) = z^n + \lambda z^{-n}$ with $\lambda \in \mathbb{C}^*$ and $n \geq 3$. We prove that the closures of escape hyperbolic components are pairwise disjoint and the boundaries of all bounded escape components (the McMullen domain and Sierpiński holes) are quasi-circles with Hausdorff dimension strictly between 1 and 2.

Key words: hyperbolic component, McMullen map, quasicircle, holomorphic motion
2020 Mathematics Subject Classification: 37F45 (Primary); 37F10, 37F15 (Secondary)

## 1. *Introduction*

1.1. *Background and main result.* In 1988, McMullen [**McM88**] introduced a rational map $f_\lambda(z) = z^2 + \lambda z^{-3}$ as a singular perturbation of $z \mapsto z^2$. In his article, McMullen showed that for $\lambda$ sufficiently small, the Julia set $J(f_\lambda)$ of $f_\lambda$ is a Cantor circle (homeomorphic to a product of a Cantor set and a circle).

In 2005, Devaney and his group [**BDL+05**, **DL05**, **DLU05**] generalized the work of McMullen and studied a more general family of McMullen maps:

$$f_\lambda(z) = z^n + \frac{\lambda}{z^m} \tag{1.1}$$

with $\lambda \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $n \geq 2, m \geq 1$. They showed that this family exhibits a very rich dynamical behavior while it has such a simple form. In [**DLU05**], the authors showed that if the free critical orbits escape to $\infty$, then its Julia set is either a Cantor set, a Cantor circle, or a Sierpiński carpet. Later on, Devaney and his group published a variety of papers on this McMullen family, see [**Dev04**, **Dev05**, **Dev06**, **Dev07**, **Dev08**, **Dev13**, **DL05**, **DLU05**, **DM07**, **DP09**]. Among these articles, lots of them devoted to the particular case of $n = m$

* The original version of this article contained an error in the name Pascale Roesch. This error has been corrected. A notice detailing this error has been published.

in equation (1.1), that is,

$$f_\lambda(z) = z^n + \frac{\lambda}{z^n} \tag{1.2}$$

with $\lambda \in \mathbb{C}^*$ and $n \geq 3$. In the following, we only study this family.

In the parameter plane, we study the escape locus of this family which consists of those maps such that all critical orbits are attracted by the super-attracting fixed point $\infty$. Every connected component of the escape locus is a hyperbolic component, called an escape component. It was proved in [**QRWY15**] that every escape component is bounded by a Jordan curve. It can also be shown that each hyperbolic component outside the escape locus belongs to a homeomorphic copy of the Mandelbrot set.

In this article, we explore the geometric regularity of the boundaries of escape components. There is a unique unbounded escape component which is called the Cantor locus. Like in the case of polynomials, a map in the Cantor locus has all of its critical values in the Fatou component of $\infty$. Hence, its Julia set is a Cantor set. In [**QRWY15**], the authors proved that cusps are dense on the boundary of the Cantor locus. In the bounded escape components, the McMullen maps behave more like non-polynomial rational maps. A map in a bounded escape component has one of its finite critical values belonging to a strict pre-image of the Fatou component of $\infty$. There is a unique escape component centered at the origin which is called the McMullen domain such that a map in it has its Julia set as a Cantor circle. All the other escape components are called Sierpiński holes since maps in them have Julia sets as Sierpiński carpets. The main purpose of this article is to characterize the geometric property of all bounded escape components. We obtain the following result.

THEOREM 1.1. *The boundary of each bounded escape component is a quasi-circle with Hausdorff dimension strictly between* 1 *and* 2*.*

We show that these geometric properties can be deduced from the following more precise topological characterization about the closures of escape components.

THEOREM 1.2. *The closures of escape components are pairwise disjoint.*

1.2. *Idea of proof.* The proof of Theorem 1.2 proceeds by seeking contradiction. The key step is to show that the boundary of each bounded escape component does not intersect with the boundary of the Cantor locus. To prove this, we first construct a local para-puzzle system in the parameter space which is the counterpart of a dynamical puzzle system introduced in [**QWY12**, **QRWY15**]. With the aid of this para-puzzle system, we can characterize the dynamical behavior of the hypothetical intersection points. We exclude all possibilities except the intersections at parabolic parameters. Then we use the parabolic implosion theory introduced in [**Lei00**, **Shi00**] to exclude the parabolic case. Indeed, on one hand, the intersection point could be accessed by a parameter ray which consists of maps with critical values in a strict pre-image of the Fatou component of $\infty$. On the other hand, according to the parabolic implosion theory, the Fatou coordinate remains stable under a perturbation within a particular sector. After showing that the parameter ray is contained in the sector, we conclude that the maps on this parameter ray should have critical points in the Fatou component of $\infty$ (not in a strict pre-image of the Fatou component of $\infty$). This is a contradiction. Finally, we show that the closures of bounded escape components are pairwise disjoint. One hidden difficulty is to exclude the intersection of two bounded escape components with potentially the same dynamical property. We solve this by establishing a rigidity result of bounded escape components.

To prove Theorem 1.1, we first observe the boundary of the Fatou component of $\infty$ is a quasi-circle with Hausdorff dimension strictly between 1 and 2 for the map which does not belong to the closure of the Cantor locus. Then, Theorem 1.2 allows us to transfer this property to the boundaries of bounded escape components in the parameter space via a holomorphic motion.

1.3. *Outline of the article.* The content is arranged as follows. Section 2 includes some basic terminology and results for the McMullen family, and the construction of the dynamical puzzle system with some relevant results. In §3, we construct a local para-puzzle system which reveals the relation between the position of the parameter and the structure of its corresponding dynamical puzzle system. Section 4 presents a rigidity result of post-critically finite McMullen maps in the escape locus. In §5, we use the local para-puzzle system to study the properties of the maps which belong to the boundaries of escape components. In §6, we use the parabolic implosion theory to exclude the possibility that any bounded escape component and the Cantor locus have no common boundary point. In §7, we finish the proofs of Theorems 1.1 and 1.2.

## 2. *Preliminary*

2.1. *Overview of McMullen family.* The McMullen family in equation (1.2) admits a symmetric conjugacy

$$e^{2\pi i/(n-1)} f_\lambda(z) = (-1)^n f_{e^{2\pi i/(n-1)}\lambda}(e^{2\pi i/(n-1)} z) \tag{2.1}$$

for all $\lambda \in \mathbb{C}^*$. Hence, it suffices to study maps whose parameters belong to the sector

$$\mathscr{F}_0 := \left\{\lambda \in \mathbb{C}^* : 0 \le \arg \lambda \le \frac{2\pi}{n-1}\right\}. \tag{2.2}$$

We denote the interior of $\mathscr{F}_0$ as $\mathscr{F}$.

The critical set of $f_\lambda$ is $\{0, \infty\} \cup C_\lambda$, where $C_\lambda = \{c \in \mathbb{C} : c^{2n} = \lambda\}$. Here, $\infty$ is a super-attracting fixed point of $f_\lambda$ which has only two pre-images 0 and $\infty$. There are only two critical values $v_\lambda^\pm = \pm 2\sqrt{\lambda}$ other than $\infty$ (here, $v_\lambda^\pm$ is well defined for $\lambda \in \mathscr{F}_0$, $v_\lambda^+$ is defined to be the one belonging to $\{z \in \mathbb{C} : 0 \le \arg z < \pi\}$). Let

$$Z_k(\lambda) := \begin{cases} \{\infty\}, & k = 0, \\ \{z \in \mathbb{C} : f_\lambda^{k-2}(z) = 0\}, & k \ge 2. \end{cases} \tag{2.3}$$

For $k \ge 2$, $Z_k(\lambda)$ is the set of all $(k-1)$th iterated pre-images of $\infty$ under $f_\lambda$ with $\infty$ itself excluded.

LEMMA 2.1. (Böttcher coordinate) *For each $z_k(\lambda) \in Z_k(\lambda)$, the Böttcher coordinate $\phi_{z_k(\lambda)}$ near $z_k(\lambda)$ is defined in the following.*

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(1) *For* $k = 0$, $z_0(\lambda) = \infty$, $\phi_{\infty(\lambda)}$ *is defined to be*

$$\phi_{\infty(\lambda)}(z) = \lim_{k\to\infty} \sqrt[n^k]{f_\lambda^k(z)} \tag{2.4}$$

*with* $\phi'_{\infty(\lambda)}(\infty) = 1$. *It satisfies the equation* $\phi_{\infty(\lambda)}(f_\lambda(z)) = (\phi_{\infty(\lambda)}(z))^n$ *and* $\phi_{\infty(\lambda)}(e^{\pi i/n}z) = e^{\pi i/n}\phi_{\infty(\lambda)}(z)$.

(2) *For* $k = 2$, $z_2(\lambda) = 0$, $\phi_{0(\lambda)}$ *is defined to be* $\phi_{0(\lambda)} := \sqrt[n]{\phi_{\infty(\lambda)} \circ f_\lambda}$ *which satisfies that* $\phi'_{0(\lambda)}(0) = \sqrt[n]{\lambda}$.

(3) *For* $k \geq 3$ *and* $z_k(\lambda) \in Z_k(\lambda) \setminus \bigcup_{l\geq 0} f_\lambda^{-l}(C_\lambda)$, $\phi_{z_k(\lambda)}$ *is defined to be* $\phi_{z_k(\lambda)} := \phi_{0(\lambda)} \circ f_\lambda^{k-2}$.

*Remark 2.1.* Let $\mathbb{D}(a, r)$ denote the disk with the center at $a$ and radius $r$, and let $\mathbb{D}_r = \mathbb{D}(0, r)$. By §9 in [**Mil11**], $\phi^{-1}_{z_k(\lambda)}$ can be extended on a maximal disk $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}_{s_{z_k}(\lambda)}}$ with $s_{z_k}(\lambda) = 1$ or with $s_{z_k}(\lambda) > 1$ such that $\partial\phi^{-1}_{z_k(\lambda)}(\mathbb{D}_{s_{z_k}(\lambda)}) \cap \bigcup_{l\geq 0} f_\lambda^{-l}(C_\lambda) \neq \emptyset$.

Each Böttcher coordinate $\phi_{z_k(\lambda)}$ introduces a system of dynamical rays and equipotential curves near $z_k(\lambda)$. The dynamical ray $R^t_{z_k(\lambda)}$ with angle $t \in \mathbb{R}/\mathbb{Z}$ is defined to be $R^t_{z_k(\lambda)} := \phi^{-1}_{z_k(\lambda)}((s_{z_k}(\lambda), \infty]e^{2\pi i t})$. The equipotential curve $E^s_{z_k(\lambda)}$ with $s \geq s_{z_k}$ is defined to be $E^s_{z_k(\lambda)} := \phi^{-1}_{z_k(\lambda)}(se^{2\pi i\mathbb{R}/\mathbb{Z}})$.

Consider the subset of the parameter plane, called the escape locus, which consists of parameters of maps all of whose critical orbits escape to infinity. That is,

$$\mathscr{H} := \{\lambda \in \mathbb{C}^* : \lim_{k\to\infty} f_\lambda^k(C_\lambda) = \infty\}.$$

The escape locus $\mathscr{H}$ is an open subset of the parameter plane. Maps inside $\mathscr{H}$ are called escape maps. Connected components of $\mathscr{H}$ are called escape components. It is worth mentioning that all escape components are hyperbolic components. According to the discussion in [**QRWY15**], hyperbolic components not in $\mathscr{H}$ have ‘renormalizable’ type. It could be proved that all hyperbolic components of this type belong to small copies of the Mandelbrot set, see Figures 1 and 2.

Let $B_\lambda$ denote the Fatou component containing $\infty$. Let $T_\lambda$ denote the Fatou component containing 0, which is the unique component of $f_\lambda^{-1}(B_\lambda)$ different from $B_\lambda$ itself. We can further distinguish those escape components by counting the number of iterations needed for $C_\lambda$ to be mapped into $B_\lambda$. For $\lambda \in \mathscr{H}$, we define its order $N(\lambda)$ to be

$$N(\lambda) = \min\{k \in \mathbb{N} : f_\lambda^k(C_\lambda) \subset B_\lambda\}.$$

It is obvious that the order $N(\lambda)$ takes a constant value on each component of $\mathscr{H}$. Hence, we can decompose the escape locus such that $\mathscr{H} = \bigcup_{k\geq 0} \mathscr{H}_k$, where $\mathscr{H}_k = \{\lambda \in \mathscr{H} : N(\lambda) = k\}$ consists of all escape maps with order $k$.

*Remark 2.2.* It is known that $\lambda \in \mathscr{H}_0$ if and only if $v_\lambda^\pm \in B_\lambda$, $\mathscr{H}_1 = \emptyset$, and $\lambda \in \mathscr{H}_k$ for $k \geq 2$ if and only if $f_\lambda^{k-1}(v_\lambda^\pm) \in T_\lambda$.

According to [**DLU05**, **Roe06**, **Ste06**], we have the following escape trichotomy and parameterization for escape components.

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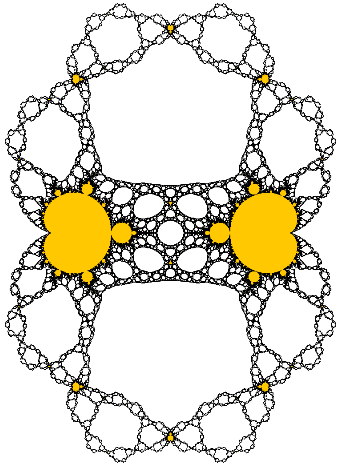

FIGURE 1. Parameter plane for $n = 3$.

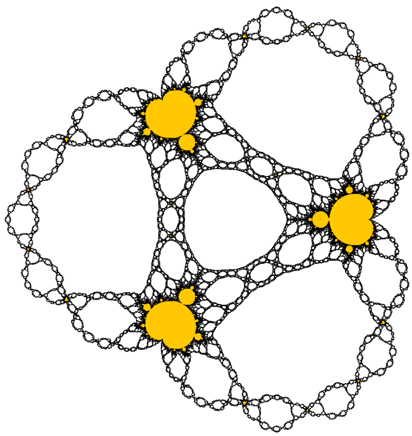

FIGURE 2. Parameter plane for $n = 4$.

THEOREM 2.2. (Escape trichotomy and parameterization) *We have the following trichotomy and the parameterization.*

(1) *The Cantor locus $\mathscr{H}_0$ is the unique unbounded hyperbolic component. The parameterization map $\Phi_{\mathscr{H}_0} : \mathscr{H}_0 \to \mathbb{C} \setminus \overline{\mathbb{D}}$ defined by $\Phi_{\mathscr{H}_0}(\lambda) := \phi_{\infty(\lambda)}(v_\lambda^+)^2$ is a holomorphic homeomorphism. For $\lambda \in \mathscr{H}_0$, its Julia set $J(f_\lambda)$ is a Cantor set, see Figure 3.*

(2) *The McMullen domain $\mathscr{H}_2$ is the unique hyperbolic component containing 0. The parameterization map $\Phi_{\mathscr{H}_2} : \mathscr{H}_2 \to \mathbb{C} \setminus \overline{\mathbb{D}}$ defined by $\Phi_{\mathscr{H}_2}(\lambda)^{n-2} := (\phi_{\infty(\lambda)} \circ f_\lambda(v_\lambda^+))^2$ is a holomorphic homeomorphism with $\lim_{\lambda\to 0} \lambda\Phi_{\mathscr{H}_2}(\lambda) = 2^{2n/(n-2)}$. For $\lambda \in \mathscr{H}_2$, its Julia set $J(f_\lambda)$ is a Cantor circle, see Figure 4.*

(3) *Sierpiński holes are connected components of $\mathscr{H}_k$ with order $k \geq 3$. For each Sierpiński hole $\mathscr{U}$, the parameterization map $\Phi_{\mathscr{U}} : \mathscr{U} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ defined by*

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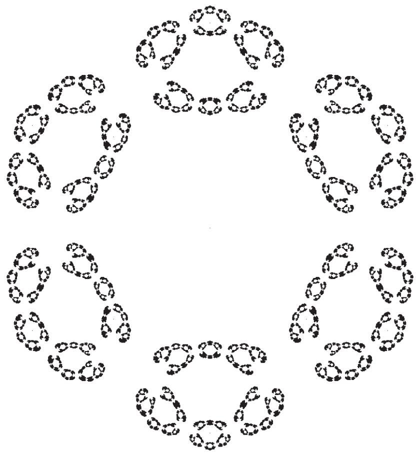

FIGURE 3. For $\lambda \in \mathscr{H}_0$ in Cantor locus, $J(f_\lambda)$ is a Cantor set.

