# THE LOCAL DIFFUSIONS OF A HARMONIC SHEAF

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1. Introduction. It is well known that the Laplacian operator is the infinitesimal generator of Brownian motion in $R^n$. Moreover, the classical harmonic measures are the hitting measures of Brownian motion. In other words, there is a natural correspondence between the Brownian motion and the classical harmonic functions. In this paper we will show that any family of abstract harmonic functions satisfying the axioms of M. Brelot [4] are annihilated by the infinitesimal generator of a diffusion, and that the corresponding harmonic measures are the hitting measures of the diffusion. This answers a question raised by P. A. Meyer [11] who has proved the existence of a Markov semi-group associated with a harmonic sheaf. Using the diffusion associated with the harmonic sheaf we will then obtain a probabilistic criterion for the regularity of boundary points, and investigate the conditions under which two harmonic sheafs have the same regular boundary points.

The main result of the paper is based on a previous paper [6] in which, for a given family of hitting measures and mean hitting times, an explicit construction of a diffusion is presented. The discussion of regular points is based on the Cartan-Brelot-Naïm fine topology [5] and its probabilistic interpretation which is due to E. B. Dynkin [8].

2. Preliminaries. Let Q be a non compact, locally compact space with a countable base. Let $\bar{Q} = Q \cup \{\infty\}$ be the one point compactification of Q, and let T designate the class of open subsets of $\bar{Q}$.

We will use the language of sheaf theory without defining all the terms. The definitions may be found in the book of R. Godement [9].

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A preharmonic presheaf is a mapping $F$ from $T$ to $\cup\{C(U \cap Q): U \in T\}$, where $C(U)$ is the vector space of continuous functions on $U$, such that the following conditions are satisfied:

(i) $F(U) \subset C(U \cap Q)$;

(ii) if $U \subset V$, there exists a homomorphism $\rho_U^V : F(V) \to F(U)$ such that $\rho_U^U$ is the identity on $F(U)$ and such that if $U \subset V \subset W$ then $\rho_U^W = \rho_U^V \rho_V^W$;

(iii) if $f \in C(U \cap Q)$ and if for every $V \subset U$, $f|V \cap Q \in F(V)$, then $f \in F(U)$;

(iv) if $U$ is connected and if $f_n \nearrow f$ and $f_n \in F(U)$, then either $f \equiv +\infty$ or else $f \in F(U)$;

(v) the constant functions belong to $F(U)$.

A function $f \in F(U)$ will be called a harmonic function on $U$.

Note that in his axiom system M. Brelot [3], [4], [5] does not include (v). However he shows that for any harmonic sheaf there is a related one in which the constant functions do belong.

An open subset $D$ of $\bar{Q}$ is called regular if it has non null boundary $\partial D$ and satisfies the conditions:

(i) for any finite continuous function $f$ on $\partial D$ there exists a unique function $H_f^D$ of $F(D)$ such that $\lim_{y \to x} H_f^D(y) = f(x)$ for every $x \in \partial D$;

(ii) if $f \geq 0$, $H_f^D \geq 0$;

(iii) $D$ is connected.

A harmonic presheaf is a preharmonic presheaf in which there is a base of relatively compact regular sets for the topology on $\bar{Q}$. The associated sheaf of germs of a harmonic presheaf is called a harmonic sheaf.

PROPOSITION 2.1. If $f \in F(U)$, then $f$ has no minimum or maximum in $U$. (M. Brelot [5].)

Let $\Delta$ designate the family of regular sets. Then if $U \in \Delta$ and $x \in U$, $H_f^U(x)$ is a positive linear functional on $C(\partial U)$. Hence $H_\cdot^U(x)$ defines a positive Radon measure on the Borel subsets of $\partial U$ which we will designate by $h_{\partial U}(x, .)$ and $H_f^U(x) = \int_{\partial U} f(y)\, h_{\partial U}(x, dy)$. We call $h_{\partial U}(x, .)$ the hitting probability on $\partial U$ with respect to the point $x$.

