# The repayment of financial debt: some mathematical considerations

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## 1. *Introduction*

This paper focuses on the mathematics behind the repayment of financial debt. The availability of credit, the inevitable accompaniment of which is debt, is an essential component of a monetised economy. Without it most people would not be able to purchase large items like homes, cars and other expensive consumer durables. Businesses would not expand and prosper without access to credit. However, the debt incurred by the availability of credit can also bring huge stress, and indeed distress, from personal and commercial insolvencies to the inhumanities of debt bondage associated with modern-day slavery and human trafficking. The historical origins of debt lie in antiquity and even pre-date the existence of monetised economies [1].

Since the global financial crash of 2008, concern has been expressed in both the UK and the USA over the level of debt burden on personal finances [2, 3, 4, 5]. According to both the IMF [6] and UNCTAD [7], the covid-19 pandemic has substantially increased debt levels across virtually all countries. The UK debt counselling charity, StepChange, reported in 2020 the increasing indebtedness of many households from loss of income due to the pandemic [8]. In 2021 the Bank of England reported that, while the UK household debt burden had not increased in general during the pandemic, the share of households reporting financial difficulty had started to increase, particularly for households with unsecured loans [9]. Then, with the war in Ukraine starting in 2022, global investment managers Fidelity International reported in May 2022 that the resulting higher energy prices and food scarcity, particularly lack of sunflower oil, could have serious implications on the creditworthiness of many already indebted borrowers [10]. In the USA, the Congressional Research Service argued that during the pandemic levels of personal debt default were mitigated by loan forbearance agreements enabling temporary suspension, but not cancellation, of repayments [11]. With increasing levels of debt there is then the attendant risk of defaulting on the debt repayment, with major implications for both lender and borrower.

In view of the adverse social consequences of unsustainable debt, it is worthwhile to examine the mathematics behind debt or loan repayment. Under UK consumer credit protection and Truth in Lending legislation in the USA, financial institutions, if approached for a loan of capital, will quote the following:

(i) *periodic repayments* required for a given interest rate and term (time duration) of repayments

(ii) *term* of the loan for a given interest rate and periodic repayments

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(iii) *cumulative interest* paid on the debt for a given repayment over a specified term and annual interest rate (the finance charge)

(iv) *annual* (*nominal*) *interest rate* levied for a given periodic repayment over a specified term.

This paper derives the equations that determine these four quantities and then continues to model the effects of repayment deferral and loan forbearance arrangements on the levels of interest paid by the borrower. While derivations of the necessary equations can be found at various places within most texts on financial or actuarial mathematics e.g. [12, 13, 14], the contribution of this paper is to bring all the relevant equations together within a single work for readers who are not familiar with the specialised notation employed in financial (actuarial) mathematics. Algebraic formulae are also developed to construct debt repayment (amortisation) schedules and numerical examples are given in the use of the derived equations.

It is recognised that different methodologies can be used to repay debts and these can result in different repayment schedules, as exemplified in [13]. The most common method used by financial institutions, and the one adopted here, is repayment by equal instalments made over a specified period of time—a fixed (or level) annuity in actuarial terms. It is further assumed that all debt repayments are made at the end of each repayment period, as is often the case in practice, and that interest rates remain the same throughout the term of the loan.

The mathematical analysis presented here is similar to that given in [15] and [16] on the financing of pensions and end-of-life care respectively. Here however, in focusing on financing debt, attention shifts to a critical social issue that can seriously impact adults in all age ranges but particularly younger people, often with long-term adverse consequences. Finally, it should be noted that none of the mathematical modelling presented should replace the need for independent professional debt counselling.

