Recurrence Formulae for the Functions which represent Solutions of the Differential Equation:

$$\frac{d^2u}{dx^2} - a^2u = \frac{p(p+1)}{x^2}u.$$

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The contents of this paper were suggested by a discussion of the equation:

$$\frac{d^2u}{dx^2} - a^2u = \frac{p(p+1)}{x^2}u. \qquad \text{(1)}$$

in a paper by Glaisher which appears in the *Philosophical Transactions*, 1881, Part III.

The solutions in series of (1) are:

$$U = x^{-p}\left\{1 - \frac{1}{p-\frac{1}{2}}\,\frac{a^2x^2}{2^2} + \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\cdot\frac{a^4x^4}{2^4\lfloor 2} - \text{etc.}\right\} \qquad \text{(2)}$$

and $$V = x^{p+1}\left\{1 + \frac{1}{p+\frac{3}{2}}\cdot\frac{a^2x^2}{2^2} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\cdot\frac{a^4x^4}{2^4\lfloor 2} + \text{etc.}\right\} \qquad \text{(3)}$$

and in the paper referred to it is shewn that the coefficients of h^{p+1} in the expansions of $e^{a(x^2+hx)^{\frac{1}{2}}}$ and of $e^{-a(x^2+hx)^{\frac{1}{2}}}$ satisfy equation (1) when p is a positive integer.

These coefficients are in fact:

$$\lambda(U - gV) \quad \text{and} \quad -\lambda(U + gV)$$

where λ and g are certain constants.

In this paper it is shown that the coefficient of h^{p+1} in the expansion of $\sinh a(x^2+hx)^{\frac{1}{2}}$ is $k_1 U$, where k_1 is a certain constant, and that the coefficient of h^{p+1} in $\cosh a(x^2+hx)^{\frac{1}{2}}$ is $k_2 V$, where k_2 is a constant.

These results are not stated in Glaisher's paper, though they seem to follow easily from § II. 10.

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They are employed in the following pages to deduce recurrence formulae for U and V for different values of p. Equation (1) is treated in the same way as Legendre's or Bessel's equation, and results analogous to those for Legendre polynomials or Bessel functions are obtained. It will then be pointed out that we easily obtain the result that $J_{p+\frac{1}{2}}(x)$ and $J_{-p-\frac{1}{2}}(x)$ for positive integral values of p are the coefficients of h^{p+1} in the expansions of

$$\frac{(-1)^{p+1}\Gamma(2p+3)}{2^{p+\frac{1}{2}}\Gamma(p+\frac{3}{2})}x^{-\frac{1}{2}}\cos(x^2+hx)^{\frac{1}{2}}$$

and of

$$\frac{(-1)^p\Gamma(p+1)\Gamma(p+2)}{\Gamma(2p+1)\Gamma(-p+\frac{1}{2})}2^{3p+\frac{5}{2}}x^{-\frac{1}{2}}\sin(x^2+hx)^{\frac{1}{2}}$$

respectively.

A proof that

$$J_{\frac{1}{2}}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x$$

and $$J_{-\frac{1}{2}}(x)=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos x$$

is added.

I. To show that the coefficients of h^{p+1} in $\sinh a(x^2+hx)^{\frac{1}{2}}$ and in $\cosh a(x^2+hx)^{\frac{1}{2}}$ satisfy equation (1).

Since the coefficients of h^{p+1} in $e^{a(x^2+hx)^{\frac{1}{2}}}$ and $e^{-a(x^2+hx)^{\frac{1}{2}}}$ satisfy the equation, the result evidently follows.

We shall establish it independently.

Write $S=\sinh a(x^2+xh)^{\frac{1}{2}}$ $C=\cosh a(a^2+hx)^{\frac{1}{2}}$.

Then
$$\frac{\partial^2 S}{\partial x^2}=\frac{a^2}{4}\cdot\frac{(2x+h)^2}{(x^2+hx)}S-\frac{a\,h^2}{4(x^2+hx)^{\frac{3}{2}}}C.$$

$$\frac{\partial^2 S}{\partial h^2}=\frac{a^2}{4}\cdot\frac{x^2}{x^2+hx}S-\frac{a}{4}\cdot\frac{x^2}{(x^2+hx)^{\frac{3}{2}}}C.$$

If C be eliminated, we obtain a differential equation satisfied by S. With some simple reduction this is seen to be :

$$x^2\frac{\partial^2 S}{\partial x^2}-h^2\frac{\partial^2 S}{\partial h^2}=a^2x^2S \quad\text{......................}\quad (4)$$

