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ON THE TOPOLOGIES OF R^∞

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ABSTRACT. Let R^∞ be the set of all the finite sequences of real numbers. The author shows that there are uncountably many distinct topologies on the set R^∞, each of which coincides with the usual topology when restricted to an R^n. However, under each of these topologies, R^∞ is always of the same homotopy type. A generalization to some other spaces is mentioned.

For each $n \geq 1$, let R^n be the set of all the infinite sequences $(x_1, x_2, x_3, \ldots)$ of real numbers such that $x_i = 0$ for all $i > n$. With this definition, we have $R^1 \subset R^2 \subset R^3 \subset \cdots$. Now, let $R^\infty = \bigcup_{n>1} R^n$. The set R^∞ is frequently of interest, for it is the smallest linear space which contains all the euclidean spaces. One may use it for instance to study the geometric complexes which contain simplexes of an arbitrarily high dimension. What topology should be put on R^∞? Naturally, we wish to consider a topology which would induce the usual topology on each R^n. Such a topology of course exists. But is it unique? In this note we shall show that there are uncountably many different topologies on R^∞, each of which induces the usual topology on R^n. We shall also compare these topologies and observe a few basic properties of them.

First note that for each p, $1 \leq p \leq \infty$, the usual l_p norm N_p is well defined on R^∞. In fact, R^∞ is a dense subset of each of the classical l_p-spaces. We shall use $\mathcal{T}_p$ to denote the topology on R^∞ induced by the l_p norm. One might also wish to consider the following two common topologies on R^∞: the product topology $\mathcal{P}$ (where R^∞ is considered as a subspace of the Cartesian product of a countable copies of real lines), and the weak topology $\mathcal{W}$ induced from the union $R^\infty = \bigcup_{n \geq 1} R^n$ (i.e. $U \in \mathcal{W}$ if and only if $U \cap R^n$ is open in R^n for each n). It is well known that all these topologies coincide when they are restricted to a euclidean space R^n. It seems somewhat surprising that these topologies are all distinct on R^∞, even though it is the set of all the *finite* sequences of real numbers. Specifically, we have the following:

THEOREM 1. *For any extended real numbers p and q with $1 \leq p < q \leq \infty$, the relations $\mathcal{P} \subset \mathcal{T}_q \subset \mathcal{T}_p \subset \mathcal{W}$ always hold. Furthermore, each inclusion is proper.*

Proof. The inclusion part is undoubtedly known. However, for completeness, we shall outline a proof here. It is well known that the topology $\mathcal{P}$ on R^∞ can

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be induced by a metric. For instance, for each $x=(x_1, x_2, \ldots)\in R^\infty$, let

$$N(x)=\sum_{i=1}^{\infty} (|x_i|/2^i(1+|x_i|)),$$

and let $\rho(x, y)=N(x-y)$ for each $x, y\in R^\infty$. Then ρ is such a metric. Let an arbitrary real number $\varepsilon>0$ be given, we may choose a positive real number δ such that $\delta/(1+\delta)<\varepsilon$. Then it is not difficult to show that for each $x\in R^\infty$, $N_\infty(x)<\delta \Rightarrow N(x)<\varepsilon$, where for each p, $1\le p\le\infty$, $N_p(x)$ denotes the l_p-norm of x. This implies that each open ball in $\mathscr{P}$ contains an open ball in the l_∞-norm, and hence, $\mathscr{P}\subset\mathscr{T}_\infty$.

For any real number $p\ge 1$, the inclusions $\mathscr{T}_\infty\subset\mathscr{T}_p$ and $\mathscr{T}_1\subset\mathscr{W}$ are immediate, for it is clear that for any $x\in R^\infty$, $N_\infty(x)\le N_p(x)$, and if a set U is open in $\mathscr{T}_1$, it is clear that $U\cap R^n$ is open in R^n for each n.

Now consider any two real numbers p and q with $1\le p<q<\infty$. Let any $\varepsilon>0$ be given. If we choose a positive $\delta<1$ such that $\delta^{p/q}<\varepsilon$, then for any $x=(x_1, x_2, \ldots)\in R^\infty$ with $N_p(x)<\delta$, we note that $|x_i|<1$ for each $i=1, 2, \ldots,$ and that

$$\begin{aligned} N_q(x)&=\left(\sum_{i=1}^{\infty} |x_i|^q\right)^{1/q}=\left(\sum_{i=1}^{\infty} |x_i|^p\cdot|x_i|^{q-p}\right)^{1/q}\\ &\le\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/q}\\ &=\left[\left(\sum_{i=1}^{\infty} |x_i|^p\right)^{1/p}\right]^{p/q}\\ &<\delta^{p/q}<\varepsilon. \end{aligned}$$

Thus, any ε-ball in N_q contains a δ-ball in N_p. Hence, $\mathscr{T}_q\subset\mathscr{T}_p$, and this finishes the proof for the inclusions.