## 2. *Notation*

| | |
|---|---|
| Capital borrowed (debt): | $A_0$ |
| Total number of repayment periods: | $n$ |
| Outstanding debt (principal) after time (repayment) periods | $A_i \quad i \in \{0,1,2\dots\}$ |
| Interest paid after time (repayment) periods | $I_i \quad i \in \{0,1,2\dots\}$ |
| Repayment periods per annum: | $k$ |
| Repayments per time period: | $P_k$ |
| Interest conversions per annum: | $K \quad 1 \leqslant K < \infty$ |
| Annual rate of interest: | $r_a$ |
| Interest rate per repayment period: | $r_k$ |

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3. *The debt repayment model*

As detailed in [15], the accumulated value of 1 unit of capital at $r_k$ per cent per interest period at the end of $k$ interest periods is $(1 + r_k)^k$ and the accumulated value of 1 unit of capital at $r_a$ per cent per annum compounded $K$ times is $\left(1 + \frac{r_a}{K}\right)^K$. When the two are equivalent

$$(1 + r_k)^k = \left(1 + \frac{r_a}{K}\right)^K, \qquad \text{from which} \qquad r_k = \left(1 + \frac{r_a}{K}\right)^{K/k} - 1 \tag{1}$$

and as $K \to \infty$ (continuous compounding) $(1 + r_k)^k \to e^{r_a}$.

It may be noted that when $k = 1$ $r_k$ is known as the *Annual Effective Rate* (AER) of interest, while $r_a$ is the nominal rate or, when expressed as a percentage, the *Annual Percentage Rate* (APR) of interest; (1) shows the relationship between the two.

Initially when $i = 0$ i.e. on debt evaluation date, the outstanding debt, or total amount borrowed, is $A_0$. Then after one, two and three successive repayment period, or $\frac{1}{k}, \frac{2}{k}, \frac{3}{k}$ years:

$$\begin{aligned} A_1 &= A_0\left(1 + r_k\right) - P_k \\ A_2 &= A_1\left(1 + r_k\right) - P_k \\ &= A_0(1 + r_k)^2 - P_k\left\{\left(1 + r_k\right) + 1\right\} \\ A_3 &= A_2\left(1 + r_k\right) - P_k \\ &= A_0(1 + r_k)^3 - P_k\left\{\left(1 + r_k\right)^2 + \left(1 + r_k\right) + 1\right\}. \end{aligned}$$

After $n$ repayments summing the resulting geometric series

$$A_n = A_0(1 + r_k)^n - P_k\left\{\frac{(1 + r_k)^n - 1}{r_k}\right\}. \tag{2}$$

The following may be noted about (2):

- it expresses the outstanding debt (principal) after $n$ repayments of $P_k$.
- lending institutions generally quote interest rates as annual (nominal) rates, so by substituting (1) into (2) $A_n$ can be expressed in terms of $r_a$.
- it requires $r_a \neq 0$, as in the case of $r_a = 0$ money has no time value and the debt is interest free, in which case $P_k = A_0/n$.
- it can be expressed as a recursive relation where after $i$ time periods:

$$A_i = A_{i-1} + A_{i-1}r_k - P_k \tag{3}$$

and, by substituting (2) into (3) and rearranging, the following useful equations are developed:

- the interest paid in each repayment period:

$$A_{i-1}r_k = r_kA_0(1 + r_k)^{i-1} - P_k\left\{(1 + r_k)^{i-1} - 1\right\} \tag{4}$$

- the debt repaid in any one repayment:

$$A_{i-1} - A_i = \left(P_k - r_kA_0\right)(1 + r_k)^{i-1} \tag{5}$$

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- the cumulative debt repaid after $i$ repayments:

$$A_0 - A_1 = P_k \left\{ \frac{(1 + r_k)^i - 1}{r_k} \right\} - A_0 \left\{ (1 + r_k)^i - 1 \right\}. \tag{6}$$

Equations (4), (5) and (6) are important for their use in the development of debt amortisation schedules as shown below in the numerical example.