Write $\quad S = U_0 + h\,U_1 + h^2\,U_2 + \ldots$ and substitute in (4)

$$x^2\,\Sigma\,h^r \frac{\partial^2 U_r}{\partial x^2} - \Sigma\,r\,(r-1)\,h^r U_r = a^2\,x^2\,\Sigma\,h^r\,U_r.$$

On equating the coefficients of h^{p+1} on each side, we find

$$x^2 \frac{\partial^2 U_{p+1}}{\partial x^2} - p\,(p+1)\,U_{p+1} = a^2\,x^2\,U_{p+1},$$

so that U_{p+1} satisfies the equation (1).

Similarly, writing

$$\cosh a\,(x^2 + hx)^{\frac{1}{2}} = V_0 + h\,V_1 + h^2\,V_2 + \ldots$$

it can be shown that V_{p+1} satisfies equation (1).

II. The value of the coefficient of h^{p+1} in the expansions of S and C will now be obtained.

$$S = \sinh a\,(x^2 + hx)^{\frac{1}{2}} = a\,(x^2 + hx)^{\frac{1}{2}} + \frac{a^3 (x^2 + hx)^{\frac{3}{2}}}{\underline{|3}} + \ldots$$

$$+ \frac{a^{2r-1}(x^2 + hx)^{\frac{2r-1}{2}}}{\underline{|2r-1}} + \ldots.$$

The coefficient of h^{p+1} in $(x^2 + hx)^{\frac{2r-1}{2}}$, *i.e.* in $x^{2r-1}\left(1 + \frac{h}{x}\right)^{\frac{2r-1}{2}}$ is

$$\frac{x^{2r-p-2}}{2^{p+1}} \frac{(2r-1)(2r-3)\ldots(2r-2p-1)}{\underline{|p+1}}.$$

Hence the coefficient of h^{p+1} in the series is:

$$\sum_{r=1}^{r=\infty} \frac{a^{2r-1}}{\underline{|2r-1}} \frac{(2r-1)(2r-3)\ldots(2r-2p-1)}{\underline{|p+1}} \frac{x^{2r-p-2}}{2^{p+1}}$$

$$= (-1)^p a . \frac{1.3.5\ldots(2p-1)}{2^{p+1}\,\underline{|p+1}}$$

$$. x^{-p}\left\{1 - \frac{1}{p-\frac{1}{2}} \frac{a^2 x^2}{2^2} + \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})} \frac{a^4 x^4}{2^4\,\underline{|2}} \ldots\right\}$$

$$= (-1)^p a . \frac{\underline{|2p}}{2^{p+1}\,\underline{|p}\,\underline{|p+1}}\,U.$$

$$C = \cosh a\,(x^2 + hx)^{\frac{1}{2}} = 1 + \frac{a^2 (x^2 + hx)}{\underline{|2}} + \frac{a^4 (x^2 + hx)^2}{\underline{|4}} + \ldots$$

h^{p+1} occurs in the terms containing :

$$(x^2+hx)^{p+1}),\ \ (x^2+hx)^{p+2},\ \ \text{etc.,}$$

and it is easily seen that its coefficient is

$$\sum_{r=p+1}^{r=a} \frac{a^{2r}}{\underline{|2r}} \cdot \frac{r(r-1)\dots(r-p)}{\underline{|p+1}} x^{2r-p-1}$$

$$= \frac{a^{2p+2}}{\underline{|2p+2}} x^{p+1} \left\{ 1 + \frac{1}{p+\frac{3}{2}} \frac{a^2 x^2}{2^2} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})} \frac{a^4 x^4}{2^4 \underline{|2}} + \text{etc.} \right\}$$

$$= \frac{a^{2p+2}}{\underline{|2p+2}} V.$$

We shall adopt the notation :

$$U_{p+1}(a\,x) = \frac{(-1)^p\, a\, \underline{|2p}}{2^{2p+1}\, \underline{|p}\, \underline{|p+1}} U.$$

$$V_{p+1}(a\,x) = \frac{a^{2p+2}}{\underline{|2p+2}} V.$$

III. Recurrence formulae :

$$S = \sinh a\,(x^2+h\,x)^{\frac{1}{2}} = U_0(a\,x) + hU_1(a\,x) + h^2U_2(a\,x) + \dots$$
$$+ h^rU_r(a\,x) + \dots$$