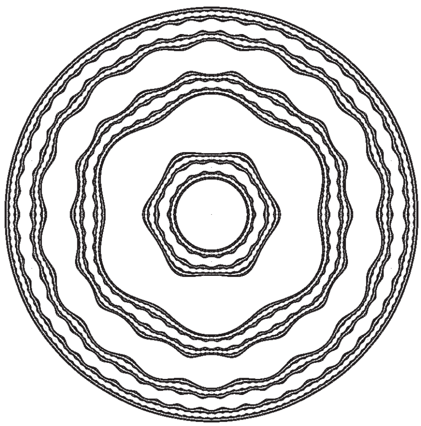

FIGURE 4. For $\lambda \in \mathscr{H}_2$ in the McMullen domain, $J(f_\lambda)$ is a Cantor circle.

*$\Phi_{\mathscr{U}}(\lambda) := \phi_{0(\lambda)} \circ f_\lambda^{k-2}(v_\lambda^+)$ is a holomorphic homeomorphism. For $\lambda \in \mathscr{U}$, its Julia set $J(f_\lambda)$ is a Sierpiński carpet, see Figure 5.*

*Remark 2.3.* Hyperbolic components not in $\mathscr{H}$ are called non-escape components. All non-escape hyperbolic components are contained in small copies of the Mandelbrot set. A map belonging to a non-escape hyperbolic component has its critical orbit $\bigcup_{k\geq 0} f_\lambda^k(C_\lambda)$ attracted by attracting periodic orbits different from $\infty$, see Figure 6.

It is easy to verify that for $\lambda \in \mathscr{H}_0$,

$$\Phi_{\mathscr{H}_0}(e^{2\pi i/(n-1)}\lambda) = e^{2\pi i/(n-1)}\Phi_{\mathscr{H}_0}(\lambda) \tag{2.5}$$

and

$$\Phi_{\mathscr{H}_0}(\overline{\lambda}) = \overline{\Phi_{\mathscr{H}_0}(\lambda)}. \tag{2.6}$$

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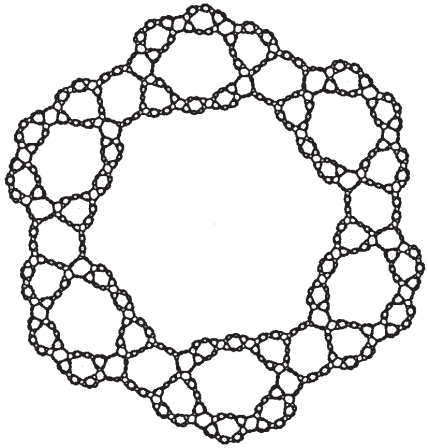

FIGURE 5. For λ in a Sierpiński hole, $J(f_\lambda)$ is a Sierpiński carpet.

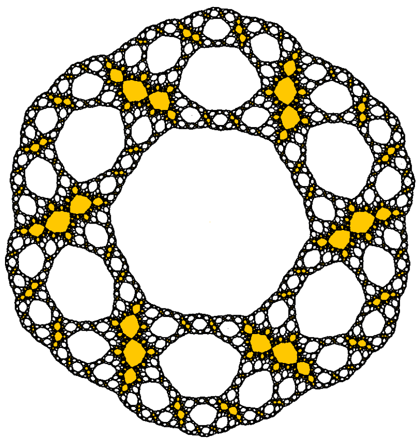

FIGURE 6. For $f_\lambda$ is a non-escape hyperbolic map, $J(f_\lambda)$ is connected.

Let $\mathscr{U} \subset \mathscr{H}$ be an escape component. The parameter ray $\mathscr{R}^t_{\mathscr{U}}$ in $\mathscr{U}$ with angle $t \in \mathbb{R}/\mathbb{Z}$ is defined to be $\mathscr{R}^t_{\mathscr{U}} := \Phi^{-1}_{\mathscr{U}}((1, \infty)e^{2\pi i t})$. The equipotential curve $\mathscr{E}^s_{\mathscr{U}}$ in $\mathscr{U}$ with level $s$ is defined to be $\mathscr{E}^s_{\mathscr{U}} := \Phi^{-1}_{\mathscr{U}}(se^{2\pi i \mathbb{R}/\mathbb{Z}})$.

2.2. *Dynamical puzzles.* Let $\Lambda \subset \mathbb{C}$ be a hyperbolic region and $X$ be a subset of $\overline{\mathbb{C}}$. Let $\pi_1 : \Lambda \times X \to \Lambda$ and $\pi_2 : \Lambda \times X \to X$ be two projections defined by $\pi_1(\lambda, z) = \lambda$ and $\pi_2(\lambda, z) = z$ respectively. A holomorphic motion of $X$, parameterized by $\Lambda$, with the base point at $\lambda_0 \in \Lambda$, is a map $h : \Lambda \times X \to \Lambda \times \overline{\mathbb{C}}$ such that:

(1) for each $x \in X$, the map $\lambda \mapsto \pi_1 \circ h(\lambda, x)$ is the identity map;

(2) for each $x \in X$, the map $\lambda \mapsto \pi_2 \circ h(\lambda, x)$ is holomorphic;

(3) for each $\lambda \in \Lambda$, the map $x \mapsto \pi_2 \circ h(\lambda, x)$ is injective;

(4) the map $x \mapsto \pi_2 \circ h(\lambda_0, x)$ is the identity map.

Let us denote $h_\lambda : X \to \overline{\mathbb{C}}$ for the map $x \mapsto \pi_2 \circ h(\lambda, x)$.

THEOREM 2.3. (Slodkowski [**Dou95**, **Slo95**]) *Suppose* $h : \Lambda \times X \to \Lambda \times \overline{\mathbb{C}}$ *is a holomorphic motion, then* $h$ *can be extended to a holomorphic motion* $h : \Lambda \times \overline{\mathbb{C}} \to \Lambda \times \overline{\mathbb{C}}$*. For the extended holomorphic motion* $h$*, for each* $\lambda \in \Lambda$*, the map* $h_\lambda : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *is a quasi-conformal homeomorphism. Furthermore, the Beltrami coefficient* $\mu_\lambda = \bar{\partial} h_\lambda / \partial h_\lambda$ *of* $h_\lambda$ *satisfies*

$$\|\mu_\lambda\|_\infty = \operatorname*{ess\,sup}_{x \in X} |\mu_\lambda(x)| \leq \frac{e^{\rho(\lambda,\lambda_0)} - 1}{e^{\rho(\lambda,\lambda_0)} + 1}, \tag{2.7}$$

*where* $\rho(\lambda, \lambda_0)$ *is the hyperbolic distance between* $\lambda$ *and* $\lambda_0$ *in* $\Lambda$*.*

Let $\tau : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be defined by $\tau(\theta) = n\theta \bmod \mathbb{Z}$. Let $\Theta_k = (k/2n, (k+1)/2n]$ for $0 \leq k \leq n$ and $\Theta_{-k} = (k/2n + \frac{1}{2}, (k+1)/2n + \frac{1}{2}]$ for $1 \leq k \leq n-1$. Obviously, $(0, 1] = \bigcup_{-n<j\leq n} \Theta_j$. Let $\Theta$ be the set of all angles $\theta \in (0, 1]$ whose orbits remain in $\bigcup_{k=1}^{n-1}(\Theta_k \cup \Theta_{-k})$ under iterations of $\tau$. Let

$$\Theta_{per} = \left( \bigcup_{p\geq 1} \{\theta \in \Theta : \tau^p \theta = \theta\} \right) \setminus \left\{1, \frac{1}{2}\right\}.$$

Then, $\Theta$ is a Cantor set and $\Theta_{per}$ is a dense subset of $\Theta$. Following [**Dev04**, **QRWY15**], for each $\theta \in \Theta$, there is a cut ray $\Omega_\lambda^\theta$ which cuts the Julia set into two parts. Instead of giving the precise definition of cut rays, we summarize the essential properties of cut rays in the following Theorem 2.4. It is the combination of Theorem 3.2, Lemma 3.3, Theorem 3.4 in [**QRWY15**].

THEOREM 2.4. (Properties of cut rays) *For any* $\theta \in \Theta_{per}$ *and* $\lambda \in \mathscr{F}_0$*, the cut ray* $\Omega_\lambda^\theta$ *with angle* $\theta$ *is a Jordan curve containing* 0 *and* $\infty$ *and symmetric with respect to* 0 *satisfies the following properties:*

(1) $f_\lambda^p(\Omega_\lambda^\theta) \subset \Omega_\lambda^\theta$*, where* $p$ *is the period of* $\theta$*;*
(2) $\Omega_\lambda^\theta \cap J(f_\lambda)$ *is a Cantor set, and* $\Omega_\lambda^\theta \cap B_\lambda = R_{\infty(\lambda)}^\theta \cup R_{\infty(\lambda)}^{\theta+1/2}$*;*
(3) *fix any* $\lambda_0 \in \mathscr{F}$*, there is a holomorphic motion* $h : \mathscr{F} \times \Omega_{\lambda_0}^\theta \to \mathscr{F} \times \overline{\mathbb{C}}$ *based at* $\lambda_0$ *such that* $h_\lambda(\Omega_{\lambda_0}^\theta) = \Omega_\lambda^\theta$*;*
(4) *fix any* $\lambda_0 \in \mathbb{R}_+$*, there exists a neighborhood* $\mathscr{W}_\theta$ *containing* $\mathbb{R}_+$ *and a holomorphic motion* $h : \mathscr{W}_\theta \times \Omega_{\lambda_0}^\theta \to \mathscr{W}_\theta \times \overline{\mathbb{C}}$ *based at* $\lambda_0$ *such that* $h_\lambda(\Omega_{\lambda_0}^\theta) = \Omega_\lambda^\theta$*.*

If for some $N \geq 1$, $(\Omega_\lambda^\theta \setminus \{0, \infty\}) \cap (\bigcup_{1\leq k\leq N} f_\lambda^k(C_\lambda)) = \emptyset$, then for any $\alpha \in \bigcup_{0\leq k\leq N} \tau^{-k}(\theta)$, there is a unique Jordan curve $\Omega_\lambda^\alpha$ containing 0 and $\infty$ such that $f_\lambda(\Omega_\lambda^\alpha) = \Omega_\lambda^{\tau(\alpha)}$ and $R_{\infty(\lambda)}^\alpha \cup R_{\infty(\lambda)}^{\alpha+1/2} = \Omega_\lambda^\alpha \cap B_\lambda$. Under this circumstance, the Jordan curve $\Omega_\lambda^\alpha$ is also called a cut ray (see Figure 7).

LEMMA 2.5. [**QRWY15**, Lemma 3.7] *Let* $\lambda \in \mathscr{F}_0$*, and* $R_{\infty(\lambda)}^{t_0}$ *and* $R_{\infty(\lambda)}^{t_1}$ *be two dynamical rays with distinct angles* $t_0$ *and* $t_1$*. If*

$$(\Omega_\lambda^\theta \setminus \{0, \infty\}) \cap \left( \bigcup_{k\geq 1} f_\lambda^k(C_\lambda) \right) = \emptyset \tag{2.8}$$

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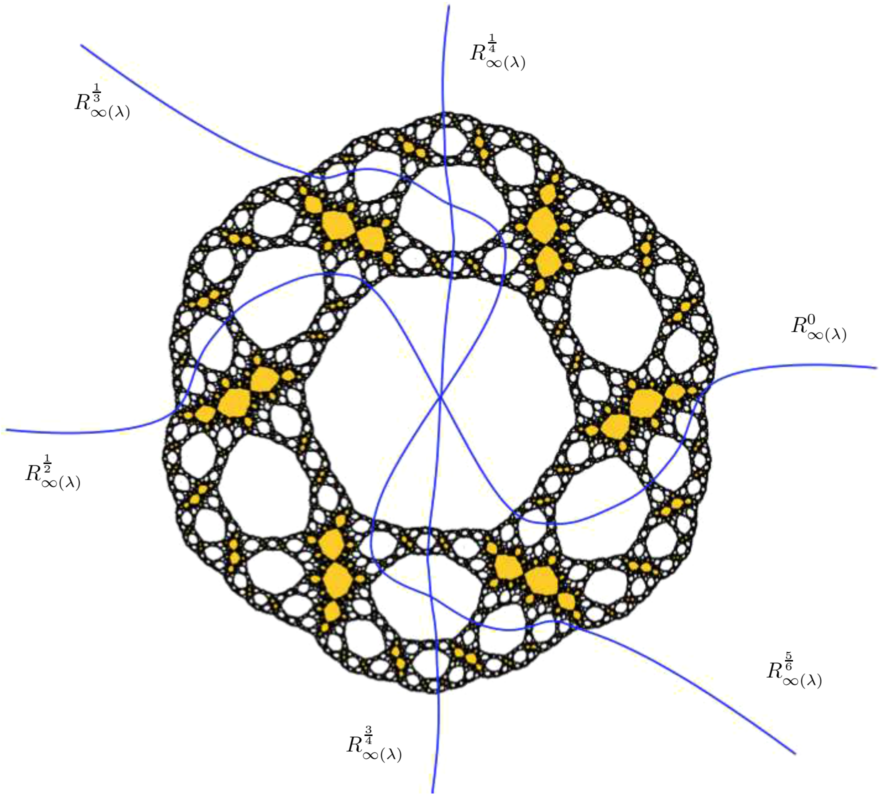


FIGURE 7. Cut ray $\Omega_\lambda^\theta$ with angle $\theta = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, (n = 3)$.

*for $\theta \in \Theta_{per}$, then there is a cut ray $\Omega_\lambda^\alpha$ with angle $\alpha \in \bigcup_{k\geq 0} \tau^{-k}(\theta)$ separating $R^{t_0}_{\infty(\lambda)}$ from $R^{t_1}_{\infty(\lambda)}$.*

In [**QWY12**], the authors used cut rays to construct a puzzle system for the McMullen family and studied the local connectivity of Julia sets. Puzzles are a regular tool in studying holomorphic dynamics, see [**BH92**, **Hub93**, **KL09**, **KSvS07**, **LvS98**, **Pet96**, **QY09**, **Roe07**, **Roe08**, **Sch04**].

For any $L > 1$, denote $X_\lambda^L := \{z \in B_\lambda : |\phi_{\infty(\lambda)}(z)| > L\}$. Given a parameter $\lambda \in \mathscr{F}_0$, we can find $\Theta_\lambda = \{\theta_1, \theta_2, \ldots, \theta_N\} \subset \Theta_{per}$ and $L > 1$ such that each cut ray $\Omega_\lambda^\theta$ with $\theta \in \Theta_\lambda$ is well defined and $C_\lambda \cap X_\lambda^L = \emptyset$. In the dynamical plane of $f_\lambda$, the graph of depth 0 associated with $\Theta_\lambda$ is defined to be

$$I_\lambda^0(\Theta_\lambda) = \partial X_\lambda^L \cup ((\mathbb{C} \setminus X_\lambda^L) \cap \bigcup_{m\geq 0} (\Omega_\lambda^{\tau^m(\theta_1)} \cup \Omega_\lambda^{\tau^m(\theta_2)} \cup \cdots \cup \Omega_\lambda^{\tau^m(\theta_N)})).$$

The graph of depth $k$ is defined to be $I_\lambda^k = f_\lambda^{-k}(I_\lambda^0(\Theta_\lambda))$. A puzzle piece $P_\lambda^k$ of depth $k$ is a connected component of $f_\lambda^{-k}((\mathbb{C} \setminus X_\lambda^L) \setminus I_\lambda^0)$. The puzzle piece of depth $k$ containing $z \in J(f_\lambda)$ is denoted by $P_\lambda^k(z)$. In [**QRWY15**, **QWY12**], the authors used the puzzle

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system $(f_\lambda, I_\lambda^k, P_\lambda^k)$ to study the dynamics of McMullen maps and obtained the following results (Theorems 1.1, 1.2, 1.4 in [**QWY12**] and Theorem 1.1 in [**QRWY15**]).

THEOREM 2.6. *For the McMullen family, $f_\lambda(z) = z^n + \lambda/z^n$ with $n \geq 3$ and $\lambda \in \mathbb{C}^*$.*

(1) *If $\lambda \notin \mathscr{H}_0$, then $\partial B_\lambda$ is a Jordan curve. Furthermore, if $\partial B_\lambda$ contains neither a parabolic point nor a recurrent critical point, then $\partial B_\lambda$ is a quasi-circle.*
(2) *Suppose $\mathscr{U} \subset \mathscr{H}_k$ is an escape component with $k \geq 2$ and $\lambda \in \partial\mathscr{U} \setminus \partial\mathscr{H}_0$, then for each $z \in J(f_\lambda)$, $\bigcap_{k\geq 0} P_\lambda^k(z) = \{z\}$ and $J(f_\lambda)$ is locally connected.*
(3) *Suppose $\mathscr{U} \subset \mathscr{H}$ is an escape component, then $\partial\mathscr{U}$ is a Jordan curve.*

By part (1) of Theorem 2.6, for $\lambda \notin \mathscr{H}_0$, the Böttcher coordinate $\phi_{\infty(\lambda)}$ can be extended to a homeomorphism $\phi_{\infty(\lambda)} : \overline{B_\lambda} \to \overline{\mathbb{C}} \setminus \mathbb{D}$. By part (3) of Theorem 2.6, for each escape component $\mathscr{U} \subset \mathscr{H}$, the parameterization map $\Phi_{\mathscr{U}}$ defined in Theorem 2.2 can be extended to a homeomorphism on its closure.