4. *Debt repayments*

The debt is fully repaid, i.e. amortised, when $A_n = 0$; hence by rearrangement of (2) repayments $P_k$ are given by

$$P_k = \frac{r_k A_0 (1 + r_k)^n}{(1 + r_k)^n - 1}. \tag{7}$$

Monetary values can only be given to two decimal places, yet the precise theoretical value of repayments arising from (7) may extend to an infinite number of decimal places. Furthermore, there are then cumulative rounding errors in the amortisation of the debt. Consequently, a common commercial practice is to round up repayments to the nearest whole unit of currency for $n - 1$ repayments. The final, $n$th repayment then is a smaller amount. If $P_{kA}$ is the adjusted (rounded up) repayment for $n - 1$ repayments, then the final repayment $P_{kF}$, with $P_{kF} < P_{kA}$ and with the accrual of one repayment period of interest, is given by

$$P_{kF} = A_{n-1} \left(1 + r_k\right),$$

which from (2) gives

$$P_{kF} = A_0 (1 + r_k)^{n-1} - P_{kA} \left\{ \frac{(1 + r_k)^{n-1} - 1}{r_k} \right\} \left(1 + r_k\right). \tag{8}$$

5. *Term (time duration) of the repayments*

The repayment periods $n$ taken to amortise the debt in full are found from rearranging (2) with $A_n = 0$

$$n = \frac{\ln (P_k) - \ln (P_n - r_k A_0)}{\ln \left(1 + r_k\right)}, \tag{9}$$

from which it is required that $P_k > r_k A_0$—otherwise the repayments would be insufficient to pay the interest on the capital loaned and so the debt would increase and never be repaid.

6. *Cumulative interest payable on the debt*

The cumulative (total) amount of debt (principal) repaid at time $i$ is given by (6) but the total amount of repayment made is $iP_k$. Hence the cumulative interest $I_i$ paid at time $i$ is given by:

$$I_i = iP_k - \left(A_0 - A_i\right)$$

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which may more usefully be rearranged as

$$I_i = A_i - A_0 + iP_k, \tag{10}$$

and substituting (2) into (10) gives

$$I_i = A_0\left\{(1 + r_k)^i - 1\right\} - P_k\left\{\frac{(1 + r_k)^i - 1}{r_k}\right\} + iP_k. \tag{11}$$

The following should be noted:

(i) on amortisation of the debt, when $i = n$ and $A_n = 0$, (10) becomes $I_n = nP_k - A_0$

(ii) since it is required that $P_{kA} > P_k$ then it is seen from (11) that the use of $P_{kA}$ will result in a slightly lower cumulative interest paid than with $P_k$ since the raised monthly repayment pays off more of the debt slightly sooner.

## 7. *The annual rate of interest*

Equation (2) has no explicit algebraic solution for $r_k$. Rearranging (2), again with $A_n = 0$, gives

$$r_kA_0(1 + r_k)^n - P_k(1 + r_k)^n + P_k = 0; \tag{12}$$

approximate graphical solutions or more accurate numerical solutions to (12) (e.g. by the Newton-Raphson method) can be found as detailed in [17, §8.4] and [18, §3.9]. Having determined $r_k$, $r_a$ is then found from rearranging (1),

$$r_a = K\left\{\sqrt[K]{(1 + r_k)^k} - 1\right\} \tag{13}$$

or, in the case of continuous compounding,

$$r_a = k\left\{\ln\left(1 + r_k\right)\right\}. \tag{14}$$

Equations (7), (9) and (11), respectively, answer the questions (i), (ii), (iii) posed in the introduction while (iv) is found from solving (12) for $r_k$ then using (1) to find the annual interest rate $r_a$. Debt amortisation schedules can be constructed from using (2), (4), (5), (6) and (11).

## 8. *Debt forbearance arrangements*

Suppose debt repayments, initially negotiated to extend over $n$ time periods, are suspended at time $m$ (i.e. after repayments) until time $t$ with $0 < m < t < n$. At time $t$ repayments resume and continue until amortisation of the debt. Of particular concern here to both borrower and lender, but maybe particularly to the former, is the interest (finance charge) paid over the full term of the debt repayments. Repayment levels of $P_k$ are used below, but equally $P_{kA}$, the upwardly adjusted level, could be used with the only difference being slightly lower interest paid.