$$\therefore\ \ \frac{\partial S}{\partial h} = \cosh a\,(x^2+h\,x)^{\frac{1}{2}} \times \frac{a\,x}{2(x^2+hx)^{\frac{1}{2}}} = U_1(a\,x) + 2h\,U_2(a\,x) + \dots$$
$$+ r\,h^{2-1}\,U_r(a\,x) + \dots$$

$$\therefore\ \ \frac{\partial^2 S}{\partial h^2} = \sinh a(x+hx)^{\frac{1}{2}} \times \frac{a^2x^2}{4(x+hx)} - \cosh a\,(x^2+h\,x)^{\frac{1}{2}} \times \frac{a\,x^2}{4(x^2+hx)^{\frac{3}{2}}}$$

$$= 2\,.\,1\,.\,U_2(a\,x) + 3\,.\,2\,.\,hU_3(a\,x) + \dots + r\,(r-1)\,h^{r-2}\,U_r(ax) + \dots$$

$$\therefore\ \ a^2x^2\{U_0(a\,x) + hU_1(a\,x) + h^2U_2(a\,x) + \dots\}$$
$$-2x\{U_1(a\,x) + 2hU_2(a\,x) + 3h^2\,U_3(a\,x) + \dots\}$$

$$= 4(x^2+hx)\{2\,.\,1\,.\,U_2(a\,x) + 3\,.\,2\,.\,hU_3(a\,x) + 4\,.\,3\,.\,hU_4(a\,x) + \dots\}$$

By equating the coefficient of h^p on each side we obtain

$$a^2x\,U_p(ax) - 2\,(p+1)\,(2\,p+1)\,U_{p+1}(ax)$$
$$-4\,(p+1)\,(p+2)\,x\,U_{p+2}(ax) = 0. \quad\dots\dots\dots\dots\dots\dots\ (5)$$

Again,

$$\frac{\partial S}{\partial x} = \frac{a(2x+h)}{(x^2+hx)^{\frac{1}{2}}} \cosh a(x^2+hx)^{\frac{1}{2}}$$

$$= \frac{d}{dx} U_0(ax) + h \frac{d}{dx} U_1(ax) + h^2 \frac{d^2}{dx^2} U_2(ax) + \dots$$

$$\frac{\partial S}{\partial h} = \frac{ax}{(x^2+hx)^{\frac{1}{2}}} \cosh a(x^2+hx)^{\frac{1}{2}}$$

$$= U_1(ax) + h\, U_2(ax) + h^2\, U_3(ax) + \dots$$

$$\therefore \qquad x\left\{\frac{d}{dx} U_0(ax) + h \frac{d^2}{dx^2} U_1(ax) + \dots\right\}$$

$$= (2x+h)\,\{U_1(ax) + h\, U_2(ax) + \dots\}.$$

Equating the coefficients of h^p on each side we find

$$x \frac{d}{dx} U_p(ax) = 2(p+1)\, x\, U_{p+1}(ax) + p\, U_p(ax). \qquad \dots\dots\dots(6)$$

These formulae may be compared with those corresponding for Bessel functions or Legendre polynomials. Thus (5) resembles

$$x J_n(x) \quad - \quad 2(n+1) J_{n+1}(x) \quad + x J_{n+2}(x) = 0$$

or $\quad (n+2) P_{n+2}(x) - (2n+3)\, x P_{n+1}(x) + (n+1) P_n(x) = 0.$

$U_0(ax)$, the term independent of h in the expansion of S, may be obtained by putting $h=0$ in S.

$$\therefore \qquad U_0(ax) = \sinh(ax) \qquad \dots\dots\dots\dots\dots\dots\dots\dots (7)$$

$$U_1(ax) = \left\{\frac{d}{dh} \sinh a(x^2+hx)^{\frac{1}{2}}\right\}_{h=0}$$

$$= \frac{a}{2} \cosh(ax). \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (8)$$

In the same way the formulae for the V-series may be shewn to be:

$$a^2 x\, V_p(ax) - 2(p+1)(2p+1)\, V_{p+1}(ax)$$
$$- 4(p+1)(p+2)\, x\, V_{p+2}(ax) = 0. \qquad \dots\dots\dots (9)$$

$$x \frac{d}{dx} V_p(ax) = 2(p+1)\, x\, V_{p+1}(ax) + p\, V_p(ax) \qquad \dots\dots\dots (10)$$

$$V_0(ax) = \cosh(ax) \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (11)$$

$$V_1(ax) = \frac{a}{2} \sinh(ax) \qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (12)$$