A hyperharmonic function $v(.)$ in an open set $D$ is a real valued function on $D$ such that

(i) $v(.)$ is lower semi-continuous,

(ii) $v(.) > -\infty$,

(iii) for every regular open set $U \subset \bar{U} \subset D$, $v(x) \geq \int_{\partial U} v(y)\, h_{\partial U}(x, dy)$ .

PROPOSITION 2.2. (M. Brelot [4].) If a hyperharmonic function is $+\infty$ in the neighborhood of a point it is $+\infty$ in the connected component containing this point.

A function $v(.)$ is said to be superharmonic in an open set $U$ if it is hyperharmonic in $U$ and it is finite on an everywhere dense subset of $U$.

Let $v(.)$ be a superharmonic function which is defined and bounded below in an open set $U$. Then there is a largest harmonic minorant. If the largest harmonic minorant is $0$ we say that $v(.)$ is a potential.

The fine topology on $Q$ is the least fine topology for which all the superharmonic functions are continuous. That is, if $x_0$ is a fine interior point of U, then U contains an open neighborhood of $x_0$ or else it contains the intersection of an open neighborhood of $x_0$ with a set of the form $\{x : v(x) < c\}$ where $v(.)$ is a nonnegative superharmonic function with $v(x_0) < c$.

3. Blankets of Regular Sets. We now give a slightly different introduction to the concept of a regular set. A boundary point $x_0$ of an open set D is said to be a regular boundary point if there exists a superharmonic function $v(.)$ such that $\lim_{x \to x_0} v(x) = 0$ and $v(x) \geq 0$ for $x \in D \cap N_{x_0}$ where $N_{x_0}$ is some neighborhood of $x_0$. $v(.)$ is called a barrier at $x_0$.

H. Bauer [1] has shown that if every boundary point of a domain D is a regular boundary point then the set D is a regular set.

A subfamily $\mathscr{S}$ of $\Delta$ is a blanket of regular sets if the following conditions are satisfied:

(i) $\mathscr{S}$ is a base for the topology of $\Omega$, and

(ii) if $D, D_1, \ldots, D_n \in \mathscr{S}$, then $D_1 \cap \ldots \cap D_n$ and $D - \bar{D}_1 \cup \ldots \cup \bar{D}_n$ belong to $\Delta$ if they are nonempty.

A set D is said to be double regular if every point $x_0 \in \partial D$ is a regular boundary point of both D and $\Omega - \bar{D}$.

THEOREM 3.1. Let $F(.)$ be a harmonic presheaf on $\Omega$. Then given any relatively compact regular subset of $\Omega$, say D, there is a double regular subset contained in D.

Proof. If D is a relatively compact regular subset of $\Omega$, there are two nonproportional harmonic functions. In this case M. Brelot [4] has shown there exists a positive potential in D. Now let $D' \subset \bar{D}' \subset D$. Another result of M. Brelot [4] implies that there exists a potential, v(.), which is bounded and continuous in D and harmonic in the complement of D'. If $c = \inf \{v(x): x \in \partial D'\}$, then $A = \{x: v(x) > c/2\}$ is a double regular set. This follows from the result of H. Bauer quoted above since v(.) forms a barrier for every boundary point of A and $Q - \bar{A}$. q. e. d.

COROLLARY. The class of double regular sets forms a blanket of double regular sets.

Proof. H. Bauer [1] has shown that if x is a regular boundary point of D and if $D' \subset D$ and if $x \in \partial D'$ then x is a regular boundary point of D'. Hence conditions (i) and (ii) are satisfied. Condition (iii) is satisfied in virtue of the above theorem and the fact that there is a base of regular sets. q. e. d.

Now by assumption every compact is contained in a regular set and hence in a double regular set.