After repayments, the outstanding debt is given by

$$A_m = A_0(1 + r_k)^m - P_k\left\{\frac{(1 + r_k)^m - 1}{r_k}\right\}, \tag{15}$$

where $P_k$ is initially determined from (7) as if the debt would fully amortise after all $n$ repayments.

The total interest $I_m$ paid on the first $m$ repayments, from (10), is given by

$$I_m = A_m - A_0 + mP_k. \tag{16}$$

Substitution of (15) into (16) gives

$$I_m = A_0\left\{(1 + r_k)^m - 1\right\} - P_k\left\{\frac{(1 + r_k)^m - 1}{r_k}\right\} + mP_k.$$

At time $t$ the outstanding debt $A_t$ is given by

$$A_t = A_m(1 + r_k)^{t - m} \tag{17}$$

and the interest $I_{t-m}$ paid during the forbearance period $t - m$ is

$$I_{t-m} = A_m\left\{(1 + r_k)^{t-m} - 1\right\}. \tag{18}$$

Since $I_t = I_m + I_{t-m}$ the accumulated interest $I_t$ at time $t$, becomes

$$I_t = A_m(1 + r_k)^{t-m} - A_0 + mP_k, \tag{19}$$

and substitution of (15) into (19) gives

$$I_t = A_0\left\{(1 + r_k)^t - 1\right\} - P_k\left\{\frac{(1 + r_k)^m - 1}{r_k}\right\}(1 + r_k)^{t-m} + mP_k.$$

Upon resumption of repayments at time $t$ the debt needs refinancing in order to amortise. Of the original debt repayment schedule there are still $n - t$ time periods remaining.

Assume a new repayment level $Q_k$ (say) within the range $A_t < Q_k \leqslant P_k$. With this new repayment level, a new loan term $N$ (say), with $N > n - t$, for the outstanding debt $A_t$ has to be calculated using (7). The resulting value of $N$ needs rounding down to the nearest integer value so as to coincide with a repayment period. There will then be $N$ repayment periods of $Q_k$, and a final, lower repayment based on the outstanding debt after $N$ repayments. Consequently, the debt amortises after a further $N + 1$ repayments made after time $t$. Using (10), after $N$ time periods the interest $I_N$ paid is

$$I_N = A_N - A_t + Q_kN,$$

and after $N + 1$ time periods the interest paid is

$$I_{N+1} = A_{N+1} - A_t + Q_k(N + 1), \tag{20}$$

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where $A_{N+1}$ is found from (2) but with $n$ replaced by $N + 1$. Consequently

$$I_{N+1} = A_t\left\{(1 + r_k)^{N+1} - 1\right\} - Q_k\left\{\frac{(1 + r_k)^{N+1} - 1}{r_k}\right\} + Q_k(N + 1).$$

Total cumulative interest paid on the entire debt is $I_t + I_{N+1}$ i.e. (19) + (20), which gives

$$I_{t+N+1} = A_m(1 + r_k)^{t-m} - A_0 + mP_k + A_{n+1} - A_t + Q_k(N + 1). \qquad (21)$$

Substituting $A_m$, $A_t$ and $A_{N+1}$ into (21) gives after some algebra,

$$I_{t+N+1} = A_0\left\{(1 + r_k)^{t+N+1} - 1\right\} - P_k(1 + r_k)^{t+N+1-m}\left\{\frac{(1 + r_k)^m - 1}{r_k}\right\} + mP_k$$

$$-Q_k\left\{\frac{(1 + r_k)^{N+1} - 1}{r_k}\right\} + Q_k(N + 1).$$

It may be noted further that:

(i) if the same level of repayment is made both pre- and post-forbearance period then $P_k = Q_k$,

(ii) if there is no forbearance period then $t = m = 0$ and hence (21) becomes (10), as required, but with $N + 1$ and $Q_k$ replaced by $n$ and $P_k$ respectively,

(iii) if demanded by either the lender or borrower, the annual interest rate $r_a$ may change at time of refinancing the outstanding debt, but this is not a necessity of the model.