IV. Extension of the above to other values of p.

If we write

$$U_{p+1}(ax) = \frac{(-1)^p a \Gamma(2p+1)}{2^{2p+1}\Gamma(p+1)\Gamma(p+2)} U \quad \ldots\ldots\ldots\ldots\ldots (13)$$

and $$V(ax) = \frac{a^{3p+2}}{\Gamma(2p+3)} V \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

it can be verified that the same recurrence formulae hold when p is not integral. This verification is quite easy and need not be given here.

It will be noticed that, for positive integral values of p, (13) and (14) are not different from the expressions at the end of § II.; (13) and (14) are, however, more general, and we proceed to examine what restrictions have still to be imposed upon p.

In the series, U and V, the radius of the circle of convergence is infinite, so that U is convergent for all real values of p except when p has the values

$$\frac{2n-1}{2}, \; (n = 1 \,.\, 2 \,.\, 3 \,.\, \ldots)$$

and V is convergent except when p has the values

$$-\left(\frac{2n+1}{2}\right), \; (n = 1 \,.\, 2 \,.\, 3 \,.\, \ldots).$$

If p has a value of either of these forms, one or other of the series has a zero factor in the denominator from a certain term, and the solution is not of the form:

$$u = AU + BV.$$

Owing to the forms of the multipliers of U and V in (13) and (14), additional restrictions have to be imposed on p.

In order to restrict ourselves to real quantities $(-1)^p$ must be real.

No difficulty arises if p is an integer, positive or negative.

Let p be a fraction $\frac{r}{s}$ in its lowest terms,

$$(-1)^p = (\cos\pi + i\sin\pi)^{\frac{r}{s}} = \cos\frac{(2k+1)}{s}r\pi + i\sin\frac{(2k+1)}{s}r\pi,$$

$$k = 0,\ 1,\ 2 \ldots (s-1).$$

In order that this may be real $\frac{2k+1}{s}r$ must be integral, *i.e.* p must be equal to a number of the form $\pm\frac{m}{2k+1}$ where m is a positive integer. Thus p must be a fraction with an odd denominator, and it may be positive or negative. This form of p gives rise to no further difficulty in the remaining parts of the constant multiplier.

None of the Gamma functions must be of the form $\Gamma(-n)$ where n is a positive integer, for in that case they are without meaning: $(2p+1)$ must not be a negative integer, thus p must not be of the form:

$$-\frac{n+1}{2} \quad (n=1\,.\,2\,.\,3\,\ldots),$$

i.e. $p \neq -1, -\frac{3}{2}, -2$, etc.

$(p+1)$ and $(p+2)$ must not be negative integers, and this requires that p should not be of the form

$$-(n+1) \quad (n=1, 2.\, 3\,\ldots)$$

or $p \neq -2, -3, -4$, etc. This is included in the last case.

Summing up these results, $U_{p+1}(ax)$ does not exist as defined in (13) and (14) for the values:

$$p=\frac{2n-1}{2} \quad (n=1\,.\,2\,.\,3\,\ldots)$$

$$p=-\frac{n+1}{2} \quad (n=1\,.\,2\,.\,3\,\ldots).$$

Moreover, in order to avoid imaginary quantities, p must be either integral or of the form $\pm\frac{m}{2k+1}$ where m and k are positive integers.

The difficulty in the case of the V-series is not so great, $(2p+3)$ must not be integral and negative, so that p must not be of the form:

$$-\frac{n+3}{2} \quad (n=1\,.\,2\,.\,3\,\ldots).$$

We have seen that p must not be of the form

$$-\frac{2n+1}{2} \quad (n=1\,.\,2\,.\,3\,\ldots),$$

so that we can include these in one case by stating that p must not have the values:

$$-\frac{n+3}{2} \quad (n=0\,.\,1\,.\,2\,.\,3\,.\,\ldots).$$

V. A relation between U and V.

Since U and V satisfy (1), we have

$$\frac{d^2U}{dx^2}-a^2U=\frac{p(p+1)}{x^2}U$$

and

$$\frac{d^2V}{dx^2}-a^2V=\frac{p(p+1)}{x^2}V.$$

Multiply these by V and U respectively, and subtract

$$V\frac{d^2U}{dx^2}-U\frac{d^2V}{dx^2}=0.$$

$$\therefore \quad V\frac{dU}{dx}-U\frac{dV}{dx}=\text{constant}.$$

On substituting the value of U and V we easily find that the constant term is $-(2p+1)$.