## 3. *Local para-puzzles*

In this section, we construct a local-puzzle system at $\lambda_0 \in \mathscr{F} \setminus \mathscr{H}$ to study the bifurcation of the puzzle systems $(f_\lambda, I_\lambda^k, P_\lambda^k)$ when the parameter $\lambda$ varies near $\lambda_0$ (see Figures 8 and 9). This is a local version of the para-puzzle constructed for polynomials, see [**ALS10**, **Fau93**, **Hub93**, **Lyu00**, **Roe00**, **Roe07**].

PROPOSITION 3.1. (Existence of para-puzzles) *For each $\lambda_0 \in \mathscr{F}_0 \setminus \mathscr{H}$, there exists a sequence of simply connected open neighborhoods $\{\mathscr{P}_{\lambda_0}^k\}_{k\geq 0}$ of $\lambda_0$, which are called para-puzzle pieces, such that the following hold.*

(1) $\mathscr{P}_{\lambda_0}^0 \supset \mathscr{P}_{\lambda_0}^1 \supset \cdots \supset \mathscr{P}_{\lambda_0}^k \supset \cdots \supset \{\lambda_0\}$.
(2) *For each $k \geq 0$, there exists a holomorphic motion $H_k : \mathscr{P}_{\lambda_0}^k \times I_{\lambda_0}^k \to \mathscr{P}_{\lambda_0}^k \times \overline{\mathbb{C}}$ such that for each $\lambda \in \mathscr{P}_{\lambda_0}^k$, $H_k(\lambda, I_{\lambda_0}^k) = (\lambda, I_\lambda^k)$.*

*Proof.* Proposition 3.1 will be proved by induction. By parts (3) and (4) of Theorem 2.4, there exists a simply connected region $\mathscr{W}$ containing $\lambda_0$ such that $h : \mathscr{W} \times \bigcup_{\theta\in\Theta_{\lambda_0}} \Omega_{\lambda_0}^\theta \to \mathscr{W} \times \overline{\mathbb{C}}$ is a holomorphic motion. Here, $\Theta_{\lambda_0}$ is a set of angles $\theta$ of cut rays $\Omega_\lambda^\theta$ we picked to construct the graph $I_{\lambda_0}^0(\Theta_{\lambda_0})$. It is not hard to find a simply connected region $\mathscr{P}_{\lambda_0}^0 \subset \mathscr{W}$ containing $\lambda_0$ such that the map $H_0 : \mathscr{P}_{\lambda_0}^0 \times I_{\lambda_0}^0 \to \mathscr{P}_{\lambda_0}^0 \times \overline{\mathbb{C}}$ defined by

$$H_0(\lambda, z) := \begin{cases} h(\lambda, z), & (\lambda, z) \in \mathscr{P}_{\lambda_0}^0 \times \bigcup_{\theta\in\Theta_{\lambda_0}} \Omega_{\lambda_0}^\theta, \\ (\lambda, \phi_{\infty(\lambda)}^{-1} \circ \phi_{\infty(\lambda_0)}(z)), & (\lambda, z) \in \mathscr{P}_{\lambda_0}^0 \times \partial X_{\lambda_0}^L \end{cases} \tag{3.1}$$

is a well-defined holomorphic motion which satisfies $H_0(\lambda, I_{\lambda_0}^0) = (\lambda, I_\lambda^0)$.

Assume the para-puzzle pieces $\mathscr{P}_{\lambda_0}^m$ and the holomorphic motion $H_m : \mathscr{P}_{\lambda_0}^m \times I_{\lambda_0}^m \to \mathscr{P}_{\lambda_0}^m \times \overline{\mathbb{C}}$ are already constructed for $0 \leq m \leq k-1$. The local parameter graph of depth $k-1$ in the para-puzzle piece $\mathscr{P}_{\lambda_0}^{k-1}$ is defined to be

$$\mathscr{I}^{k-1} := \{\lambda \in \mathscr{P}_{\lambda_0}^{k-1} : v_\lambda^+ \in I_\lambda^{k-1}\}.$$

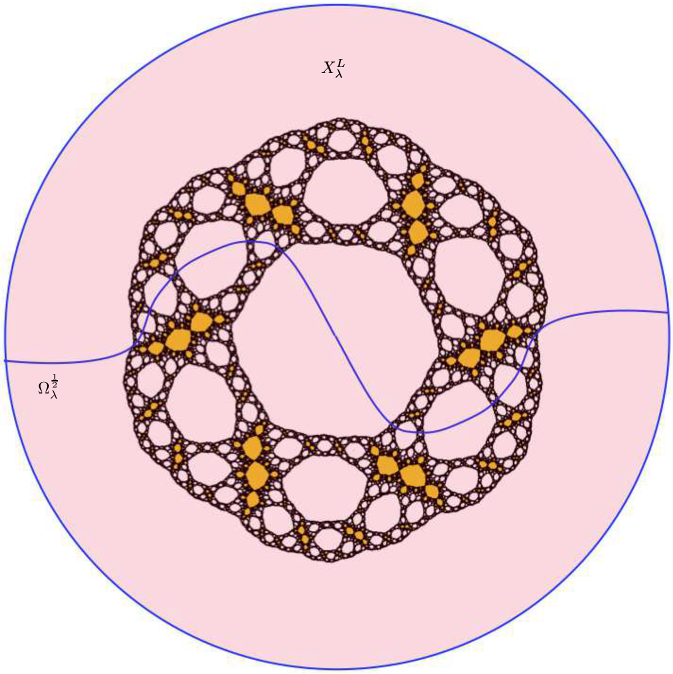


FIGURE 8. The graph $I^0_\lambda(\{\frac{1}{2}\})$ for $n = 3$.

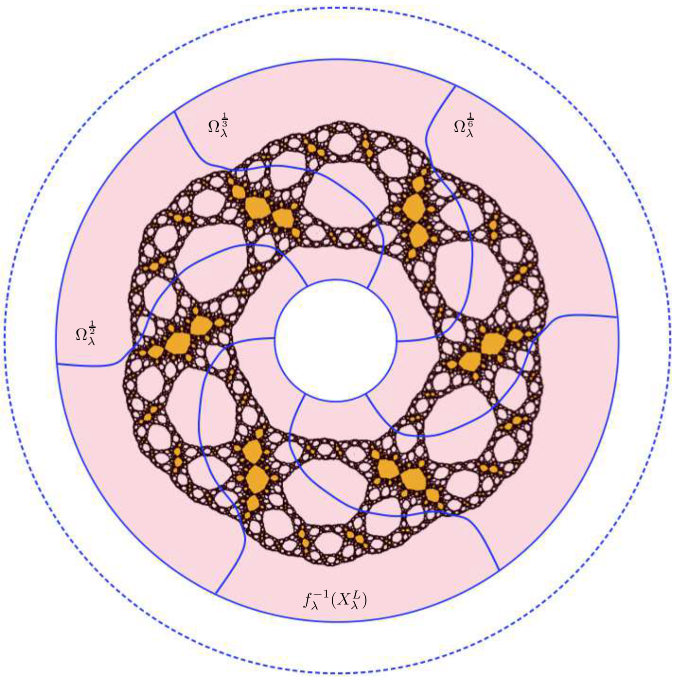


FIGURE 9. The graph $I^1_\lambda(\{\frac{1}{2}\})$ for $n = 3$.

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By the construction of puzzle system $(f_{\lambda_0}, I^k_{\lambda_0}, P^k_{\lambda_0})$, $v^+_{\lambda_0} \notin I^{k-1}_{\lambda_0}$. Hence, $\lambda_0 \notin \mathscr{I}^{k-1}$. Let $\lambda_j \to \lambda \in \mathscr{P}^{k-1}_{\lambda_0}$ as $j \to \infty$ with $\{\lambda_j : j \geq 1\} \subset \mathscr{I}^{k-1}$. Since $H_{k-1}$ is a holomorphic motion, $v^+_{\lambda_j} \in I^{k-1}_{\lambda_j}$ for each $j \geq 1$ implies that $v^+_\lambda \in I^{k-1}_\lambda$. Hence, $\mathscr{I}^{k-1}$ is relatively closed in $\mathscr{P}^{k-1}_{\lambda_0}$. Define $\mathscr{P}^k_{\lambda_0}$ to be a simply connected open subset of $\mathscr{P}^{k-1}_{\lambda_0} \setminus \mathscr{I}^{k-1}$ containing $\lambda_0$ ($\mathscr{P}^k_{\lambda_0} = \mathscr{P}^{k-1}_{\lambda_0}$ if $\mathscr{I}^{k-1} = \emptyset$). Since $v^+_\lambda \notin I^{k-1}_\lambda$ for $\lambda \in \mathscr{P}^k_{\lambda_0}$, there exists a holomorphic motion $H_k$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{P}^k_{\lambda_0} \times I^k_{\lambda_0} & \xrightarrow{H_k} & \mathscr{P}^k_{\lambda_0} \times \overline{\mathbb{C}} \\ \Big\downarrow i.\times f_{\lambda_0} & & \Big\downarrow i.\times f_\lambda \\ \mathscr{P}^{k-1}_{\lambda_0} \times I^{k-1}_{\lambda_0} & \xrightarrow{H_{k-1}} & \mathscr{P}^{k-1}_{\lambda_0} \times \overline{\mathbb{C}}, \end{array} \tag{3.2}$$

where $i. : \mathscr{P}^k_{\lambda_0} \to \mathscr{P}^{k-1}_{\lambda_0}$ denotes the inclusion map. By diagram (3.2), $H_k(\lambda, I^k_{\lambda_0}) = (\lambda, I^k_\lambda)$. Hence, Proposition 3.1 holds for $m = k$. □

COROLLARY 3.2. *There exists a holomorphic motion* $h_k : \mathscr{P}^k_{\lambda_0} \times (I^{k-1}_{\lambda_0} \cup \{v^+_{\lambda_0}\}) \to \mathscr{P}^k_{\lambda_0} \times \overline{\mathbb{C}}$ *such that* $h_k|_{\mathscr{P}^k_{\lambda_0} \times I^{k-1}_{\lambda_0}} = H_{k-1}$ *and* $h_k(\lambda, v^+_{\lambda_0}) = (\lambda, v^+_\lambda)$.

*Proof.* This follows immediately from $v^+_\lambda \notin I^{k-1}_\lambda$ for $\lambda \in \mathscr{P}^k_{\lambda_0}$. □

## 4. *Centers of* $\mathscr{H}$

In this section, we discuss the relation between the centers of hyperbolic components of $\mathscr{H}$ and the iterated pre-images of $\infty$ in the dynamical plane.

Suppose that $\mathscr{U}_k$ is a connected component of $\mathscr{H}_k$, $k \geq 2$. Recall the parameterization map $\Phi_{\mathscr{U}_k}$ defined in Theorem 2.2. The center of $\mathscr{U}_k$ is defined to be $\lambda_{\mathscr{U}_k} := \Phi^{-1}_{\mathscr{U}_k}(\infty)$ (for the McMullen domain $\mathscr{H}_2$, define $\lambda_{\mathscr{H}_2} := 0$), which satisfies the equation $f^{k-2}_\lambda(v^+_\lambda) = 0$. Let $\Lambda_k := \{\lambda \in \mathbb{C}^* : f^{k-2}_\lambda(v^+_\lambda) = 0\}$. Then $\Lambda_k$ is a finite set since $f^{k-2}_\lambda(v^+_\lambda) = 0$ is an algebraic equation. It follows that $\mathscr{H}_k$ has finitely many connected components since the number of connected components of $\mathscr{H}_k$ is equal to $\#\Lambda_k$.

LEMMA 4.1. *Let* $\mathscr{V} \subset \mathbb{C}^* \setminus \bigcup_{2\leq j\leq k-1} \Lambda_j$ *be a simply connected region, then there exist* $(2n)^{k-2}$ *distinct holomorphic functions* $z^i_k(\lambda)$ *defined on* $\mathscr{V}$ *such that* $f^{k-2}_\lambda(z^i_k(\lambda)) = 0$, $i = 1, 2, \ldots, (2n)^{k-2}$.

*Proof.* Consider the algebraic equation

$$f^{k-2}_\lambda(z) = 0. \tag{4.1}$$

For $k = 2$, Lemma 4.1 is trivial. For $k \geq 3$, since $\mathscr{V} \subset \mathbb{C}^* \setminus \bigcup_{2\leq j\leq k-1} \Lambda_j$, for each $\lambda \in \mathscr{V}$, the solution $z$ of equation (4.1) cannot be a critical point of $f^{k-1}_\lambda$. For otherwise, by

$$(f^{k-2}_\lambda)'(z) = \prod_{0\leq i\leq k-3} (f_\lambda)'(f^i_\lambda(z)),$$

there is an $0 \leq i \leq k-3$ such that $f^i_\lambda(z) \in C_\lambda$ which implies $f^{i+1}_\lambda(z) = v^\pm_\lambda$. Hence, $f^{k-3-i}_\lambda(v^+_\lambda) = 0$, which implies that $\lambda \in \Lambda_{k-i-1}$. This is a contradiction. It follows that

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equation (4.1) has $(2n)^{k-2}$ distinct roots $z_i$ for $1 \le i \le (2n)^{k-2}$. By the implicit function theorem, there exist $(2n)^{k-2}$ distinct holomorphic functions $z_k^i(\lambda)$ defined near $\lambda_0$ such that $f_\lambda^{k-2}(z_k^i(\lambda)) = 0$ and $z_k^i(\lambda_0) = z_i$ for $1 \le i \le (2n)^{k-2}$. Since $\mathscr{V}$ is simply connected, the $(2n)^{k-2}$ holomorphic functions $z_k^i(\lambda)$ can be extended to the whole region $\mathscr{V}$. □

Define $z_0 : \mathbb{C}^* \to \mathbb{C}$ to be the constant map $z_0(\lambda) = \infty$. The functions $z_0$ and each $z_k^i$ for $1 \le i \le (2n)^{k-2}$ defined in Lemma 4.1 are called root functions.

Recall the definition of Thurston's combinatorial equivalence and Thurston's rigidity theorem, see [**DH93**, **Hub06**].

*Definition 4.2.* (Thurston's combinatorial equivalence) Let $f, g : S^2 \to S^2$ be post-critically finite branched coverings with post-critical sets $P(f)$ and $P(g)$, respectively. Suppose that there exist two orientation-preserving homeomorphisms $\phi, \psi$ from $S^2$ to itself such that the following diagram commutes.

$$\begin{array}{ccc} (S^2, P(f)) & \xrightarrow{\psi} & (S^2, P(g)) \\ \downarrow f & & \downarrow g \\ (S^2, P(f)) & \xrightarrow{\phi} & (S^2, P(g)) \end{array} \tag{4.2}$$

Furthermore, $\phi$ and $\psi$ satisfy that $\phi|_{P(f)} = \psi|_{P(f)}$, and $\phi$ and $\psi$ are isotopic to each other relatively to the post-critical set $P(f)$. Then we say the maps $f$ and $g$ are Thurston combinatorially equivalent.

*Remark 4.1.* If $\phi$ and $\psi$ are both quasi-conformal homeomorphisms, then the condition of isotopy in Definition 4.2 can be replaced by homotopy, see [**Hub06**].

THEOREM 4.3. (Thurston's rigidity) *If two post-critically finite rational maps $f, g$ are Thurston combinatorially equivalent, then $f, g$ are conformally conjugated.*

PROPOSITION 4.4. *Let $k \ge 3$ and $\mathscr{V} \subset \mathbb{C}^* \setminus \bigcup_{2\le j\le k-1} \Lambda_j$ be a simply connected region. For $\lambda_1, \lambda_2 \in \mathscr{V} \cap \mathscr{F}_0$, if there exists a root function $z_k^i$ such that $z_k^i(\lambda_1) = v_{\lambda_1}^+$ and $z_k^i(\lambda_2) = v_{\lambda_2}^+$, then $\lambda_1 = \lambda_2$.*

*Proof.* Let $\mathscr{W} := \{\lambda \in \mathbb{C}^* : f_\lambda^l(v_\lambda^+), 0 \le l \le k-3, \text{are k-2 distinct points}\}$. It is obvious that $\mathbb{C}^* \setminus \mathscr{W}$ is a finite set, so we may assume that $\mathscr{V} \subset \mathscr{W}$. Define $\Phi : \mathscr{V} \times P(f_{\lambda_1}) \to \mathscr{V} \times \overline{\mathbb{C}}$ by

$$\Phi(\lambda, f_{\lambda_1}^l(v_{\lambda_1}^\pm)) := \begin{cases} (\lambda, f_\lambda^l(v_\lambda^\pm)), & 0 \le l \le k-3, \\ (\lambda, \mathrm{id}), & l \ge k-2. \end{cases} \tag{4.3}$$

It is easy to check that $\{f_\lambda^l(v_\lambda^\pm) : 0 \le l \le k-3\} \cup \{0, \infty\}$ contains exactly $2k-2$ (or $k+1$) distinct points for $\lambda \in \mathscr{W} \setminus \bigcup_{j\le k-1} \Lambda_j$ if $n$ is odd (or even). Each of them is holomorphically dependent on the parameter $\lambda$. This implies that $\Phi$ is a holomorphic motion based at $\lambda_1$. By Theorem 2.3, $\Phi$ can be extended to a holomorphic motion $\Phi : \mathscr{V} \times \overline{\mathbb{C}} \to \mathscr{V} \times \overline{\mathbb{C}}$. Let $F(\lambda, z) := (\lambda, f_\lambda(z))$ for $(\lambda, z) \in \mathscr{V} \times \overline{\mathbb{C}}$.