Figure 1 illustrates the time line for debt repayments under a forbearance arrangement and shows the interest paid on the debt at different stages in the extended repayment schedule.

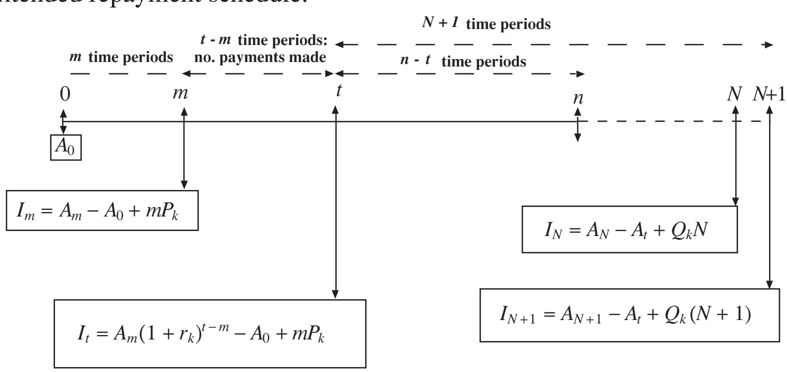


FIGURE 1: Extended debt repayment time line in a debt forbearance arrangement

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9. *Debt deferral*

If $m = 0$ but $t > 0$ then (17) becomes

$$A_t = A_0 (1 + r_k)^t, \tag{22}$$

meaning that commencement of debt repayment is deferred by $t$ time periods: a deferred annuity. After a further $n$ repayment periods

$$A_{n+t} = A_0 (1 + r_k)^{n+t} - P_k \left\{ \frac{(1 + r_k)^n - 1}{r_k} \right\} \tag{23}$$

and the interest $I_{n+t}$ paid over the subsequent $n$ time periods of repayment, using (10) and (22), is given by

$$\begin{aligned} I_{n+t} &= A_{n+t} - A_0 + nP_k \\ &= A_0 \left\{ (1 + r_k)^{n+t} - 1 \right\} - P_k \left\{ \frac{(1 + r_k)^n - 1}{r_k} \right\} + nP_k. \end{aligned} \tag{24}$$

Since, under the assumptions of the model, repayments are at the end of each time period (annuity in arrears), the first repayment made under this arrangement is $t + 1$ periods after the time of debt evaluation. It may finally be noted that the interest paid in the above cases of forbearance and deferral arrangements may usefully be compared with the situation in which uninterrupted repayments are made from evaluation date through to amortisation. Such a comparison enables the cost to the borrower of these deferral or forbearance arrangements to be fully appreciated and is undertaken in the following numerical example.

10. *Numerical example*

A borrower owes £10,000 ($A_0 = £10{,}000$) to be repaid monthly ($k = 12$). Table 1, with the appropriate equation number, gives

(i) the monthly repayment level $P_k$ at 5% annual interest ($r_a = 0.05$) over a term of 5 years ($n = 60$),

(ii) the term $n$ of debt repayment with monthly repayments of £200 ($P_k = 200$) at 5% annual interest,

(iii) the cumulative interest $I_n$ paid with 5% annual interest and monthly repayment level as set in (i), over a term of 5 years,

(iv) the annual percentage interest rate (APR), which is $100 \times r_a$, with monthly repayments of £200 over a term of 5 years.

Each result is given for four different interest conversion periods: annual, quarterly, monthly and continuous.