$$\therefore \quad U\frac{dV}{dx}-V\frac{dU}{dx}=(2p+1) \qquad \text{(15)}$$

From (13) and (14) we have

$$U_{p+1}(ax)\,.\,\frac{d}{dx}V_{p+1}(ax)-V_{p+1}(ax)\frac{d}{dx}U_{p+1}(ax)$$

$$=\frac{(-1)^p a^{2p+3}}{(2p+2)\,\Gamma(p+1)\,\Gamma(p+2)} \qquad \text{(16)}$$

$$=K \text{ (say)}.$$

From (15)

$$\frac{d}{dx}\left(\frac{V}{U}\right)=\frac{2p+1}{U^2}$$

$$\therefore \qquad V=(2p+1)\,U\int\frac{dx}{U^2}$$

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In order to find the proper limits we need only examine the lowest power of x on the right, which is x^{2p} under the integral sign. Hence the lowest term on the right is x^{p+1}.

Thus $$V=(2p+1)\,U\int_0^x \frac{dx}{U^2}. \quad \text{.....................} \quad (17)$$

Similarly it can be shown that

$$U=(2p+1)\,V\int_x^\infty \frac{dx}{V^2}. \quad \text{.....................} \quad (18)$$

(17) and (18) do not hold for all values of p. In (17) $(2p+1)$ must be positive, or $p > -\frac{1}{2}$.

In (18) $(2p+1)$ must also be positive.

If $p \not> -\frac{1}{2}$, instead of (17) we have

$$V=(2p+1)\,U\int_\infty^x \frac{dx}{U^2}, \quad \text{...................} \quad (17')$$

and instead of (18)

$$U=(2p+1)\,V\int_x^0 \frac{dx}{V^2}. \quad \text{...................} \quad (18')$$

There are, of course, similar formulae for U_{p+1} and V_{p+1}, but these differ only by some constant factor and can be obtained by applying (13) and (14).

We may compare these results with

$$Q_n(x)=P_n(x)\int_x^\infty \frac{dx}{(x^2-1)\{P_n(x)\}^2}$$

and

$$\frac{d}{dx}\left\{\frac{J_{-n}(x)}{J_n(x)}\right\} = -\frac{2\sin n\pi}{\pi x\{J_n(x)\}^2}.$$

VI. In all the above we might write $a=1$, but the advantage of keeping in a is seen in the fact that if ia be written for a we obtain similar relations for the equation:

$$\frac{d^2u}{dx^2}+a^2u=\frac{p(p+1)}{x^2}\,u \quad \text{.....................} \quad (19)$$

It follows at once that the coefficients of h^{p+1} in the expansions of $\sin a(x^2+hx)^{\frac{1}{2}}$ and of $\cos a(x^2+hx)^{\frac{1}{2}}$ satisfy (19).

The series U and V become:

$$W = x^{-p}\left\{1 + \frac{1}{p-\frac{1}{2}}\,\frac{a^2x^2}{2^2} + \frac{1}{(p-\frac{1}{2})(p-\frac{3}{2})}\,\frac{a^4x^4}{2^4\,\underline{|2}} + \text{etc.}\right\}$$

$$X = x^{p+1}\left\{1 - \frac{1}{p+\frac{3}{2}}\,\frac{a^2x^2}{2^2} + \frac{1}{(p+\frac{3}{2})(p+\frac{5}{2})}\,\frac{a^4x^4}{2^4\,\underline{|2}} + \text{etc.}\right\}$$

and the coefficient of h^{p+1} in $\sin a\,(x^2+hx)^{\frac{1}{2}}$ is

$$\frac{(-1)^p\,a\,\Gamma(2p+1)}{2^{2p+1}\,\Gamma(p+1)\,\Gamma(p+2)}\cdot \quad W = W_{p+1}(ax), \text{ say,}$$

and of h^{p+1} in $\cos a\,(x^2+hx)^{\frac{1}{2}}$ is

$$\frac{(-1)^{p+1}\,a^{2p+2}}{\Gamma(2p+3)}\cdot \quad X = X_{p+1}(ax).$$

Evidently $W = \dfrac{1}{2^{p+\frac{1}{2}}}\,\Gamma(-p+\tfrac{1}{2})\,x^{\frac{1}{2}}\,a^{p+\frac{1}{2}}\,J_{-p-\frac{1}{2}}(a\,x)$

and $\quad X = \dfrac{2^{p+\frac{1}{2}}}{a^{p+\frac{1}{2}}}\,\Gamma(p+\tfrac{3}{2})\,x^{\frac{1}{2}}\,J_{p+\frac{1}{2}}(a\,x).$