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CLAIM 1. *There exists a lifting mapping $\Psi : \mathscr{V} \times \overline{\mathbb{C}} \to \mathscr{V} \times \overline{\mathbb{C}}$ of $\Phi$ which is also a holomorphic motion such that the following diagram commutes.*

$$\begin{array}{ccc} \mathscr{V} \times \overline{\mathbb{C}} & \xrightarrow{\Psi} & \mathscr{V} \times \overline{\mathbb{C}} \\ \downarrow{\scriptstyle id\times f_{\lambda_1}} & & \downarrow{\scriptstyle F} \\ \mathscr{V} \times \overline{\mathbb{C}} & \xrightarrow{\Phi} & \mathscr{V} \times \overline{\mathbb{C}}. \end{array} \tag{4.4}$$

*Furthermore, $\Psi$ satisfies the following property on the post-critical set $P(f_{\lambda_1})$ of $f_{\lambda_1}$:*

$$\Psi(\lambda, f_{\lambda_1}^l(v_{\lambda_1}^{\pm})) = \begin{cases} (\lambda, f_\lambda^l(v_\lambda^{\pm})), & 0 \le l \le k-4, \\ \lambda, f_\lambda^l(\pm z_k^i(\lambda))), & l = k-3, \\ (\lambda, \mathrm{id}), & l \ge k-2. \end{cases} \tag{4.5}$$

*Proof.* Since the map $\Phi(\lambda, v_{\lambda_1}^{\pm}) = (\lambda, v_\lambda^{\pm})$ and $\Phi(\lambda, \infty) = (\lambda, \infty)$, there exists a unique lifting map $\Psi : \mathscr{V} \times \overline{\mathbb{C}} \to \mathscr{V} \times \overline{\mathbb{C}}$ such that it is a holomorphic motion and satisfies the diagram (4.4).

For $l = k-3$, denote $h^{\pm}(\lambda) = \Psi_\lambda \circ f_{\lambda_1}^{k-3}(v_{\lambda_1}^{\pm})$. By equation (4.3) and diagram (4.4), $f_\lambda(h^{\pm}(\lambda)) = f_{\lambda_1}^{k-2}(v_{\lambda_1}^{\pm}) = 0$. Since $z_k^i(\lambda_1) = v_{\lambda_1}^{+}$, then $h^{\pm}(\lambda_1) = f_{\lambda_1}^{k-3}(v_{\lambda_1}^{\pm}) = f_{\lambda_1}^{k-3}$ $(\pm z_k^i(\lambda_1))$. By Lemma 4.1, there are $2n$ distinct pre-images of 0 which are holomorphic with respect to $\lambda \in \mathscr{V}$. Hence, $h^{\pm}(\lambda) = f_\lambda^{k-3}(\pm z_k^i(\lambda))$ for $\lambda$ near $\lambda_1$. Since $\mathscr{V}$ is connected, then $h^{\pm}(\lambda) = f_\lambda^{k-3}(\pm z_k^i(\lambda))$ for all $\lambda \in \mathscr{V}$. The proofs for $l \le k-4$ and $l \ge k-2$ are very similar and will be omitted here. □

The proof of Proposition 4.4 continues. By Claim 1 and the condition $z_k^i(\lambda_2) = v_{\lambda_2}^{+}$, we have $\Psi_{\lambda_2} \circ f_{\lambda_1}^{k-3}(v_{\lambda_1}^{\pm}) = f_{\lambda_2}^{k-3}(\pm z_k^i(\lambda_2)) = f_{\lambda_2}^{k-3}(v_{\lambda_2}^{\pm})$. Hence, $\Phi_{\lambda_2} \circ f_{\lambda_1}^l(v_{\lambda_1}^{\pm}) = \Psi_{\lambda_2} \circ f_{\lambda_1}^l(v_{\lambda_1}^{\pm}) = f_{\lambda_2}^l(v_{\lambda_2}^{\pm})$ for all $l \ge 0$. By diagram (4.4), we have

$$\begin{array}{ccc} \overline{\mathbb{C}} & \xrightarrow{\Psi_{\lambda_2}} & \overline{\mathbb{C}} \\ \downarrow{\scriptstyle f_{\lambda_1}} & & \downarrow{\scriptstyle f_{\lambda_2}} \\ \overline{\mathbb{C}} & \xrightarrow{\Phi_{\lambda_2}} & \overline{\mathbb{C}}. \end{array} \tag{4.6}$$

Let $H : [0,1] \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be defined by

$$H_t = (1-t)\Phi_{\lambda_2} + t\Psi_{\lambda_2}.$$

It is easy to check that $H_1 = \Psi_{\lambda_2}$, $H_0 = \Phi_{\lambda_2}$, and $H_t|_{P(f_{\lambda_1})} = \Psi_{\lambda_2}$. Hence, $\Psi_{\lambda_2}$ and $\Phi_{\lambda_2}$ are homotopic relative to $P(f_{\lambda_1})$. Since $\Phi_{\lambda_2}$ and $\Psi_{\lambda_2}$ are both quasi-conformal homeomorphisms, by Remark 4.1 and Theorem 4.3, $f_{\lambda_1}$ and $f_{\lambda_2}$ are conformally conjugated. Since $\lambda_1, \lambda_2 \in \mathscr{F}_0$, it is easy to check that $\lambda_1 = \lambda_2$. □

COROLLARY 4.5. *Let $\mathscr{U} \subset \mathscr{H}_k$ be an escape component and $\lambda = \Phi_{\mathscr{U}}^{-1}(re^{2\pi it}) \cap \mathscr{F}_0$. Let $\mathscr{V} \subset \mathbb{C}^* \setminus \bigcup_{2\le j\le k-1} \Lambda_j$ be a simply connected region containing $\overline{\mathscr{U}}$, then there exists a unique root function $z_k^i(\lambda)$ defined on $\mathscr{V}$ such that $v_\lambda^{+} = \phi_{z_k(\lambda)}^{-1}(\rho_k(r)e^{2\pi i\theta_k(t)})$, where*

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$$\theta_k(t) := \begin{cases} \dfrac{t}{2}, & k = 0, \\ \dfrac{2nt}{n-2}, & k = 2, \\ t, & k \geq 3, \end{cases} \tag{4.7}$$

*and*

$$\rho_k(r) := \begin{cases} \sqrt{r}, & k = 0, \\ \sqrt[n-2]{r^{2n}}, & k = 2, \\ r, & k \geq 3. \end{cases} \tag{4.8}$$

*Proof.* First, let us show the existence and uniqueness of the root function. Since $f_\lambda^{k-2}(v_{\lambda_{\mathscr{U}}}^+) = 0$, by Lemma 4.1 and Proposition 4.4, there exists a unique root function $z_k^i(\lambda)$ defined on $\mathscr{V}$ such that $z_k^i(\lambda_{\mathscr{U}}) = v_{\lambda_{\mathscr{U}}}^+$ (the uniqueness is trivial for $k \leq 2$ and insured by Proposition 4.4 for $k \geq 3$).

It follows easily that if $\lambda \in \mathscr{U}$, $v_\lambda^+$ is contained in the Fatou component containing $z_k^i(\lambda)$. By Remark 2.1, the Böttcher coordinates $\phi_{z_k^i(\lambda)}$ defined in Lemma 2.1 can be extended to $v_\lambda^+$. The rest of the proof follows easily from the asymptotic behavior of the parameterization maps defined in Theorem 2.2 and dynamical Böttcher coordinates defined in Lemma 2.1. □

## 5. *Dynamics of maps on* $\partial\mathscr{H}$

Let $\mathscr{U} \subset \mathscr{H}$ be an escape component. In this section, we describe the dynamics of $f_\lambda$ for $\lambda = \Phi_{\mathscr{U}}^{-1}(e^{2\pi i t}) \in \partial\mathscr{U} \cap \mathscr{F}_0$, where $t \in \mathbb{R}/\mathbb{Z}$ via the puzzle system $(f_\lambda, I_\lambda^k, P_\lambda^k)$.

LEMMA 5.1. *If* $\lambda \in \mathscr{F}_0 \setminus \mathscr{H}_0$, *then for any sequence of puzzle pieces* $\{P_\lambda^k\}_{k\geq 0}$, $(\bigcap_{k\geq 0} P_\lambda^k) \cap \partial B_\lambda$ *is either empty or a singleton.*

*Proof.* Suppose $(\bigcap_{k\geq 0} P_\lambda^k) \cap \partial B_\lambda \neq \emptyset$. Then, $\{P_\lambda^k\}$ is a nested sequence of puzzle pieces. By part (1) in Theorem 2.6, every point in $(\bigcap_{k\geq 0} P_\lambda^k) \cap \partial B_\lambda$ is a landing point of a dynamical ray. Suppose that $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t_0}) \in (\bigcap_{k\geq 0} P_\lambda^k) \cap \partial B_\lambda$. It suffices to show for $t \neq t_0$, $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t}) \notin (\bigcap_{k\geq 0} P_\lambda^k) \cap \partial B_\lambda$. Since each $\theta \in \Theta_\lambda$ satisfies equation (2.8) in Lemma 2.5, by the construction of graphs, there is an angle $\alpha \in \bigcup_{k\geq 0} \tau^{-k}(\theta)$ such that the cut ray $\Omega_\lambda^\alpha$ separates $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t})$ from $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t_0})$. Note that there must be a $k$ such that $\alpha \in \tau^{-k}(\theta)$ which implies that $\Omega_\lambda^\alpha \subset I_\lambda^k$. We get that $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t})$ and $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t_0})$ are separated by the graph $I_\lambda^k$ of depth $k$. This follows that $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t}) \notin P_\lambda^k(\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t_0})) = P_\lambda^k$, and hence $\phi_{\infty(\lambda)}^{-1}(e^{2\pi i t}) \notin (\bigcap_{k\geq 0} P_\lambda^k) \cap \partial B_\lambda$. □

Let us denote $K_\lambda^\pm := \bigcap_{k\geq 0} P_\lambda^k(v_\lambda^\pm)$.

PROPOSITION 5.2. *Let* $\mathscr{U} \subset \mathscr{H}_k$ *be an escape component with* $k \geq 0$. *If* $\lambda_0 = \Phi_{\mathscr{U}}^{-1}(e^{2\pi i t}) \in \mathscr{F}_0$, *then there exists* $z_k^i(\lambda_0) \in Z_k(\lambda_0)$ *such that*

$$w_k^t(\lambda_0) := \lim_{s\to 1} \phi_{z_k^i(\lambda_0)}^{-1}(s e^{2\pi i \theta_k(t)}) \in K_{\lambda_0}^+,$$

*where* $\theta_k(t)$ *is defined in equation (4.7).*

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*Proof.* For $k=0$, $z_k^i(\lambda_0)=\infty$. For $k\geq 2$, there is a simply connected region $\mathscr{U}^*\supset\overline{\mathscr{U}}$ such that $\mathscr{U}^*\cap(\bigcup_{2\leq l\leq k-1}\Lambda_l)=\emptyset$. Let $z_k^i(\lambda)$ denote the root function obtained in Corollary 4.5. In the following, we prove that for any $m\geq k+2$,

$$\phi^{-1}_{z_k^i(\lambda_0)}(\sqrt[n^{m-k+2}]{L}e^{2\pi i\theta_k(t)})\in P^{m-1}_{\lambda_0}(v^+_{\lambda_0}). \tag{5.1}$$

Here, $L$ is the number we choose for $X^L_{\lambda_0}$ to construct puzzles in §2.2. For each $m\geq k+2$, choose $r_m$ such that $1<\rho_k(r_m)\leq\sqrt[n^{m-k+2}]{L}$ and $\lambda_m=\Phi^{-1}_{\mathscr{U}}(r_me^{2\pi it})\in\mathscr{P}^{m+2}_{\lambda_0}\cap\mathscr{R}^t_{\mathscr{U}}$. Here, $\rho_k$ is the function we defined in equation (4.8). By Corollary 4.5, $v^+_{\lambda_m}=\phi^{-1}_{z_k^i(\lambda_m)}(\rho_k(r_m)e^{2\pi i\theta_k(t)})$. By Corollary 3.2, there is no critical value in $f_\lambda^{-m-1}(X^L_\lambda)\setminus\{\infty\}$ for $\lambda\in\mathscr{P}^{m+1}_{\lambda_0}$ since there are no critical values of $f_{\lambda_0}$ in $f_{\lambda_0}^{-m-1}(X^L_{\lambda_0})\setminus\{\infty\}$. Hence, $\phi^{-1}_{z_k^i(\lambda)}(\sqrt[n^{m-k+2}]{L}e^{2\pi i\theta_k(t)})\in I^{m+2}_\lambda$ is well defined for $\lambda\in\mathscr{P}^{m+2}_{\lambda_0}$. Since both $v^+_{\lambda_m}=\phi^{-1}_{z_k^i(\lambda_m)}(\rho_k(r_m)e^{2\pi i\theta_k(t)})$ and $\phi^{-1}_{z_k^i(\lambda_m)}(\sqrt[n^{m-k+2}]{L}e^{2\pi i\theta_k(t)})$ belong to the dynamical ray $R^{\theta_k(t)}_{z_k^i(\lambda_m)}$, and $I^{m-1}_\lambda$ does not contain the equipotential curve $E^s_{z_k^i(\lambda_m)}$ with $1<s<\sqrt[n^{m-k+2}]{L}$, we have

$$\phi^{-1}_{z_k^i(\lambda_m)}(\sqrt[n^{m-k+2}]{L}e^{2\pi i\theta_k(t)})\in P^{m-1}_{\lambda_m}(v^+_{\lambda_m}). \tag{5.2}$$

By Corollary 3.2,

$$h_m:\mathscr{P}^m_{\lambda_0}\times(\partial P^{m-1}_{\lambda_0}(\phi^{-1}_{z_k^i(\lambda_0)}(\sqrt[n^{m-k+2}]{L}e^{2\pi i\theta_k(t)}))\cup\{v^+_{\lambda_0}\})\to\mathscr{P}^m_{\lambda_0}\times\overline{\mathbb{C}}$$

is a holomorphic motion. Hence, equation (5.2) implies that

$$\phi^{-1}_{z_k^i(\lambda)}(\sqrt[n^{m-k+2}]{L}e^{2\pi i\theta_k(t)})\in P^{m-1}_{\lambda}(v^+_{\lambda})$$

for all $\lambda\in\mathscr{P}^m_{\lambda_0}$. Then equation (5.1) follows by setting $\lambda=\lambda_0$.

Certainly, for $1<s<\sqrt[n^{m-k+2}]{L}$, we also have

$$\phi^{-1}_{z_k^i(\lambda_0)}(se^{\pi i\theta_k(t)})\in P^{m-1}_{\lambda_0}(v^+_{\lambda_0}). \tag{5.3}$$

Let $s\to 1$. Then, $w_k^t(\lambda_0):=\lim_{s\to 1}\phi^{-1}_{z_k^i(\lambda_0)}(se^{2\pi i\theta_k(t)})\in P^{m-1}_{\lambda_0}(v^+_{\lambda_0})$ for all $m\geq k+2$. This follows that $w_k^t(\lambda_0)\in\bigcap_{m\geq k+2}P^{m-1}_{\lambda_0}(v^+_{\lambda_0})=K^+_{\lambda_0}$. □

For $\mathscr{H}_0$, $\lambda=\Phi^{-1}_{\mathscr{H}_0}(e^{2\pi it})\in\mathscr{F}_0$ if and only if $t\in(0,1/(n-1))$ and $\Phi^{-1}_{\mathscr{H}_0}(1)\in\partial\mathscr{F}_0$.