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| | Equations used | Interest conversion periods | | | |
|---|---|---|---|---|---|
| | | $K = 1$ (annual) | $K = 4$ (quarterly) | $K = 12$ (monthly) | $K \to \infty$ (continuous) |
| (i) $P_k$ | 7 | 188.20 | 188.62 | 188.71 | 188.76 |
| (ii) $n$ | 9 | 56.02 | 56.15 | 56.18 | 56.20 |
| (iii) $I_n$ | 11 | 1292.24 | 1317.05 | 1322.74 | 1325.61 |
| (iv) $100r_a$ | 12, 13, 14 | 7.68 | 7.47 | 7.42 | 7.40 |

TABLE 1: Repayment details on a debt of £10,000 repaid monthly

Table 2 gives the first six iterations of the Newton-Raphson method for solving (12) from which, by use of (13) and/or (14) the annual interest rates can be determined. In each case the results are given in Table 1 for annual, quarterly, monthly, and instantaneous interest conversion periods: $K = 1$, $K = 4$, $K = 12$ and $K \to \infty$, respectively.

| $r_k$ | $\Delta_1$ | $\Delta_2$ | $\Delta_3$ |
|---|---|---|---|
| 0.00550 | -1.50810 | 1872.75824 | 0.00631 |
| 0.00631 | 0.31831 | 2675.09390 | 0.00619 |
| 0.00619 | 0.00734 | 2552.05447 | 0.00618 |
| 0.00618 | 0.00000 | 2549.10172 | 0.00618 |
| 0.00618 | 0.00000 | 2549.10001 | 0.00618 |
| 0.00618 | 0.00000 | 2549.10001 | 0.00618 |

TABLE 2: Newton-Raphson iterations for finding $r_k$ where $\Delta_1$, $\Delta_2$ and $\Delta_3$ are

$$\Delta_1 : f(r_k) = r_k A_0 (1 + r_k)^n - P_n (1 - r_k)^n + P_k,$$
$$\Delta_2 : f(r_k) = A_0 (1 + r_k)^n + n r_k A_0 (1 + r_k)^{n-1} - nP_k (1 + r_k)^{n-1},$$
$$\Delta_3 : r_k - \frac{f(r_k)}{f'(r_k)}.$$

If the monthly repayments in Table 1 corresponding to each of the four interest conversion periods are rounded up from £188.20 to £189.00 ($P_{kA} = 189.00$) then there will be 59 repayments of this amount with final monthly repayments $P_{kF}$ being given in Table 3 for the four interest conversion periods under consideration.

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| | Equations used | *Interest conversion periods* | | | |
|---|---|---|---|---|---|
| | | $K = 1$ | $K = 4$ | $K = 12$ | $K \to \infty$ |
| $P_k$ | 7 | 188.20 | 188.62 | 188.71 | 188.76 |
| $P_{kA}$ | | 189.00 | 189.00 | 189.00 | 189.00 |
| $P_{kF}$ | 8 | 135.02 | 163.01 | 169.44 | 172.68 |

TABLE 3: Final $n$th monthly repayments when previous $n - 1$ repayments are rounded up.

Tables 1 and 3 show that the compounding frequency $K$ per annum makes only a marginal difference to the repayment amounts $P_K$. The utility however of providing such information is that it allows the borrower, faced with a given repayment level, to ascertain what interest compounding frequency is being applied by the lender even though such information might not be provided by the lender in the loan agreement. By contrast, Table 1 shows that, the cumulative interest $I_n$ paid on the debt rises more substantially with increasing frequency of compounding. A final point to note regarding the levels of final repayments $P_{kF}$ shown in Table 3 is that the value of £135.02 for $K = 1$ appears inconsistently lower than other values of $P_{kF}$. The reason for this is that $P_{kF}$ is not a linear function of $K$ but rather an inverse exponential (logarithmic) function.

In Figure 2, the increasing repayment levels, derived from using equations (1) and (7), on a £10,000 debt are shown for annual interest rates $r_a$ in the range $1 \leqslant r_a \leqslant 10$ taken at 0.5% intervals. Only annual ($K = 1$), and instantaneous interest conversions $(K \to \infty)$ are shown and the closeness of the two lines indicates the marginal difference interest compounding frequency makes to repayment amounts.