The J's denoting Bessel's functions.

$$\therefore\; W_{p+1}(a\,x) = \frac{(-1)^p\,a^{p+\frac{3}{2}}\,\Gamma(2p+1)\,\Gamma(-p+\frac{1}{2})}{2^{3p+\frac{3}{2}}\,\Gamma(p+1)\,\Gamma(p+2)}\,x^{\frac{1}{2}}J_{-p-\frac{1}{2}}(ax) \quad (20)$$

$$\text{and } X_{p+1}(a\,x) = \frac{(-1)^{p+1}\,2^{p+\frac{1}{2}}\,a^{p+\frac{3}{2}}\,\Gamma(p+\frac{3}{2})}{\Gamma(2p+3)}\,x^{\frac{1}{2}}\,J_{p+\frac{1}{2}}(a\,x) \ldots\ldots (21)$$

Thus $J_{p+\frac{1}{2}}(x) = \dfrac{(-1)^{p+1}\,\Gamma(2p+3)}{2^{p+\frac{1}{2}}\,\Gamma(p+\frac{3}{2})}\,x^{-\frac{1}{2}}\,X_{p+1}(x)$, (substituting $a = 1$).

It follows that $J_{p+\frac{1}{2}}(x)$, where p is a positive integer, is the coefficient of h^{p+1} in

$$\frac{(-1)^{p+1}\,\Gamma(2p+3)}{2^{p+\frac{1}{2}}\,\Gamma(p+\frac{3}{2})}\,x^{-\frac{1}{2}}\cos(x^2+h\,x)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots\; (22)$$

Similarly $J_{-p-\frac{1}{2}}(x)$ is the coefficient of h_{p+1} in

$$\frac{(-1)^p\,2^{3p+\frac{3}{2}}\,\Gamma(p+1)\,\Gamma(p+2)}{\Gamma(2p+1)\,\Gamma(-p+\frac{1}{2})}\,x^{-\frac{1}{2}}\sin(x^2+hx)^{\frac{1}{2}} \quad (23)$$

At the end of Glaisher's paper above referred to it is stated that the solution of the equation

$$\frac{d^2 y}{dx^2}+\frac{1}{x}\frac{dy}{dx}+\left\{1-\frac{(p+\frac{1}{2})^2}{x^2}\right\}y=0$$

may be written

$$y=A\,x^{-\frac{1}{2}}\,\{\text{ coefficient of } h^{p+1} \text{ in } \sin(x^2+h\,x)^{\frac{1}{2}}\,\}$$
$$+B\,x^{-\frac{1}{2}}\,\{\text{ coefficient of } h^{p+1} \text{ in } \cos(x^2+h\,x)^{\frac{1}{2}}\,\}.$$

VII. If we write $p=0$ we have from (22) that $J_{\frac{1}{2}}(x)$ is the coefficient of h in $-\dfrac{\Gamma(3)}{2^{\frac{1}{2}}\,\Gamma(\frac{3}{2})}x^{-\frac{1}{2}}\cos(x^2+hx)^{\frac{1}{2}}$.

$$\therefore\quad J_{\frac{1}{2}}(x)=-2\left(\frac{2}{\pi x}\right)^{-\frac{1}{2}}\left\{\frac{d}{dh}\cos(x^2+hx)^{\frac{1}{2}}\right\}_{h=0}=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\sin x \quad \ldots\ldots(24)$$

and

$$J_{-\frac{1}{2}}(x)=\frac{2^{\frac{3}{2}}\,\Gamma(1)\,\Gamma(2)}{\Gamma(1)\,\Gamma(\frac{1}{2})}x^{-\frac{1}{2}}\left\{\frac{d}{dh}\sin(x^2+hx)^{\frac{1}{2}}\right\}_{h=0}=\left(\frac{2}{\pi x}\right)^{\frac{1}{2}}\cos x. \quad (25)$$

(24) and (25) will be found in a table on page 42 of Gray and Mathews' "Bessel Functions," and are there obtained by another method.