COROLLARY 5.3. *Suppose $\mathscr{U}\subset\mathscr{H}_k$ is an escape component. Then for $\lambda=\Phi^{-1}_{\mathscr{U}}(e^{2\pi it})\cap\mathscr{F}_0$,*

$$f_\lambda^{k-1}(K^+_\lambda)\cap\partial B_\lambda=\{f_\lambda^{k-1}(w_k^t(\lambda))\}.$$

*Proof.* By Proposition 5.2, $f_\lambda^{k-1}(w_k^t(\lambda))\in f_\lambda^{k-1}(K^+_\lambda)$. Note that

$$f_\lambda^{k-1}(\phi^{-1}_{z_k^i(\lambda_0)}(se^{2\pi i\theta_k(t)}))\in B_\lambda$$

yields $f_\lambda^{k-1}(w_k^t(\lambda)) \in f_\lambda^{k-1}(K_\lambda^+) \cap \partial B_\lambda$. Conversely, by Lemma 5.1, $f_\lambda^{k-1}(K_\lambda^+) \cap \partial B_\lambda = (\bigcap_{n\geq 0} P_\lambda^n(f_\lambda^{k-1}(v_\lambda^+))) \cap \partial B_\lambda$ contains at most one point. This implies that $f_\lambda^{k-1}(K_\lambda^+) \cap \partial B_\lambda = \{f_\lambda^{k-1}(w_k^t(\lambda))\}$. □

## 6. *No bounded escape components attached on $\partial\mathscr{H}_0$*

The purpose of this section is to prove that the boundary of the Cantor locus is disjoint with the boundary of any bounded escape component. This result is also the key step of proving Theorem 1.2.

6.1. *Assumptions.* Let $\mathscr{U} \subset \mathscr{H}_k$ be an escape hyperbolic component with order $k \geq 2$. To prove that $\partial\mathscr{U} \cap \partial\mathcal{H}_0 = \emptyset$, it suffices to show that $\overline{\mathscr{R}_{\mathscr{H}_0}^{t_0}} \cap \overline{\mathscr{R}_{\mathscr{U}}^{t_1}} \cap \mathscr{F}_0 = \emptyset$ for any $t_0, t_1 \in \mathbb{R}/\mathbb{Z}$. Here, $\mathscr{R}_{\mathscr{H}_0}^{f_0}$ and $\mathscr{R}_{\mathcal{U}}^{t_1}$ are parameter rays in $\mathscr{H}_0$ and $\mathscr{U}$, respectively. We prove this by seeking a contradiction under the assumption $\overline{\mathscr{R}_{\mathscr{H}_0}^{t_0}} \cap \overline{\mathscr{R}_{\mathscr{U}}^{t_1}} \cap \mathscr{F}_0 \neq \emptyset$. Let us use the notation in the proof of Proposition 5.2. The function $z_k^i$ is the root function defined on $\mathscr{U}^* \supset \overline{\mathscr{U}}$, such that $v_{\lambda_{\mathscr{U}}}^+ = z_k^i(\lambda_{\mathscr{U}})$.

PROPOSITION 6.1. *Let $\mathscr{U} \subset \mathscr{H}_k$ be an escape hyperbolic component with order $k \geq 2$. If $\overline{\mathscr{R}_{\mathscr{H}_0}^{t_0}} \cap \overline{\mathscr{R}_{\mathscr{U}}^{t_1}} \cap \mathscr{F}_0 \neq \emptyset$, then $\tau^{k-1}(t_0) = t_0$.*

*Proof.* Suppose $\lambda_0 \in \overline{\mathscr{R}_{\mathscr{H}_0}^{t_0}} \cap \overline{\mathscr{R}_{\mathscr{U}}^{t_1}} \cap \mathscr{F}_0$. By Proposition 5.2, $\{w_0^{t_0}(\lambda_0), w_k^{t_1}(\lambda_0)\} \subset K_{\lambda_0}^+$. Then,

$$\{f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0)), f_{\lambda_0}^{k-2}(w_k^{t_1}(\lambda_0))\} \subset f_{\lambda_0}^{k-2}(K_{\lambda_0}^+) \tag{6.1}$$

and

$$\{f_{\lambda_0}^{k-1}(w_0^{t_0}(\lambda_0)), f_{\lambda_0}^{k-1}(w_k^{t_1}(\lambda_0))\} \subset f_{\lambda_0}^{k-1}(K_{\lambda_0}^+) \cap \partial B_\lambda.$$

By Corollary 5.3, we get

$$f_{\lambda_0}^{k-1}(w_k^{t_1}(\lambda_0)) = f_{\lambda_0}^{k-1}(w_0^{t_0}(\lambda_0)). \tag{6.2}$$

If $f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0)) \neq f_{\lambda_0}^{k-2}(w_k^{t_1}(\lambda_0))$, then by equations (6.1), (6.2), and the fact

$$f_{\lambda_0}^{k-2}(K_{\lambda_0}^+) \subset P_{\lambda_0}^m(f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0)))$$

for all integers $m \geq 0$, we have that

$$f_{\lambda_0} : P_{\lambda_0}^{m+1}(f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0))) \to P_{\lambda_0}^m(f_{\lambda_0}^{k-1}(w_0^{t_0}(\lambda_0)))$$

is a ramified covering of degree at least two. Since the puzzle pieces are all simply connected, $P_{\lambda_0}^m(f_{\lambda_0}^{k-1}(w_0^{t_0}(\lambda_0)))$ must contain a critical value for all $m \geq 0$. It follows that $f_{\lambda_0}^{k-1}(K_{\lambda_0}^+)$ contains a critical value $v_{\lambda_0}^+$ or $v_{\lambda_0}^-$, that is, $f_{\lambda_0}^{k-1}(K_{\lambda_0}^+) = K_{\lambda_0}^+$ or $= K_{\lambda_0}^-$. This implies that $\tau^{k-1}(t_0/2) = t_0/2$ or $\tau^{k-1}(t_0/2) = t_0/2 + \frac{1}{2}$. Both lead to the consequence that $\tau^{k-1}(t_0) = t_0$.

If $f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0)) = f_{\lambda_0}^{k-2}(w_k^{t_1}(\lambda_0))$, then both dynamical rays $f_{\lambda_0}^{k-2}(R_{z_k^i(\lambda_0)}^{\theta_k(t_1)}) \subset T_{\lambda_0}$ and $f_{\lambda_0}^{k-2}(R_{\infty(\lambda_0)}^{\theta_0(t_0)}) \subset B_{\lambda_0}$ land on the common point $f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0))$. Note that

$f_{\lambda_0}^{k-1}(R_{z_k^i(\lambda_0)}^{\theta_k(t_1)}) = f_{\lambda_0}^{k-1}(R_{\infty(\lambda_0)}^{\theta_0(t_0)}) \subset B_{\lambda_0}$ is the external ray landing on $f_{\lambda_0}^{k-1}(w_0^{t_0}(\lambda_0))$. It follows that $f_{\lambda_0}^{k-2}(w_0^{t_0}(\lambda_0))$ is a critical point and $f_{\lambda_0}^{k-1}(w_0^{t_0}(\lambda_0)) \in f_{\lambda_0}^{k-1}(K_{\lambda_0}^+)$ is a critical value $v_{\lambda_0}^+$ or $v_{\lambda_0}^-$. We again obtain that $\tau^{k-1}(t_0/2) = t_0/2$ or $\tau^{k-1}(t_0/2) = t_0/2 + \frac{1}{2}$, and then $\tau^{k-1}(t_0) = t_0$. □

Suppose that $t_0 \in [0, 1/(n-1))$ satisfies $\tau^p t_0 = t_0$ for an integer $p \geq 1$. Let $\lambda_0 = \Phi_{\mathscr{H}_0}^{-1}(e^{2\pi i t_0})$. It follows from [QRWY15] (Lemmas 4.6, 4.8, Theorem 6.2, and Remark 6.3) that there is a quadratic-like map $g_{\lambda_0} : U \to V$ with a parabolic fixed point $\beta_{\lambda_0} \in U$ satisfying

$$g_{\lambda_0}(\beta_{\lambda_0}) = \beta_{\lambda_0},\ g'_{\lambda_0}(\beta_{\lambda_0}) = 1, \text{ and } g''_{\lambda_0}(\beta_{\lambda_0}) \neq 0,$$

which is defined according to the following three cases.

(1) If $\tau^p(t_0/2) = t_0/2$, then $g_{\lambda_0} = f_{\lambda_0}^p$, $U = P_{\lambda_0}^N(v_{\lambda_0}^+)$, and $V = P_{\lambda_0}^{N-p}(v_{\lambda_0}^+)$ for an $N$ large, $\beta_{\lambda_0} = w_0^{t_0}(\lambda_0)$, and $v_{\lambda_0}^+$ is the unique critical value.

(2) If $\tau^p(t_0/2) = t_0/2 + \frac{1}{2}$ and $\tau^p(t_0/2 + \frac{1}{2}) = t_0/2 + \frac{1}{2}$, then $g_{\lambda_0} = f_{\lambda_0}^p$, $U = P_{\lambda_0}^N(v_{\lambda_0}^-)$, and $V = P_{\lambda_0}^{N-p}(v_{\lambda_0}^-)$ for an $N$ large, $\beta_{\lambda_0} = -w_0^{t_0}(\lambda_0)$, and $v_{\lambda_0}^-$ is the unique critical value.

(3) If $\tau^p(t_0/2) = t_0/2 + \frac{1}{2}$ and $\tau^p(t_0/2 + \frac{1}{2}) = t_0/2$, then $g_{\lambda_0} = -f_{\lambda_0}^p$, $U = P_{\lambda_0}^N(v_{\lambda_0}^+)$, and $V = P_{\lambda_0}^{N-p}(v_{\lambda_0}^+)$ for an $N$ large, $\beta_{\lambda_0} = w_0^{t_0}(\lambda_0)$, and $v_{\lambda_0}^+$ is the unique critical value.

*Remark 6.1.* In the above, we have assumed that $\lambda_0 \notin \mathbb{R}^+$. When $\lambda_0 \in \mathbb{R}^+$, the regions $U, V$ will be taken as those constructed in the proof of Lemma 7.2 in [QWY12].

In the following discussion, without loss of generality, we may assume that we are in case (1). Hence, we have the following Assumption 6.2.

*Assumption 6.2.* Suppose that $\mathscr{U} \subset \mathscr{H}_k$ is an escape component with $k \geq 2$, $\lambda_0 \in \overline{\mathscr{R}_{\mathscr{H}_0}^{t_0}} \cap \overline{\mathscr{R}_{\mathscr{U}}^{t_1}} \cap \mathscr{F}_0$, and $p|(k-1)$ is a positive integer such that $\tau^p(t_0/2) = t_0/2$. Then:

(1) there exists $N$ such that $g_{\lambda_0} = f_{\lambda_0}^p : P_{\lambda_0}^N(v_{\lambda_0}^+) \to P_{\lambda_0}^{N-p}(v_{\lambda_0}^+)$ is a quadratic-like map with unique critical value $v_{\lambda_0}^+$;

(2) $g_{\lambda_0}(w_0^{t_0}(\lambda_0)) = w_0^{t_0}(\lambda_0)$, $g'_{\lambda_0}(w_0^{t_0}(\lambda_0)) = 1$, and $g''_{\lambda_0}(w_0^{t_0}(\lambda_0)) \neq 0$;

(3) $g_{\lambda_0}^{(k-1)/p}(w_k^{t_1}(\lambda_0)) = w_0^{t_0}(\lambda_0)$, where $w_0^{t_0}(\lambda_0)$ is the landing point of ray $R_{\infty(\lambda_0)}^{\theta_0(t_0)}$ and $w_k^{t_1}(\lambda_0)$ is the landing point of ray $R_{z_k^i(\lambda_0)}^{\theta_k(t_1)}$.

6.2. *Rational family with parabolic implosion.* The original parabolic implosion theory is established by Douady and Lavaurs [Lav89], see [DH84]. This theory is further developed by Shishikura, see [Shi98, Shi00]. Here, we use the terminology and results given in [Lei00, Shi00].

Let $\Delta_{\rho,\theta}$ be a bounded connected open set with 0 on its boundary, and $F : \overline{\Delta_{\rho,\theta}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $(u, z) \mapsto F_u(z)$ be a map satisfying the following conditions.

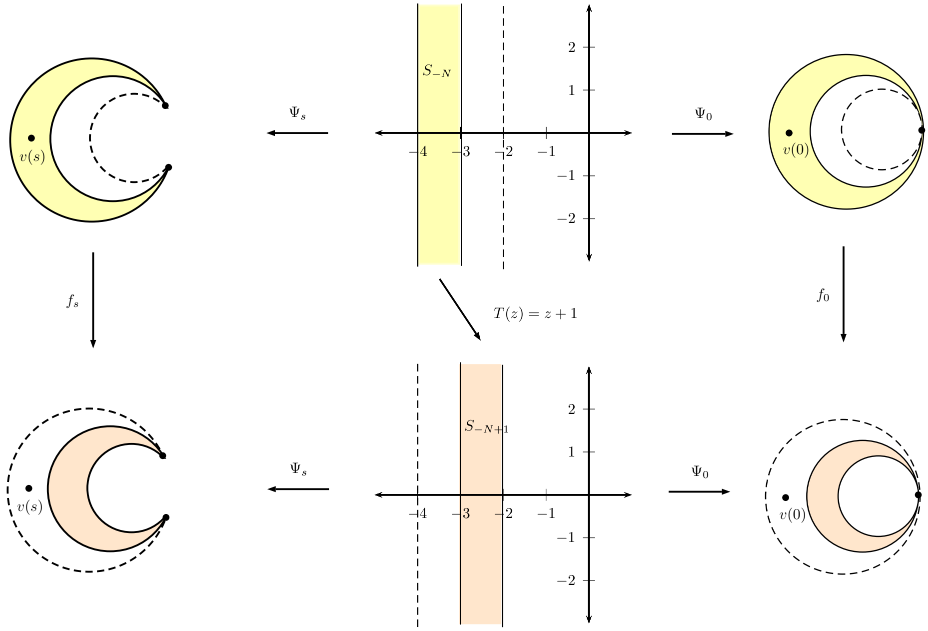


FIGURE 10. Illustration of Theorem 6.3.

(1) $F : (u, z) \mapsto F_u(z)$ is continuous on $\overline{\Delta_{\rho,\theta}} \times \overline{\mathbb{C}}$, holomorphic on $\Delta_{\rho,\theta} \times \overline{\mathbb{C}}$, and $F_u(z)$ is a rational map for each $u \in \overline{\Delta_{\rho,\theta}}$.

(2) $F_u(z) = m(u)z + O(z^2)$ as $z \to 0$ with $m(0) = 1$, $F_0''(0) \neq 0$, and $F_0$ has a unique simple critical value $v_0$ contained in the parabolic basin of 0.

(3) $\sigma(u) := (m(u) - 1)/2\pi i$ maps $\Delta_{\rho,\theta}$ univalently onto $\{z \in \mathbb{C} : |z| < \rho, |\arg z| < \theta\}$ with $\rho > 0$ small and $\theta \in (0, \pi)$.

By equation (2), $F_u$ has two fixed points 0 and $q(u)$ when $u \in \overline{\Delta_{\rho,\theta}} \setminus \{0\}$. The parabolic implosion phenomenon for such a family $F_u$ was given as the Douday–Lavaurs–Shishikura theorem (see Theorem 2.1 in [**Lei00**]). Since we need only to use the attracting part of Theorem 2.1 in [**Lei00**], we translate it to the following Theorem 6.3. Set $S_M = \{z \in \mathbb{C} : 0 < \Re(z) < M + 1\}$ for any $M \in \mathbb{N}$.

THEOREM 6.3. (Douady–Lavaurs–Shishikura) *Let $F : \overline{\Delta_{\rho,\theta}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, $(u, z) \mapsto F_u(z)$ satisfy the conditions given above. For any given $M \in \mathbb{N}$ large, when $\rho$ is small enough, there exists a continuous map $\Psi : \overline{\Delta_{\rho,\theta}} \times S_M \to \mathbb{C}$, $(u, z) \mapsto \Psi_u(z)$ satisfying the following properties (see Figure 10).*

(1) *For $u = 0$, $\Psi_0 : S_M \to \Psi_0(S_M) \subset \Omega_0$ is a univalent map where $\Omega_0$ is the attracting petal of $F_0$ with the parabolic fixed point 0 on its boundary, and $\Psi_0^{-1}$ is the restriction of the usual Fatou coordinate $\varphi_0 : \Omega_0 \to \{z : \Re(z) > 0\}$.*

(2) *For $u \in \overline{\Delta_{\rho,\theta}} \setminus \{0\}$, $\Psi_u : S_M \to \mathbb{C}$ is a univalent map such that $\Psi_u(S_M)$ is a Jordan region containing two fixed points 0 and $q(u)$ of $F_u$ on its boundary.*

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(3) *Whenever* $z, z+1 \in S_M$,

$$\Psi_u(z+1) = F_u(\Psi_u(z)).$$

*Furthermore,* $\Psi_u(S_0)$ *contains the critical value* $v(u)$ *of* $F_u$ *with* $v(0) = v_0$.