THEOREM 3.2. Given any harmonic sheaf there exists a nonstationary strict Markov process with continuous paths up to the boundary of any compact subset, $\Gamma$, of Q. If we designate the process by $X = (x_t, \zeta, F_t^s, P_{s,x})$ where $x_t$ is the position at time $t$, $\zeta$ is the time at which the boundary of $\Gamma$ is first reached, $F_t^s$ is the $\sigma$-field generated by $x_u$, $s \le u \le t$, and $P_{s,x}$ is the probability measure on $F_\zeta^s$ given that the process starts at x at time s, then the following condition is satisfied: $h_{\partial D}(x, A) = P_{s,x}(x_\tau \in A)$ where D is any regular subset, $x \in D$, A is a Borel subset of $\partial D$ and $\tau$ is the time at which the boundary of D is first hit, that is $\tau = \inf(t: x(t) \in \partial D)$. (Recall that the Markov property is that if $0 \le s \le t \le u$, A a Borel subset of Q, then

$$P_{s,x}\{x_u \in A \mid F_t^s\} = P_{t,x_t}(x_u \in A),$$ except for a set of $P_{s,x}$-probability zero. The other technical conditions as well as the definition of the strict Markov property are found in [6, section 1].)

Proof. The theorem is an immediate consequence of theorem 3.1 above and [6, theorem 5.4].

4. The Existence of Mean Hitting times. If $D \in \Delta$, then $e_D(.)$ is a mean hitting time for $D$ if:

(i) $0 < e_D(x) < \infty$, $x \in D$,

(ii) $e_D(.)$ is a continuous function on $D$,

(iii) if $D' \in \Delta$, $D' \subset \bar{D}' \subset D$, then

$$e_D(x) - \int_{\partial D'} h_{\partial D'}(x, dy)\, e_D(y) = e_{D'}(x) ,$$

(iv) if $N_x$ is a fine neighborhood of $x$ and $D_n \in \Delta$, $D_n \searrow N_x$, then there exists an $\epsilon > 0$ such that $e_{D_n}(x) > \epsilon$ for all $n$,

(v) $e_D(x) \to 0$ as $x \to \partial D$.

Let $D$ be a given relatively compact regular set and $\mathscr{S}$ the class of double regular subsets of $\Omega$. Now we can find a countable subclass $D_1, D_2, \ldots$ of $\mathscr{S}$ which is a base for open subsets of $\Gamma$. We define $e_i(x)$ as follows:

$e_i(x) = 1$ if $x \in D_i$,

= the solution of the Dirichlet problem in $\Gamma - D_i$ with boundary value $1$ on $\partial D_i$ and zero on $\partial \Gamma$.

Finally, let $e_\Gamma(x) = \Sigma_{i=1}^{\infty} \lambda_i e_i(x)$ where $\Sigma_{i=1}^{\infty} \lambda_i < \infty$, $\lambda_i > 0$.

THEOREM 4.1. $e_\Gamma(.)$ is a mean hitting time for $\Gamma$.

Proof. Let $F$ be a fine neighborhood of $x$ and let $0_n$ be a sequence of regular sets decreasing to $F$. Now if $x$ is not an interior point of $F$, there exist sets $A$ and $U$ such that $x \in A' = A \cap U \subset F$, where $U$ is a relatively compact set in which $f(y) > f(x) - \epsilon/2$ and $A$ is a set of the form $A = \{y: f(y) < f(x) + \epsilon\}$ where $f(.)$ is some superharmonic function. If $0'_n = 0_n \cap U$ then $0'_n \searrow A'$. We must show that $e_{0'_m}(x) \geq \eta > 0$ for all $m$. Now because $f$ is a superharmonic function $f(x) \geq \int_{\partial 0'_m} f(y)\, h_{\partial 0'_m}(x, dy) = I_m$. Let $p_m = h_{\partial 0'_m}(x, \partial 0'_m \cap A)$. Then $f(x) \geq I_m \geq p_m(f(x) - \epsilon/2) + (1-p_m)(f(x) + \epsilon)$ and hence $p_m \geq 2/3$ for all $m$. Now there is some $i$ such that $D_i \subset \bar{D}_i \subset U$. Because $0'_m \cap A \subset \partial U$, $e_i(x) - \int_{0'_m} e_i(y) h_{\partial 0'_m}(x, dy) > \eta > 0$ for some $\eta$ and hence $e_\Gamma(x) - \int_{\partial 0'_m} e_\Gamma(y)\, h_{\partial 0'_m}(x, dy) > \eta > 0$ which completes the proof of the theorem.

THEOREM 4.2. Given any harmonic sheaf containing the constants there is a strict stationary Markov process with continuous paths, that is, a diffusion, whose infinitesimal generator annihilates all the functions of the sheaf.