We now show that all these topologies are distinct. In the following, for each $x\in R^\infty$, we shall use x_i to denote the ith component of x. Consider the set $U_w=\{x\in R^\infty \mid |x_i|<2^{-i} \text{ for each } i\}$. U_w is an open neighborhood of the origin $0\in R^\infty$ with respect to the topology $\mathscr{W}$. We contend that U_w is not open in $\mathscr{T}_1$. Let any $\varepsilon>0$ be given, we can fix a positive integer k such that $2^{-k}<\varepsilon$. Then consider the point $x\in R^\infty$ such that $x_i=2^{-k}$ if $i=k$, and $x_i=0$ if $i\ne k$. Clearly, $N_1(x)=2^{-k}<\varepsilon$ but $x\notin U_w$. Thus each ε-ball of 0 with respect to N_1 contains a point not in U_w. Therefore, U_w is not open in $\mathscr{T}_1$.

Now, let any real numbers p and q be given with $1\le p<q<\infty$. Let $U_p=\{x\in R^\infty \mid N_p(x)<1\}$ be the unit open ball in $\mathscr{T}_p$. We shall show that U_p is not open in $\mathscr{T}_q$. Let any $\varepsilon>0$ be given. We may choose an integer n so large that $(\varepsilon/2)\cdot n^{(q-p)/pq}>1$. Then consider a point $x\in R^\infty$ defined by

$$x_i=\begin{cases} \dfrac{\varepsilon}{2n^{1/q}}, & \text{for } 1\le i\le n\\ 0\quad, & \text{for } n<i \end{cases}.$$

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It is straightforward to show that

$$N_q(x)=\left(n\cdot\left(\frac{\varepsilon}{2n^{1/q}}\right)^q\right)^{1/q}=\left(n\cdot\left(\frac{\varepsilon}{2}\right)^q\cdot\frac{1}{n}\right)^{1/q}=\frac{\varepsilon}{2}<\varepsilon.$$

But

$$N_p(x)=\left(n\cdot\left(\frac{\varepsilon}{2n^{1/q}}\right)^p\right)^{1/q}=\left[\left(\frac{\varepsilon}{2}\right)^p\cdot n^{1-(p/q)}\right]^{1/p}=\left(\frac{\varepsilon}{2}\right)n^{(q-p)/pq}>1.$$

Thus, each ε-ball with respect to N_q contains some point whose N_p norm is greater than 1. Therefore, the unit open ball U_p cannot be an open set in $\mathcal{T}_q$.

To show $\mathcal{T}_\infty\neq\mathcal{T}_p$ for any $1\leq p<\infty$, consider the unit open ball $U_p=\{x\in R^\infty \mid N_p(x)<1\}$. For each $\varepsilon>0$, fix an integer k such that $k^{1/p}(\varepsilon/2)>1$. Then the point $x\in R^\infty$ defined by

$$x_i=\begin{cases}\dfrac{\varepsilon}{2} & \text{for} \quad 1\leq i\leq k\\ 0 & \text{for} \quad k<i\end{cases}$$

clearly belongs to the ε-ball with respect to N_∞, but it does not belong to U_p. Thus, $\mathcal{T}_\infty\neq\mathcal{T}_p$.

Finally, let $U_\infty=\{x\in R^\infty \mid N_\infty(x)<1\}$. For each $\varepsilon>0$, choose an integer n such that $2^{-n}<\varepsilon$. Then the point $x\in R^\infty$ defined by

$$x_i=\begin{cases}2 & \text{if} \quad i=n\\ 0 & \text{if} \quad i\neq n\end{cases}$$

belongs to the ε-ball with respect to N but not in the ball U_∞. This finishes the proof.

REMARK 1. Using the idea of our proof, we can actually show that each ε-ball in N is unbounded in the norm N_∞, no matter how small ε is. Likewise, for each p and q with $1\leq p<q\leq\infty$, each ε-ball in N_q is unbounded in the norm N_p.