By this theorem, we call the map $F$ or the family $F_u$ the rational family with parabolic implosion (RFPI).

*Remark 6.2.* From equation (3), $\Psi_u(S_0)$ is a fundamental region of $F_u$, and $F_u^m$ is well defined and univalent on $\Psi_u(S_0)$ for any $1 \le m \le M$.

6.3. *Application to McMullen family.* Set $g_\lambda := f_\lambda^p$, where $p$ is given in Assumption 6.2. Consider the equation $g_\lambda(z) - z = 0$. By part (2) of Assumption 6.2, the fixed points of $g_\lambda(z)$ for $(\lambda, z)$ near $(\lambda_0, w_0^{t_0}(\lambda_0))$ can be uniformized by setting $\lambda = \lambda_0 + u^2$. That is, there exists $r > 0$ small enough such that $\mathbb{D}(\lambda_0, r) \subset \mathscr{U}^*$ ($\mathscr{U}^*$ is defined in Proposition 5.2), a local change of coordinates $\lambda : \mathbb{D}(0, \sqrt{r}) \to \mathbb{D}(\lambda_0, r)$ given by $u \mapsto \lambda_0 + u^2$, and a holomorphic function $p : \mathbb{D}(0, \sqrt{r}) \to \mathbb{C}$ such that $p(u)$ and $p(-u)$ are the two local fixed points of $g_{\lambda_0+u^2}$ satisfying $p(u) = w_0^{t_0}(\lambda_0 + u^2)$ for $\lambda_0 + u^2 \in \mathbb{D}(0, \sqrt{r}) \setminus \mathscr{H}_0$. Let $m : \mathbb{D}(0, \sqrt{r}) \to \mathbb{C}$, $u \mapsto g'_{\lambda_0+u^2}(p(u))$ be the multiplier of the fixed point $p(u)$. Then, $m(u)$ is holomorphic on $\mathbb{D}(0, \sqrt{r})$ and $m(0) = 1$. It is evident that

$$g_{\lambda_0+u^2}(z) = p(u) + m(u)(z - p(u)) + O(z - p(u))^2 \quad \text{as } z \to p(u). \tag{6.3}$$

Let $T_u(z) := z - p(u)$ and $F_u(\omega) := T_u \circ g_{\lambda_0+u^2} \circ T_u^{-1}(\omega)$. Then, equation (6.3) becomes

$$F_u(\omega) = m(u)\omega + O(\omega^2) \quad \text{as } \omega \to 0. \tag{6.4}$$

*Remark 6.3.* It is obvious that another fixed point $p(-u)$ of $g_{\lambda_0+u^2}$ has the multiplier $m(-u)$. So, except the fixed point 0, the other fixed point of $F_u$ is $q(u) = p(-u) - p(u)$ with the multiplier $m(-u)$ when $u \neq 0$.

Let $v(u) := T_u(v^+_{\lambda_0+u^2})$ be the critical value of $F_u(\omega)$. Let $U_0 = T_0(P^N_{\lambda_0}(v^+_{\lambda_0}))$ and $V_0 := T_0(P^{N-p}_{\lambda_0}(v^+_{\lambda_0}))$. Then by equation (1) of Assumption 6.2, $F_0 : U_0 \to V_0$ is a quadratic-like map with the unique critical value $v(0) \in U_0$, which is obviously contained in the parabolic basin of 0. When $r > 0$ is small enough, then by the continuity of $F_u$ with respect to $u \in \mathbb{D}(0, \sqrt{r})$, it is easy to find a simply connected region $U_u \subset U_0$ such that $F_u : U_u \to V_u := F_u(U_u)$ is also a quadratic-like map with the unique critical value $v(u) \in U_u$. Furthermore $U_u \to U_0$ as $u \to 0$ in Hausdorff topology.

Let $\Gamma_\pm := \lambda^{-1}(\mathscr{R}^{t_1}_{\mathscr{U}})$ denote the two pre-images of $\mathscr{R}^{t_1}_{\mathscr{U}}$ in the $u$-plane which can be parameterized by $\Gamma_\pm(s) := \lambda^{-1}(\Phi_{\mathscr{U}}(se^{2\pi i t_1}))$. Obviously, $\Gamma_\pm(s) \to 0$ as $s \to 1$.

PROPOSITION 6.4. *For any* $\theta \in (0, \pi/2)$*, there exist* $\rho > 0$ *and* $\Delta_{\rho,\theta} \subset \mathbb{D}(0, \sqrt{r})$ *which is mapped univalently onto* $\{z \in \mathbb{C} : |z| < \rho, |\arg z| < \theta\}$ *by* $\sigma = (m-1)/(2\pi i)$ *such that* $\Gamma_+(s) \in \Delta_{\rho,\theta}$ *or* $\Gamma_-(s) \in \Delta_{\rho,\theta}$ *for s sufficiently close to* 1.

The proof of Proposition 6.4 will be given in the next subsection. By Proposition 6.4 and the discussion above, $F : \overline{\Delta_{\rho,\theta}} \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ defined by $F(u, z) := F_u(z)$ is an RFPI.

PROPOSITION 6.5. *For any escape component $\mathscr{U} \subset \mathscr{H}_k$ with $k \geq 2$, $\overline{\mathscr{U}} \cap \overline{\mathscr{H}_0} = \emptyset$.*

*Proof.* If the proposition is not true, we may assume Assumption 6.2 holds. By Proposition 6.4, without loss of generality, let us assume $\Gamma_+(s) \in \Delta_{\rho,\theta}$ for $s(> 1)$ close to 1.

Let $R_u := T_u(R^{\theta_0(t_0)}_{\infty(\lambda_0)})$ be parameterized by $R_u(s) := T_u(\phi^{-1}_{\infty(\lambda)}(se^{2\pi i\theta_0(t_0)}))$ and $\tilde{R}_u := T_u(R^{\theta_k(t_1)}_{z^i_k(\lambda_0+u^2)})$ be parameterized by $\tilde{R}_u(s) := T_u(\phi^{-1}_{z^i_k(\lambda_0+u^2)}(se^{2\pi i\theta_k(t_1)}))$. Then we have $F_u(R_u) = R_u$ and $F_u^{(k-1)/p}(\tilde{R}_u) = R_u$ by equation (3) of Assumption 6.2. Note that by Corollary 4.5, the critical value $v(u) = \tilde{R}_u(\rho_k(s))$ for $u = \Gamma_+(s) \in \Delta_{\rho,\theta}$ is close to 0. Hence, $F_u^{(k-1)/p}(v(u)) = R_u(s_0) \in R_u$ for some $s_0 > 1$. Take $M = (k-1)/p$ and $\Psi : \overline{\Delta_{\rho,\theta}} \times S_M \to \mathbb{C}$, as given in Theorem 6.3. Then by part (3) of Theorem 6.3 (see Remark 6.2), $v(u) \in \Psi_u(S_0)$ and $R_u(s_0) = F_u^M(v(u)) \in \Psi_u(S_M)$.

However, in Remark 2.1, it is pointed out that $R_u$ can be extended continuously to $R_u(\sqrt[n^M]{s_0})$ such that $F_u^M(R_u(\sqrt[n^M]{s_0})) = R_u(s_0)$. The invariance and continuity of $R_u$ ensure that $R_u(\sqrt[n^M]{s_0}) \in \Psi_u(S_0)$. By Remark 6.2, $F_u^M$ is univalent on $\Psi_u(S_0)$. Hence, $F_u^M(v(u)) = F_u^M(R_u(\sqrt[n^M]{s_0})) = R_u(s_0)$ implies that $v(u) = R_u(\sqrt[n^M]{s_0}) \in R_u$. This contradicts with $v(u) = \tilde{R}_u(\rho_k(s)) \in \tilde{R}_u$, where $\rho_k$ is defined in equation (4.8) (see Figure 11). □

6.4. *Proof of Proposition 6.4.* Let $X_{\rho,\theta} := \{|z-1| < 2\pi\rho : |\arg(z-1)| \leq \pi/2 - \theta\}$ and $Y^{\pm}_{\rho,\theta} := \{z \in \mathbb{H}_{\pm} : |z-1| < 2\pi\rho\} \setminus (-X_{\rho,\theta} \cup X_{\rho,\theta})$. By part (3) of the definition of RFPI, it suffices to show that either $m(\Gamma_+(s)) \in Y^+_{\rho,\theta}$ or $m(\Gamma_-(s)) \in Y^+_{\rho,\theta}$ for $s$ large. Recall that by the conjugacy under affine transformation $T_u(z) = z - p(u)$, $g_{\lambda_0+u^2}$ is conjugated to

$$F_u(\omega) = m(u)\omega + O(\omega^2).$$

Since the multiplier $m(u)$ is a non-constant holomorphic function in $\mathbb{D}(0, \sqrt{r})$ and $m(0) = 1$, $m(u)$ has the following expansion:

$$m(u) = 1 + au^{\ell} + o(u^{\ell}), \tag{6.5}$$

where $\ell$ is a positive integer and $a \neq 0$.

LEMMA 6.6. *$\ell$ is an odd number.*

*Proof.* It is known that for $u \in \mathbb{D}(0, \sqrt{r})$ with $r > 0$ small, $F_u : U_0 \to V_u := F_u(U_0)$ is a quadratic-like map with the unique critical value $v(u) \in U_0$. From Remark 6.3, $F_u$ has two fixed points 0 and $q(u) = p(-u) - p(u)$ contained in $U_0$ with the multipliers $m(u)$ and $m(-u)$, respectively. Suppose $\ell$ is even, then from equation (6.5), we have $m(u) - m(-u) = o(u^{\ell})$, which implies that $|m(u) - m(-u)| < |m(u) - 1|$ for $u$ small since $m(u) - 1 \sim au^{\ell}$. It follows that

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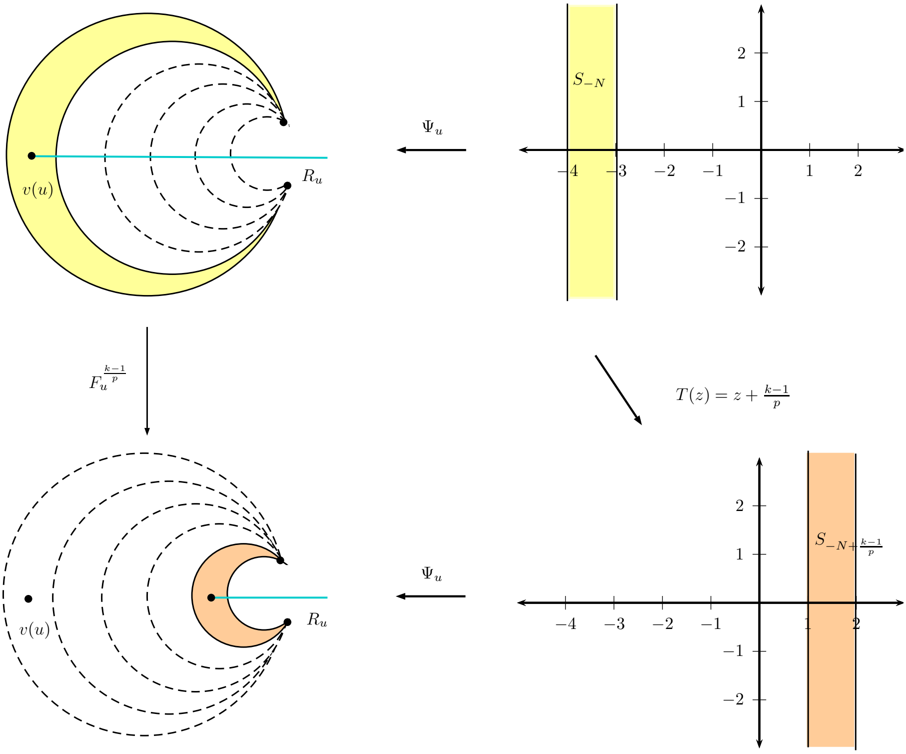


FIGURE 11. Illustration for the proof of Proposition 6.5.

$$\begin{aligned} |m(-u)| &\leq |m(u) - m(-u)| + |m(u)| \\ &< |1 - m(u)| + |m(u)| \\ &= 1 - |m(u)| + |m(u)| = 1. \end{aligned} \tag{6.6}$$

Hence, $q(u) = p(-u) - p(u)$ is an attracting fixed point of $F_u$.

However, by the open mapping theorem, we can find $u$ close to 0 and $q(u) \in U_u$ such that $|m(u)| < 1$. So 0 is also an attracting fixed point of $F_u$ for such $u$. This is impossible since $F_u$ is a quadratic-like map on $U_u$. □

LEMMA 6.7. *For any $\theta \in (0, \pi/2)$, if $u$ is small and $|\arg(m(u) - 1)| < \pi/2 - \theta$, then $m(-u) \in \mathbb{D}$.*

*Proof.* By equation (6.5) and Lemma 6.6 (see Figure 12), $m(u) + m(-u) - 2 = o(u^\ell)$, and then

$$|m(u) + m(-u) - 2| = o(|m(u) - 1|) \tag{6.7}$$

for $u$ small. Let $r(u) := |m(u) - 2|$ and $\theta(u) := \arg(m(u) - 1)$. It follows that for $u$ small,

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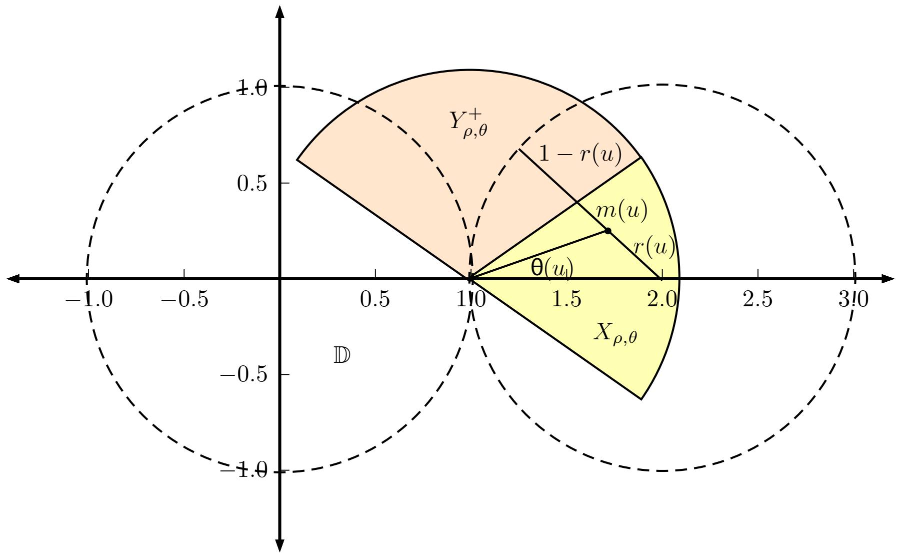


FIGURE 12. Illustration for the proof of Lemma 6.7.

$$\begin{aligned}|m(u)-1| &= \cos(\theta(u)) - \sqrt{(r(u))^2 - \sin^2(\theta(u))} \\ &= \cos(\theta(u)) - \sqrt{(r(u)-1)^2 + 2(r(u)-1) + \cos^2(\theta(u))} \\ &= \cos(\theta(u)) - \cos(\theta(u))\sqrt{1 + \frac{2r(u)-2+(r(u)-1)^2}{\cos^2(\theta(u))}} \\ &= \cos(\theta(u)) - \cos(\theta(u))\left(1 + \frac{r(u)-1}{\cos^2(\theta(u))} + o(r(u)-1)\right) \\ &= \frac{1-r(u)}{\cos(\theta(u))} + o(r(u)-1).\end{aligned}$$

Hence, for $u$ small and $|\theta(u)| \le \pi/2 - \theta < \pi/2$, we have

$$|m(u)-1| = O(1 - |2 - m(u)|). \tag{6.8}$$

Coupling it with equation (6.7) gives that $|m(u)+m(-u)-2| < 1 - |2-m(u)|$ for $u$ small. Hence,

$$\begin{aligned}|m(-u)| &\le |m(u)+m(-u)-2| + |m(u)-2| \\ &< 1 - |2-m(u)| + |m(u)-2| = 1.\end{aligned} \tag{6.9}$$

□

PROPOSITION 6.8. *For any $\theta \in (0, \pi/2)$ and $\rho > 0$, one of the following is true:*

- *$m(\Gamma_+(s)) \in Y^+_{\rho,\theta}$ for all $s > 1$ sufficiently close to 1;*
- *$m(\Gamma_-(s)) \in Y^+_{\rho,\theta}$ for all $s > 1$ sufficiently close to 1.*

*Proof.* Fix any $\theta \in (0, \pi/2)$ and $\rho > 0$. Since $\Gamma_\pm(s) \to 0$ and $m(\Gamma_\pm(s)) \to 1$ as $s \to 1$, we have $|m(\Gamma_\pm(s)) - 1| < \rho$ for $s > 1$ sufficiently close to 1. Since $f_{\lambda_0+u^2}$ is an escape

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map for $u = \Gamma_\pm(s)$, $F_u$ has no attracting fixed points in $\mathbb{C}$. Hence, it follows from Lemma 6.7, $m(\Gamma_\pm(s)) \in (Y^+_{\rho,\theta} \cup Y^-_{\rho,\theta})$ for all $s$ close to 1.