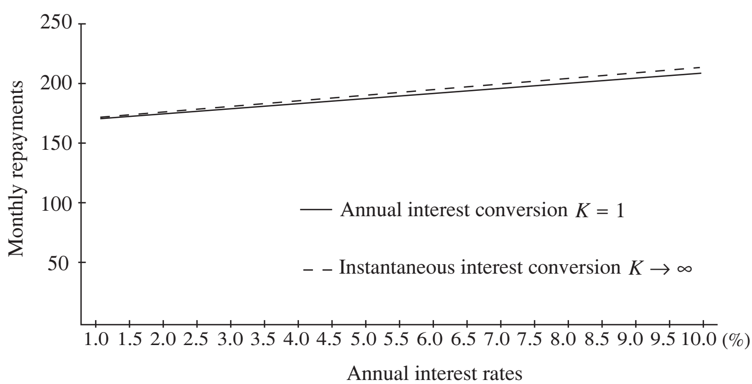


FIGURE 2: Monthly repayments on a £10,000 debt over a 5 year term with annual interest rates between 1 and 10 percent. Interest conversion periods shown are annual and instantaneous.

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For the case of annual interest conversion ($K = 1$), the most commonly used in commercial practice, the first and last few lines of a debt repayment (amortisation) schedule are shown in Table 4. Note that the outstanding debt on the 60th repayment is negative because a 60th repayment of £189.00 would overpay the debt by £53.98, hence the reduced final monthly repayment of £189.00 − £53.98 = £135.02.

<table>
<tr><td colspan="2">59 monthly payments:</td><td colspan="4">189.00</td></tr>
<tr><td colspan="2">60th (final) payment:</td><td colspan="4">135.02</td></tr>
<tr><td>Payment</td><td>Cumulative interest at 5% pa</td><td>Cumulative principal repaid</td><td>Principal repaid</td><td>Payment period interest at 5% pa</td><td>Outstanding principal</td></tr>
<tr><td>$n$</td><td>$I_j = A_i - A_0 + iP_k$ (eq. 10)</td><td>$A_0 - A_i$ (eq.6)</td><td>$A_{i-1} - A_i$ (eq. 5)</td><td>$r_k * A_{i-1}$ (eq. 4)</td><td>$A_n$ (eq. 2)</td></tr>
<tr><td>0</td><td></td><td></td><td></td><td></td><td>10000.00</td></tr>
<tr><td>1</td><td>40.74</td><td>148.26</td><td>148.26</td><td>40.74</td><td>9851.74</td></tr>
<tr><td>2</td><td>80.88</td><td>297.12</td><td>148.86</td><td>40.14</td><td>9702.88</td></tr>
<tr><td>3</td><td>120.41</td><td>446.59</td><td>149.47</td><td>39.53</td><td>9553.41</td></tr>
<tr><td>:</td><td>:</td><td>:</td><td>:</td><td>:</td><td>:</td></tr>
<tr><td>58</td><td>1284.16</td><td>9677.84</td><td>186.93</td><td>2.07</td><td>322.16</td></tr>
<tr><td>59</td><td>1285.47</td><td>9865.53</td><td>187.69</td><td>1.31</td><td>134.47</td></tr>
<tr><td>60</td><td>1286.02</td><td>10053.98</td><td>188.45</td><td>0.55</td><td>-53.98</td></tr>
</table>

TABLE 4: Amortisation schedule for debt of £10,000 at 5% annual interest repaid over 60 months

Suppose in the above example that the borrower arranges for a cessation of repayments after two years ($m = 24$) for six months ($t - m = 6$): a forbearance arrangement. Figure 3 shows the increase in cumulative interest paid (finance charge) on the debt, again for annual interest rates $r_a$ in the range $1 \leqslant r_a \leqslant 10$ and taken at 0.5% intervals. Only annual interest conversion ($K = 1$) is considered. The four situations exhibited in Figure 3 are as follows:

(i) there is neither a deferral nor a forbearance arrangement in place ($m = t = 0$). This is found from (10) and presented just for comparative purposes with (ii), (iii) and (iv) below,

(ii) post forbearance, the term $n$ of the original repayment schedule is recalculated, using (9), and extended but keeping the same level of repayments ($P_k = Q_k = £188.20$),

(iii) the repayment level $Q_k$ after the forbearance period is reduced to £150.00 per month with, again, the term being recalculated from (9)

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$(Q_k = £150{,}00)$,

(iv) there is no forbearance period but the commencement of repayment is deferred for six months ($m = 0$, $t = 5$) from debt evaluation date.