Even though there are uncountably many distinct topologies on R^∞ yet from the homotopy point of view, all these topologies described in Theorem 1 are essentially the same on R^∞ in the sense that under each of these topologies, R^∞ is always of the same homotopy type. This is shown in the following

THEOREM 2. *Under each of the topologies $\mathcal{P}$, $\mathcal{W}$ and $\mathcal{T}_p$ with $1\leq p\leq\infty$, the space R^∞ is always contractible.*

Proof. Let I be the closed interval $[0, 1]$ on the real line. We need only show that the function $F: R^\infty\times I\to R^\infty$ defined by $F(x, t)=tx$ is continuous with respect to each given topology on R^∞. That it is continuous with respect to the topology $\mathcal{P}$ and to each $\mathcal{T}_p$ with $1\leq p\leq\infty$ is a consequence of the following two relations, each of which is easy to establish: for each x and $y\in R^\infty$ and each

$t \in [0, 1]$,

(1) $N_p(tx - ty) = t \cdot N_p(x - y)$, and
(2) $N(tx - ty) \leq N(x - y)$.

As for the topology $\mathcal{W}$ on R^∞, since each R^n is locally compact, it is known that the product topology on $R^\infty \times I$ can be considered as the weak topology induced by the union $R^\infty \times I = \bigcup_{n \geq 1} (R^n \times I)$ (see [3, (H)], or [2, Theorem 1.9]). Thus, the function F is continuous with respect to $\mathcal{W}$ because each restriction

$$F \mid (R^n \times I) : (R^n \times I) \to R^\infty$$

is clearly continuous.

REMARK 2. One may also show that R^∞ under each of the topologies $\mathcal{P}$, $\mathcal{W}$ and $\mathcal{T}_p$, $1 \leq p \leq \infty$, is a topological vector space. Among these topologies, $\mathcal{W}$ is the only one which is not metrizable (see for instance [1, Ex. 6(a), p. 175]). However, $\mathcal{W}$ does have one advantage over the other topologies. It is well known that with respect to $\mathcal{W}$, a set K is compact in R^∞ if and only if K is compact subset of some R^n. But with respect to each of the other topologies considered here one can show easily that the following set $K = \{0\} \cup \{x^{(n)} \mid n = 1, 2, \ldots\}$, where for each n, the point $x^{(n)} \in R^\infty$ is defined by

$$x_i^{(n)} = \begin{cases} \dfrac{1}{n} & \text{if} \quad i = n \\ 0 & \text{if} \quad i \neq n, \end{cases}$$

is a compact subset of R^∞ which is not contained in any R^n. Thus, $\mathcal{W}$ is the only topology considered here for which the compact sets of R^∞ are contained in some finite dimensional subspaces.

REMARK 3. Our theorems can be proved for some other spaces also. We get an immediate extension if the space R of the real numbers is replaced by the complex numbers, the quaternions, or the Cayley's numbers. In fact, our Theorem 1 can be generalized to the following situation. Consider an infinite collection $\{((X_i, a_i), \tau_i)\}_{i=1}^\infty$ of spaces X_i with a distinguished point a_i and a metric τ_i which satisfy the following assumption:

A. There exists a positive number r such that for each positive $\varepsilon < r$, and for each $i = 1, 2, \ldots$, there exists a point $x_i \in X_i$ with $\tau_i(x_i, a_i) = \varepsilon$.

For such a collection, we may consider the set

$$X^\infty = \left\{(x_1, x_2, \ldots) \in \prod_{i=1}^\infty X_i \mid x_i = a_i \quad \text{for all except finitely many } i\text{'s}\right\}.$$

On the set X^∞, we may again define the topologies $\mathcal{P}$, $\mathcal{W}$ and $\mathcal{T}_p$ for each $1 \leq p \leq \infty$ where a l_p-metric on X^∞ is defined by

$$N_p(x, y) = \left(\sum_{i=1}^{\infty} (\tau_i(x_i, y_i))^p \right)^{1/p} \quad \text{if} \quad 1 \leq p < \infty,$$

and

$$N_\infty(x, y) = \operatorname{Sup}\{\tau_i(x_i, y_i) \mid i = 1, 2, \ldots\}$$

for each $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots) \in X^\infty$.

Our Theorem 1 is true for such an X^∞, and each of these topologies does induce the usual product topology on each of the finite product space $\prod_{i=1}^n X_i$. The assumption A is essentially what we need in carrying out the proof for our Theorem 1. Note that if there exists a number $r > 0$ such that each a_i has a pathwise connected neighborhood in X_i which contains at least one point x_i with $\tau_i(x_i, a_i) \geq r$, then the assumption A will be satisfied for such a collection.

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