Assume that $m(\Gamma_-(s)) \in Y^-_{\rho,\theta}$ for an $s > 1$, then by continuity, $m(\Gamma_-(s)) \in Y^-_{\rho,\theta}$ for all $s > 1$ close to 1. It follows that

$$\Im(1 - m(\Gamma_-(s))) \geq |1 - m(\Gamma_-(s))| \cos\theta. \tag{6.10}$$

Since $\Gamma_+(s) = -\Gamma_-(s)$, we have from equations (6.7) and (6.10) that

$$m(\Gamma_+(s)) + m(\Gamma_-(s)) - 2 = o(\Im(1 - m(\Gamma_-(s))). \tag{6.11}$$

It follows that

$$\begin{aligned}\Im(m(\Gamma_+(s))) &= \Im(2 - m(\Gamma_-(s))) - \Im(2 - m(\Gamma_-(s)) - m(\Gamma_+(s)))\\ &= \Im(1 - m(\Gamma_-(s))) - \Im(2 - m(\Gamma_-(t)) - m(\Gamma_+(s)))\\ &> \Im(1 - m(\Gamma_-(s))) - |2 - m(\Gamma_-(s)) - m(\Gamma_+(s))|\\ &> \Im(1 - m(\Gamma_-(s))) - \Im(1 - m(\Gamma_-(s))) = 0.\end{aligned} \tag{6.12}$$

Hence, $m(\Gamma_+(s)) \in Y^+_{\rho,\theta}$ for all $s > 1$ close to 1. □

*Proof of Proposition 6.4.* Take $\rho$ small enough. By Proposition 6.8, we can choose an inverse branch $m^{-1}$ of $m$ such that $m^{-1}(Y^+_{\rho,\theta})$ contains $\Gamma_+(s)$ or $\Gamma_-$ for $s$ close to 1. Let $\Delta_{\rho,\theta} := m^{-1}(Y^+_{\rho,\theta}) \subset \mathbb{D}(0, \sqrt{r})$. Then $m : \Delta_{\rho,\theta} \to Y^+_{\rho,\theta}$ is a univalent map. According to the relation of $\sigma$ and $m$, the result follows directly. □

## 7. *Proofs of Theorems 1.1 and 1.2*

In this section, we will present the proofs of Theorems 1.2 and 1.1.

### 7.1. *Closures of escape components are pairwise disjoint.*

In this subsection, we finish the proof of Theorem 1.2. By Proposition 6.5, it remains to show that the closures of escape components with order at least 2 are pairwise disjoint.

LEMMA 7.1. *There is no parameter $\lambda \in \partial\mathcal{H}$ such that $f^m_\lambda(v^+_\lambda) \in C_\lambda$ with $m \geq 0$.*

*Proof.* Suppose that $\lambda$ satisfies $f^m_\lambda(v^+_\lambda) \in C_\lambda$ with $m \geq 0$. Then, either $f^{m+1}_\lambda(v^+_\lambda) = v^+_\lambda$ or $f^{m+1}_\lambda(v^+_\lambda) = v^-_\lambda$. In the first case, we get $f^m_\lambda(v^+_\lambda)$ is a periodic critical point, which is clearly impossible.

In the second case, recall the McMullen map satisfies $f^{m+1}_\lambda(-z) = -f^{m+1}_\lambda(z)$ or $f^{m+1}_\lambda(-z) = f^{m+1}_\lambda(z)$ depending on whether $n$ is odd or even. So if $n$ is odd, then $f^{m+1}_\lambda(v^-_\lambda) = -f^{m+1}_\lambda(v^+_\lambda) = v^+_\lambda$. This means $f^{2m+2}_\lambda(v^+_\lambda) = v^+_\lambda$, which deduces to the first case. If $n$ is even, then $f^{m+1}_\lambda(v^-_\lambda) = f^{m+1}_\lambda(v^+_\lambda) = v^-_\lambda$. So $f^m_\lambda(v^-_\lambda)$ is a periodic critical point with period $m + 1$, which is also clearly impossible. □

PROPOSITION 7.2. *Suppose $\mathcal{U}_1 \subset \mathcal{H}_{k_1}$ and $\mathcal{U}_2 \subset \mathcal{H}_{k_2}$ are two distinct escape components with order $k_1, k_2 \geq 2$, then $\partial\mathcal{U}_1 \cap \partial\mathcal{U}_2 = \emptyset$.*

*Proof.* Suppose $\lambda_0 = \overline{\mathcal{R}^{t_1}_{\mathcal{U}_1}} \cap \overline{\mathcal{R}^{t_2}_{\mathcal{U}_2}} \in \partial\mathcal{U}_1 \cap \partial\mathcal{U}_2$. Without loss of generality, we may assume that $\lambda_0 \in \mathcal{F}_0$. Let $\mathcal{V}_j$ be a simply connected open subset of $\mathbb{C}^* \setminus \bigcup_{2\leq\ell\leq k_j-1} \Lambda_\ell$

containing $\lambda_0$ and $\lambda_{\mathscr{U}_j}$ for $j = 1, 2$. Here, $\Lambda_l$ for $l \geq 2$ is defined in §4. By Lemma 4.1, there exist two root functions $z_{k_j}^{i_j}(\lambda)$ defined on $\mathscr{V}_j$ for $j = 1, 2$ such that $v_{\lambda_{\mathscr{U}_j}}^+ = z_{k_j}^{i_j}(\lambda_{\mathscr{U}_j})$. Then both $z_{k_1}^{i_1}(\lambda_0)$ and $z_{k_2}^{i_2}(\lambda_0)$ are well defined.

Let $U_{k_j}^{i_j}(\lambda_0)$ denote the Fatou component containing $z_{k_j}^{i_j}(\lambda_0)$ and let

$$w_{k_j}^{t_j}(\lambda_0) := \lim_{s\to 1} \phi_{z_{k_j}^{i_j}(\lambda_0)}^{-1}(se^{2\pi i\,\theta_{k_j}(t_j)})$$

for $j = 1, 2$. That is, $w_{k_j}^{t_j}(\lambda_0)$ is the landing point of dynamical ray $R_j := R^{t_j}_{z_{k_j}^{i_j}(\lambda_0)}$ in $U_{k_j}^{i_j}(\lambda_0)$. By Proposition 5.2, $\{w_{k_1}^{t_1}(\lambda_0), w_{k_2}^{t_2}(\lambda_0)\} \subset K_{\lambda_0}^+$. By part (2) of Theorem 2.6, $K_{\lambda_0}^+ = \{v_{\lambda_0}^+\}$. It follows that $w_{k_1}^{t_1}(\lambda_0) = w_{k_2}^{t_2}(\lambda_0) = v_{\lambda_0}^+$, which implies the dynamical rays $R_1$ and $R_2$ land together at the common point $v_{\lambda_0}^+ \in \partial U_{k_1}^{i_1}(\lambda_0) \cap \partial U_{k_2}^{i_2}(\lambda_0)$.

Suppose $z_{k_1}^{i_1}(\lambda_0) \neq z_{k_2}^{i_2}(\lambda_0)$. Notice that both $U_{k_1}^{i_1}(\lambda_0)$ and $U_{k_2}^{i_2}(\lambda_0)$ will be mapped eventually to the Fatou component $B_{\lambda_0}$, and $B_{\lambda_0}$ is invariant. We get that there exists $m > 0$ such that $f_{\lambda_0}^m(U_{k_1}^{i_1}(\lambda_0)) \neq f_{\lambda_0}^m(U_{k_2}^{i_2}(\lambda_0))$, but $f_{\lambda_0}^{m+1}(U_{k_1}^{i_1}(\lambda_0)) = f_{\lambda_0}^{m+1}(U_{k_2}^{i_2}(\lambda_0))$. (In fact, if $k_1 \neq k_2$, then $m = \max\{k_1, k_2\} - 2$; if $k_1 = k_2$, then $m < k_1 - 2$.) Thus, $f_{\lambda_0}^m(R_1)$ and $f_{\lambda_0}^m(R_2)$ are two different dynamical rays which land together at $f_{\lambda_0}^m(v_{\lambda_0}^+)$, but $f_{\lambda_0}^{m+1}(R_1) = f_{\lambda_0}^{m+1}(R_2)$ is a dynamical ray landing on $f_{\lambda_0}^{m+1}(v_{\lambda_0}^+)$ since the Fatou component $f_{\lambda_0}^{m+1}(U_{k_1}^{i_1}) = f_{\lambda_0}^{m+1}(U_{k_2}^{i_2})$ is a Jordan domain (see part (1) of Theorem 2.6) and there is only one dynamical ray landing on $f_{\lambda_0}^{m+1}(v_{\lambda_0}^+)$. This implies $f_{\lambda_0}^m(v_{\lambda_0}^+) \in C_{\lambda_0}$, which contradicts with Lemma 7.1.

Suppose $z_{k_1}^{i_1}(\lambda_0) = z_{k_2}^{i_2}(\lambda_0)$. In this case, $k_1 = k_2 \geq 3$ and $U_{k_1}^{i_1}(\lambda_0) = U_{k_2}^{i_2}(\lambda_0)$. Since $\partial U_{k_1}^{i_1}(\lambda_0)$ is a Jordan curve and $w_{k_1}^{t_1}(\lambda_0) = w_{k_2}^{t_2}(\lambda_0)$, we have $t_1 = t_2$. By the discreteness of pre-images of 0, it is not hard to see that $z_{k_1}^{i_1}(\lambda) = z_{k_2}^{i_2}(\lambda)$ for $\lambda$ near $\lambda_0$. Hence, we can find a simply connected region $\mathcal{V} \subset \mathscr{V}_1 \cup \mathscr{V}_2$ containing $\lambda_{\mathscr{U}_1}$, $\lambda_{\mathscr{U}_2}$, and $\lambda_0$ such that $z_{k_1}^{i_1}(\lambda) = z_{k_2}^{i_2}(\lambda)$ for $\lambda \in \mathcal{V}$. By Proposition 4.4, we have $\lambda_{\mathscr{U}_2} = e^{2m\pi i/(n-1)}\lambda_{\mathscr{U}_1}$ for some $m \in \mathbb{N}$. It follows that $\overline{\mathscr{R}_{\mathscr{U}_1}^{t_1}} \cup \overline{\mathscr{R}_{\mathscr{U}_2}^{t_2}} \cap \mathbb{R}^+ \neq \emptyset$. Without loss of generality, we may suppose $\overline{\mathscr{R}_{\mathscr{U}_1}^{t_1}} \cap \mathbb{R}^+ \neq \emptyset$ and $\lambda' \in \overline{\mathscr{R}_{\mathscr{U}_1}^{t_1}} \cap \mathbb{R}^+$. It is not hard to check that for $\lambda \in \mathbb{R}^+$, both $B_\lambda \cap \mathbb{R}^+$ and $T_\lambda \cap \mathbb{R}^+$ are connected. Hence, we may suppose $B_{\lambda'} \cap \mathbb{R}^+ = (z_0, \infty)$, $T_{\lambda'} \cap \mathbb{R}^+ = (0, z_1)$. Since $\lambda' \in \overline{\mathscr{R}_{\mathscr{U}_1}^{t_1}}$, then $v_{\lambda'}^+ \in [z_1, z_0]$ and $f_{\lambda'}^{k_1}(v_{\lambda'}^+) \in [0, z_1]$. Notice that $v_{\lambda'}^+ = \min_{z\in\mathbb{R}^+} f_{\lambda'}(z)$. It follows that $\lambda' = \lambda_0$ $v_{\lambda'}^+ = z_1 \in \partial T_{\lambda'} \cap \partial U_{k_1}^{i_1}(\lambda')$ which can be deduced to the previous case. □

A combination of Propositions 6.5 and 7.2 completes the proof of Theorem 1.2. The proof of Proposition 7.2 also implies the following.

COROLLARY 7.3. *Suppose $\mathscr{U} \subset \mathscr{H}_k$ is an escape component with $k \geq 2$. If $\lambda = \Phi_{\mathscr{U}}^{-1}(e^{2\pi i t}) \in \partial\mathscr{U}$, then $v_\lambda^+ = \phi_{z_k^i(\lambda)}^{-1}(e^{2\pi i\theta_k(t)}) \in \partial U_k^i(\lambda)$, where $z_k^i$ is the root function defined on a neighborhood of $\overline{\mathscr{U}}$ satisfying that $v_{\lambda_{\mathscr{U}}}^+ = z_k^i(\lambda_{\mathscr{U}})$ and $U_k^i(\lambda)$ is the Fatou component containing $z_k^i(\lambda)$. In particular, $v_\lambda^\pm \notin \partial B_\lambda$.*

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7.2. *McMullen domain and Sierpiński holes are quasi-disks.* In this section, we prove that the McMullen domain and all Sierpiński holes are all bounded by quasi-circles. Recall that $\mathscr{U}$ is a McMullen domain or a Sierpiński hole if and only if $\mathscr{U} \subset \mathscr{H}_k$ is an escape component with $k \geq 2$.

LEMMA 7.4. *Let $\mathscr{W} = \mathbb{C} \setminus \overline{\mathscr{H}_0}$. Then for any $\lambda_0 \in \mathscr{W}$, there is a holomorphic motion based at $\lambda_0$:*

$$H : \mathscr{W} \times B_{\lambda_0} \to \mathscr{W} \times \overline{\mathbb{C}}. \tag{7.1}$$

*Proof.* When $\lambda \in \mathscr{W} \setminus \{0\}$, $C_\lambda \cap B_\lambda = \emptyset$ and $B_\lambda$ is a Jordan region. The Böttcher coordinate $\phi_{\infty(\lambda)}$ is a holomorphic homeomorphism from $B_\lambda$ to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$. When $\lambda = 0$, set $B_0 = \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\phi_{\infty(0)} = \mathrm{id}$. By the usual construction of the Böttcher coordinates, it is easy to check $\phi_{\infty(\lambda)}$ is holomorphic with respect to $\lambda \in \mathscr{W}$, even at $\lambda = 0$. Hence,

$$H_\lambda(z) := \phi^{-1}_{\infty(\lambda)} \circ \phi_{\infty(\lambda_0)}(z), \quad z \in B_{\lambda_0}$$

is well defined for $\lambda \in \mathscr{W}$. Then it is direct to verify that $H : \mathscr{W} \times B_{\lambda_0} \to \mathscr{W} \times \overline{\mathbb{C}}$, $(\lambda, z) \mapsto (\lambda, H_\lambda(z))$ is a holomorphic motion. □

By Theorem 2.3, the holomorphic motion defined in Lemma 7.4 can be extended to a holomorphic motion $H : \mathscr{W} \times \overline{\mathbb{C}} \to \mathscr{W} \times \overline{\mathbb{C}}$. It follows that the following map:

$$\Phi_k(\lambda) := \begin{cases} H_\lambda^{-1}(f_\lambda^{k-1}(v_\lambda^+)), & \lambda \in \mathscr{W} \setminus \{0\}, \\ \infty, & \lambda = 0 \end{cases} \tag{7.2}$$

is well defined on $\mathscr{W}$ for all $k \geq 2$. Noting that when $\lambda \to 0$, $H_\lambda^{-1}(f_\lambda^{k-1}(v_\lambda^+)) \to \infty$, $\Phi_k(\lambda)$ is continuous even at $\lambda = 0$.

In the following, we always assume that $\lambda_0 \in \mathscr{W}$ is given.

LEMMA 7.5. *The map $\Phi_k : \mathscr{W} \to \overline{\mathbb{C}}$ defined by equation (7.2) is quasi-regular on any region $\mathscr{W}^* \Subset \mathscr{W}$.*

*Proof.* Consider the derivative of equation $H_\lambda \circ \Phi_k(\lambda) = f_\lambda^{k-1}(v_\lambda^+)$. Since $H_\lambda$ and $f_\lambda^{k-1}(v_\lambda^+)$ are all holomorphic in $\lambda$, therefore, $\partial H_\lambda/\partial\overline{\lambda} = \partial f_\lambda^{k-1}(v_\lambda^+)/\partial\overline{\lambda} = 0$, and we have

$$\frac{\partial H_\lambda}{\partial z}\bigg|_{\Phi_k(\lambda)} \frac{\partial \Phi_k}{\partial \overline{\lambda}}\bigg|_\lambda + \frac{\partial H_\lambda}{\partial \overline{z}}\bigg|_{\Phi_k(\lambda)} \frac{\partial \overline{\Phi}_k}{\partial \overline{\lambda}}\bigg|_\lambda = 0, \tag{7.3}$$

where $\partial H_\lambda/\partial z$ and $\partial H_\lambda/\partial \overline{z}$ exist almost everywhere since $H_\lambda$ is quasi-conformal. Thus,

$$\left|\frac{\partial \Phi_k/\partial\overline{\lambda}}{\partial \Phi_k/\partial\lambda}\bigg|_\lambda\right| = \left|\frac{\partial \Phi_k/\partial\overline{\lambda}}{\partial \overline{\Phi}_k/\partial\overline{\lambda}}\bigg|_\lambda\right| = \left|\frac{\partial H_\lambda/\partial\overline{z}}{\partial H_\lambda/\partial z}\bigg|_{\Phi_k(\lambda)}\right| = |\mu_\lambda(\Phi_k(\lambda))|, \tag{7.4}$$

where $\mu_\lambda$ is the Beltrami coefficient of $H_\lambda$.