Situations (i) and (iv) are shown in the Figure 3 using (11) and (24) respectively while situations (ii) and (iii) both use (21). It is readily seen in Figure 3 how the amount of interest paid (finance charge) rises with both deferral and forbearance arrangements as well as with increasing annual interest rates. While these rises in amounts of interest paid might be expected, the example shown here illustrates the extent of the increases using the mathematical model developed. Any interruption of debt repayment, whether through deferral or forbearance, increases the cumulative interest paid and this, as Figure 3 shows, becomes particularly marked as annual interest rates rise.

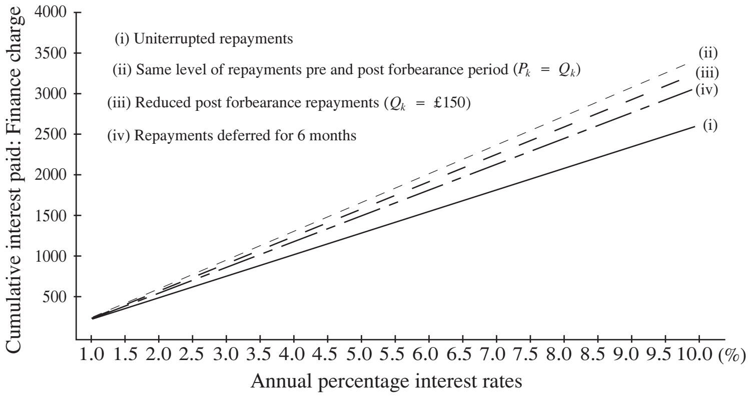


FIGURE 3: Cumulative interest (finance charge, £) paid on debt repayments under forbearance, deferral and uninterrupted arrangements on a debt of £10,000.

## 11. *Discussion and conclusion*

With the debt repayment model derived above, the periodic repayments, term, cumulative interest and APR can be found for any size of debt. The equations that underpin debt amortisation schedules have also been derived. The utility of these equations lies in the ability they give to intending borrowers to cross-check their repayment schedule with that provided by the lending institution. As stated earlier, the debt repayment methods presented in this paper are the most usual standard ones applied in the financial industry but they are not the only ones. Nevertheless, through the use of the equations presented here, borrowers will be able to gain a clearer perspective on the type of financial commitment which they are about to undertake.

While constant interest rates have been assumed throughout the term of the debt repayments, the use of a range of annual rates in the numerical example shows how the interest paid (finance charge) increases with the higher levels of interest levied on the debt. The higher rates of interest may be particularly pertinent to those seeking unsecured credit given the greater

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risk of debt default, as indicated in the introduction.

The paper has shown further how measures to ease the burden of debt through deferral of repayments and/or the arrangement of debt forbearance periods (payment holidays) only increase the borrowing cost through the raised levels of interest that have to be paid on any outstanding debt. Application of the equations derived above enable the borrower to provide a numerical calculation of the cost of extending the term of the indebtedness. Hence the financial consequences of prolonged indebtedness become apparent while recognising that this may be the only remedy to avoid complete debt default.

The financial upheavals from the covid-19 pandemic and the war in Ukraine referred to earlier are indeed only likely to exacerbate the adverse consequences of prolonged financial indebtedness. With almost everyone across the world encountering financial debt at some stage in their lives and given the onerous burden debt can place on people, the ability to calculate the full financial liability incurred can only be of assistance to understanding personal finance. However, it must again be emphasised that the ability to undertake these calculations should not be taken as any indication of the advisability of incurring debt.

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