Proof. Given a mean hitting time it is shown in ([6], sections 6 and 7) that there exists a random time change which transforms the nonstationary process $X = (x_t, \zeta, F^s_t, P_{s,x})$ into a stationary strict Markov process with continuous paths, that is, a diffusion. This diffusion will be designated by $\tilde{X} = (\tilde{x}_t, \tilde{\zeta}, F^s_t, P_x)$ where $\tilde{x}_t$ is the position of the path at the time $t$, $\tilde{\zeta}$ is the time at which the boundary of $\Gamma$ is first hit, $F^s_t$ is the $\sigma$-field of subsets of paths generated by $\tilde{x}_u$, $s \leq u \leq t$, and $P_x$ is the probability measure on $F^s$ given that the process starts at $x$. Moreover, $h_{\partial D}(x, A) = P_x(\tilde{x}_T \in A)$ if $D$ is any regular subset and $A \subset \partial D$, and therefore if $h(.)$



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is a function in the sheaf $G\,h(x) = \lim_{U \searrow x} \frac{E_x(h(\check{x}_T)) - h(x)}{e_U(x)} = 0$,

where $E_x$ designates integration with respect to the measure $P_x$ and $G$ is the infinitesimal generator of the diffusion. q. e. d.

Note that actually there is a whole family of time changes correspondong to different possible mean hitting times for the harmonic sheaf. Blumenthall, Getoor and McKean have discussed this class of time changes in their paper [2].

5. Probabilistic Interpretation of Regular Boundary Points The following is a generalization to general harmonic sheaves of a theorem of J. L. Doob [7] which he stated and proved for the classical case.

THEOREM 5.1. A point $x$ is an irregular boundary point of an open set $D$ if and only if $P_x(\check{x}_s \in D,\ 0 < s \leq t$ for some $t) = 1$.

Proof. M. Brelot [5] has shown that a point $x$ is irregular if and only if it is a fine interior point of $\{x\} \cup D$. The result then follows immediately from E. B. Dynkin's theorem [8] which states that $x$ is a fine interior point of $D'$ if and only if $P_x(\check{x}_s \in D',\ 0 \leq s < t$ for some $t) = 1$. q. e. d.

A harmonic sheaf of functions in $R^n$ is said to be probabilistically elliptic if:

(i) it has a base of regular sets in common with the harmonic sheaf of Brownian motion in $R^n$, and

(ii) on these regular sets the harmonic measures are equivalent to those of Brownian motion in the measure theoretic sense. Furthermore the sheaf is said to be uniformly probabilistically elliptic if the Radon Nikodym derivatives between the sheaf harmonic measures and the Brownian motion harmonic measures are uniformly bounded.

LEMMA. A point $x$ is a regular boundary point of an open set D if an only if $\limsup_{y \to x} P_y(\tilde{x}_\tau \in \partial N) = 0$ for all sufficiently small neighborhoods N of x where $\tau = \inf \{ t : \tilde{x}_t \notin N \cap D \cup \{x\} \}$.

Proof. If $\limsup_{y \to x} P_y(\tilde{x}_\tau \in \partial N) = 0$, then as a function of y this forms a barrier at x. Now if x is an irregular point it has been shown in the proof of theorem 4.1 that $P_x(\tilde{x}_\tau \in \partial N) > 0$ and hence $\limsup_{y \to x} P_y(\tilde{x}_\tau \in \partial N) > 0$. q.e.d.

From this lemma we may easily deduce the following result.

THEOREM 5.2. A point x is a regular boundary point of an open set D with respect to a uniformly probabilistically elliptic harmonic sheaf if and only if it is a regular point with respect to Brownian motion.

Theorem 5.2 is the analogy for probabilistically uniformly elliptic harmonic sheaves of the theorem of W. Littman, G. Stampacchia and H. F. Weinberger [10] which states that all uniformly elliptic differential operators of a certain form have the same regular points as the Laplacian operator. However the latter result is a rather deep one. It appears to be an open question as to whether uniformly elliptic differential operators are probabilistically uniformly elliptic.

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