Let $\rho(\cdot, \cdot)$ denote the hyperbolic distance of $\mathscr{W}$. Then,

$$\rho^* = \sup_{\lambda \in \mathscr{W}^*} \rho(\lambda, \lambda_0) < \infty.$$

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By Theorem 2.3, for any $\lambda \in \mathscr{W}^*$,

$$\operatorname*{ess\,sup}_{z\in\overline{\mathbb{C}}} |\mu_\lambda(z)| \le \frac{e^{\rho(\lambda,\lambda_0)}-1}{e^{\rho(\lambda,\lambda_0)}+1} \le k := \frac{e^{\rho^*}-1}{e^{\rho^*}+1} < 1. \tag{7.5}$$

Therefore,

$$\left\| \frac{\partial\Phi_k/\partial\overline{\lambda}}{\partial\Phi_k/\partial\lambda} \right\|_\infty = \operatorname*{ess\,sup}_{\lambda\in\mathscr{W}^*} \left| \frac{\partial\Phi_k/\partial\overline{\lambda}}{\partial\Phi_k/\partial\lambda} \right| = \operatorname*{ess\,sup}_{\lambda\in\mathscr{W}^*} |\mu_\lambda(\Phi_k(\lambda))| \le k < 1. \tag{7.6}$$

It means that $\Phi_k$ is a quasi-regular map on the region $\mathscr{W}^*$. □

From the proof of Lemma 7.5, we have the following corollary.

COROLLARY 7.6. *Let $\mu_{\Phi_k}(\lambda) = (\partial\Phi_k/\partial\overline{\lambda})/(\partial\Phi_k/\partial\lambda)$ be the Beltrami coefficient of $\Phi_k(\lambda)$. Let $\mathscr{D}_m = \mathbb{D}(\lambda_0, 1/m)$ for $m > 0$ large such that $\mathscr{D}_m \subset \mathscr{W}$. Then,*

$$\|\mu\|_{m,k} := \operatorname*{ess\,sup}_{\lambda\in\mathscr{D}_m} |\mu_{\Phi_k}(\lambda)| \to 0 \quad \textit{as } m \to \infty. \tag{7.7}$$

*Proof.* It follows immediately from equations (7.5), (7.6), and the fact that $\sup_{\lambda\in\mathscr{D}_m} \rho(\lambda, \lambda_0) \to 0$ as $m \to \infty$. □

PROPOSITION 7.7. *Let $\mathscr{U}$ be an escape component of $\mathscr{H}_k$ with $k \ge 2$. Then there is a neighborhood $\mathscr{V}$ of $\overline{\mathscr{U}}$ and a quasi-conformal homeomorphism $\Psi_k : \mathscr{V} \to \overline{\mathbb{C}}$ such that $\Psi_k(\overline{\mathscr{U}}) = \overline{B_{\lambda_0}}$.*

*Proof.* Set $\Psi_k : \mathscr{W} \to \overline{\mathbb{C}}$ be defined by

$$\Psi_k := \begin{cases} \phi^{-1}_{\infty(\lambda_0)} \circ (\phi_{\infty(\lambda_0)} \circ \Phi_k)^{2/(n-2)}, & k = 2, \\ \phi^{-1}_{\infty(\lambda_0)} \circ (\phi_{\infty(\lambda_0)} \circ \Phi_k)^{1/n}, & k \ge 3, \end{cases} \tag{7.8}$$

or more clearly, using equation (7.2), $\Psi_k(0) = \infty$ and for $\lambda \ne 0$,

$$\Psi_k(\lambda) = \begin{cases} \phi^{-1}_{\infty(\lambda_0)} \circ (\phi_{\infty(\lambda_0)} \circ H_\lambda^{-1}(f_\lambda(v_\lambda^+)))^{2/(n-2)}, & k = 2, \\ \phi^{-1}_{\infty(\lambda_0)} \circ (\phi_{\infty(\lambda_0)} \circ H_\lambda^{-1}(f_\lambda^{k-1}(v_\lambda^+)))^{1/n}, & k \ge 3. \end{cases} \tag{7.9}$$

When $\lambda \in \mathscr{U}$, it is easy to check that

$$\Psi_k(\lambda) = \phi^{-1}_{\infty(\lambda_0)} \circ \Phi_{\mathscr{U}}(\lambda),$$

where $\Phi_{\mathscr{U}}(\lambda)$ is defined in Theorem 2.2 which is a holomorphic homeomorphism from $\mathscr{U}$ to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ (here for $k = 2$, we need to extend the definition of $\Phi_{\mathscr{U}}$ such that $\Phi_{\mathscr{U}}(0) = \infty$). It follows that $\Psi_k : \mathscr{U} \to B_{\lambda_0}$ is a holomorphic homeomorphism. By parts (1), (3) of Theorem 2.6, both $\partial\mathscr{U}$ and $\partial B_{\lambda_0}$ are Jordan curves. Hence, $\Psi_k : \overline{\mathscr{U}} \to \overline{B_{\lambda_0}}$ is a homeomorphism with $\Psi_k(\partial\mathscr{U}) = \partial B_{\lambda_0}$.

By Propositions 6.5 and 7.2, we have $\overline{\mathscr{U}} \cap \overline{\mathscr{U}'} = \emptyset$ for any escape component $\mathscr{U}'$ different from $\mathscr{U}$. Since for fixed $\ell \le k$, $\mathscr{H}_\ell$ has only finitely many components, there exists a simply connected region $\mathscr{W}^*$ such that $\mathscr{U} \Subset \mathscr{W}^* \Subset \mathscr{W}$ and $\overline{\mathscr{W}^*} \cap \overline{\mathscr{U}'} = \emptyset$ for all components $\mathscr{U}'$ of $\mathscr{H}_\ell$ with $0 \le \ell \le k$ which are different from $\mathscr{U}$.

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Now, we restrict $\Psi_k$ on $\mathscr{W}^*$ and then show that $\Psi_k^{-1}(\overline{B_{\lambda_0}}) = \overline{\mathscr{U}}$. That is, if $\lambda \in \mathscr{W}^*$ such that $\Psi_k(\lambda) \in \overline{B_{\lambda_0}}$, then $\lambda \in \overline{\mathscr{U}}$.

We first prove that $\Psi_k(\lambda) \in B_{\lambda_0}$ implies $\lambda \in \mathscr{U}$. From equation (7.9) and the definition of $\phi_{\infty(\lambda)}$, we have that $\Psi_k(\lambda) \in B_{\lambda_0}$ implies $H_\lambda^{-1}(f_\lambda^{k-1}(v_\lambda^+)) \in B_{\lambda_0}$. Since $H_\lambda : \mathbb{C} \to \mathbb{C}$ is a quasi-conformal homeomorphism and $H_\lambda(B_{\lambda_0}) = B_\lambda$, we get that $f_\lambda^{k-1}(v_\lambda^+) \in B_\lambda$. This shows that $\lambda \in \mathscr{H}_\ell$ for some $\ell \le k$. However, by the definition of $\mathscr{W}^*$, $\mathscr{W}^* \cap \bigcup_{\ell \le k} \mathscr{H}_\ell = \mathscr{U}$. We get that $\lambda \in \mathscr{U}$.

Suppose that $\Psi_k(\lambda) \in \partial B_{\lambda_0}$. Since $\Phi_k$ is quasi-regular by Lemma 7.5, $\Psi_k$ is also quasi-regular and obviously non-constant. Thus, $\Psi_k$ is an open map. It follows that for any neighborhood $\mathscr{N}$ of $\lambda$, $\Psi_k(\mathscr{N}) \cap B_{\lambda_0} \neq \emptyset$. Hence, $\mathscr{N} \cap \mathscr{U} \neq \emptyset$, which implies that $\lambda \in \partial\mathscr{U}$.

Finally, we take a Jordan region $\mathscr{V}'$ satisfying $\mathscr{U} \Subset \mathscr{V} \Subset \mathscr{W}^*$. Let $\gamma = \partial\mathscr{V}$ and $\Gamma = \Psi_k(\partial\mathscr{V})$. Then, by the discussion above, $\Gamma \cap \overline{B_{\lambda_0}} = \emptyset$. Let $B'$ be the component of $\mathbb{C} \setminus \Gamma$ which contains $B_{\lambda_0}$. Noting that a quasi-regular map is the composition of a holomorphic map and a quasi-conformal homeomorphism, the argument principle can be applied. Since the point $z \in B_{\lambda_0} \subset B'$ has only one pre-image of $\Psi_k$ in $\mathscr{U} \subset \mathscr{V}'$, we get that every $z \in B'$ has only one pre-image in $\mathscr{V}'$. Take $\mathscr{V} = \Psi_k(U') \subset \mathscr{V}'$. Then, $\mathscr{V}$ is a neighborhood of $\mathscr{U}$ and $\Psi_k : \mathscr{V} \to \Psi_k(\mathscr{V}) = B'$ is a quasi-conformal homeomorphism. □

COROLLARY 7.8. *$\partial\mathscr{U}$ is a quasi-circle.*

*Proof.* Take $\lambda_0 = \lambda_{\mathscr{U}}$ as the center of $\mathscr{U}$. Then, $f_{\lambda_0}$ is hyperbolic and hence $\lambda_0$ satisfies the condition of part (1) of Theorem 2.6. Hence, $\partial B_{\lambda_0}$ is a quasi-circle. By Proposition 7.7, $\partial\mathscr{U}$ is also a quasi-circle. □

7.3. *Hausdorff dimension of the boundary of escape component.* Let $\dim_H X$ denote the Hausdorff dimension of a Borel subset $X$ of $\mathbb{C}$. The following results are well known, see [**Fal04**].

LEMMA 7.9. *Let $X \subset \mathbb{C}$ be a Borel subset. If $f : X \to \mathbb{C}$ satisfies the Hölder condition*

$$|f(z_1) - f(z_2)| \le C|z_1 - z_2|^\alpha,$$

*then*

$$\dim_H f(X) \le \frac{1}{\alpha} \dim_H X.$$

LEMMA 7.10. *If $f$ is a non-constant holomorphic map defined on a neighborhood of $X \subset \mathbb{C}$, then*

$$\dim_H f(X) = \dim_H X.$$

Recall that for a $K$-quasi-conformal homeomorphism, we have the following Mori's theorem, see [**Ahl06**].

THEOREM 7.11. (Mori) *Suppose $f : \mathbb{D} \to \mathbb{D}$ is a $K$-quasi-conformal homeomorphism. Then for each $z_1, z_2 \in \mathbb{D}$,*

$$|f(z_1) - f(z_2)| \le 16|z_1 - z_2|^{1/K}. \tag{7.10}$$

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We also need the following theorem due to Przytycki [**Prz06**] as we know.

THEOREM 7.12. (Przytycki) *Let $f:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational map of degree $d\ge 2$ and $\Omega$ be a simply connected immediate basin of attraction to a periodic attracting point. Then, provided f is not a Blaschke product in some holomorphic coordinates or a quotient of a Blaschke product by a rational function of degree 2, the Hausdorff dimension of $\partial\Omega$ is greater than 1.*

COROLLARY 7.13. *Let $\mathscr{U}\subset\mathscr{H}_k$ be an escape component with order $k\ge 2$. If $\lambda_0\in\partial\mathscr{U}$ and $z_0\in\partial B_{\lambda_0}$, then for any neighborhood U of $z_0$,*

$$\dim_H(U\cap\partial B_{\lambda_0})=\dim_H\partial B_{\lambda_0}>1.$$

*Proof.* From Theorem 7.12, we have that $\dim_H\partial B_{\lambda_0}>1$ as long as $\lambda_0\in\mathscr{W}\setminus\{0\}$. So $\dim_H\partial B_{\lambda_0}>1$ if $\lambda_0\in\partial\mathscr{U}$. For any neighborhood $U$ of $z_0\in\partial B_{\lambda_0}$,

$$f_{\lambda_0}^m(U\cap\partial B_{\lambda_0})=\partial B_{\lambda_0}$$

as $m$ is sufficiently large. By Lemma 7.10,

$$\dim_H(U\cap\partial B_{\lambda_0})=\dim_H f_{\lambda_0}^m(U\cap\partial B_{\lambda_0})=\dim_H\partial B_{\lambda_0}>1.\qquad\square$$

PROPOSITION 7.14. *Let $\mathscr{U}\subset\mathscr{H}_k$ be an escape component with order $k\ge 2$. Then the Hausdorff dimension of $\partial\mathscr{U}$ satisfies*

$$1<\dim_H\partial\mathscr{U}<2. \tag{7.11}$$

*Proof.* Astala [**Ast94**] proved that the image of a set of Hausdorff dimension 1 under a $K$-quasi-conformal homeomorphism has the Hausdorff dimension at most $1+k$, where $k=(K-1)/(K+1)<1$. It follows that any quasi-circle has the Hausdorff dimension less than 2. So, $\dim_H\partial\mathscr{U}<2$ since $\partial\mathscr{U}$ is a quasi-circle by Corollary 7.8. It remains to show that $\dim_H\partial\mathscr{U}>1$.

Choose a $\lambda_0\in\partial\mathscr{U}$ and let $\mathscr{D}_m=\mathbb{D}(\lambda_0,1/m)$ for $m>0$ large such that $\mathscr{D}_m\subset\mathscr{V}$, where $\mathscr{V}$ is given in Proposition 7.7. By Corollary 7.6, equation (7.7) holds, that is,

$$\|\mu\|_{m,k}:=\operatorname*{ess\,sup}_{\lambda\in\mathscr{D}_m}|\mu_{\Phi_k}(\lambda)|\to 0\quad\text{as } m\to\infty.$$

Then, $\Phi_k$, as a quasi-regular map restricted on $\mathscr{D}_m$, has its maximal dilatation

$$K_{m,k}:=\frac{1+\|\mu\|_{k,m}}{1-\|\mu\|_{k,m}}\to 1\quad\text{as } m\to\infty. \tag{7.12}$$

Let $\Psi_k$ be the quasi-conformal homeomorphism defined in equation (7.8), let $\eta_{m,k}$ be the Riemann map from $\Psi_k(\mathscr{D}_m)$ onto $\mathbb{D}$, and let $\xi_m(\lambda):\mathscr{D}_m\to\mathbb{D}$ be the affine map defined by $\xi_m(\lambda)=m(\lambda-\lambda_0)$. Then, $\widetilde{\Psi}_{m,k}:\mathbb{D}\to\mathbb{D}$ defined by $\widetilde{\Psi}_{m,k}:=\eta_{m,k}\circ\Psi_k\circ\xi_m^{-1}$ is a quasi-conformal homeomorphism. It has the same maximal dilatation as $\Phi_k$ since $\phi_{\infty(\lambda_0)}$, $\eta_{m,k}$, and $\xi_m$ are all conformal. By Theorem 7.11 and Lemmas 7.9, 7.10,

$$\dim_H(\partial\mathscr{U}\cap\mathscr{D}_m)\ge\frac{1}{K_m}\mathrm{dim_H}(\Psi_k(\partial\mathscr{U}\cap\mathscr{D}_m)). \tag{7.13}$$

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By Corollary 7.13, there exists a constant $c > 1$ such that

$$\dim_H(\Psi_k(\partial\mathscr{U} \cap \mathscr{D}_m)) = \dim_H(U_{m,k} \cap \partial B_{\lambda_0}) = \dim_H \partial B_{\lambda_0} \geq c > 1,$$

where $U_{m,k} = \Psi_k(\mathscr{D}_m)$ is a neighborhood of $z_0 := \Psi_k(\lambda_0) \in \partial B_{\lambda_0}$. Since $K_m \to 1$ as $m \to \infty$, then $\dim_H(\partial\mathscr{U} \cap \mathscr{D}_m) > 1$ for $m$ large enough. Hence,

$$\dim_H \partial\mathscr{U} \geq \dim_H(\partial\mathscr{U} \cap \mathscr{D}_m) > 1. \qquad \square$$

*Proof of Theorem 1.1.* The proof is a combination of Corollary 7.8 and Proposition 7.14. □

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