THE HOSE-PIPE INSTABILITY IN STELLAR SYSTEMS

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1. Introduction

Indications are that instabilities play an important role in many of the phenomena of stellar dynamics. Examples of such phenomena are: the formation of spiral arms, and the evolution of stellar clusters at a rate faster than one would expect from normal two-body collisions. The analogy with the situation in plasma physics, where similar phenomena are known to be dominated by instabilities, is very suggestive that one might seek for instabilities in stellar dynamics that correspond to similar ones in plasma physics.

However, there is a difficulty. Most plasma instabilities only occur for $\lambda > D$, where D is the Debye length. (For shorter lengths, collective forces are insufficient to hold particles bunched against dispersive thermal forces.) On the other hand, plasma instabilities are usually treated under the simplifying assumption of an infinite uniform medium. This assumption is valid for many plasma systems whose size L is many times larger than D. In gravitating systems, it cannot be strictly valid, since one always has the approximate relation $L \sim D$, where D here refers to the Jeans length. Thus, in treating collective instabilities in stellar systems one is forced to study them on the basis of an inhomogeneous theory. This prevents a bodily carrying over of all plasma instabilities into stellar dynamics.

Therefore, in searching for collisionless instabilities one cannot rely on those already discovered in plasma theory, but must start from scratch, working with inhomogeneous theory. One must take as simple an equilibrium as possible and treat its possible instabilities exactly, by the full Poisson-Vlasov equations in order to be certain as to the correctness of a decision about stability or instability. Two such simple equilibria are the spherically symmetric one, characteristic of stellar clusters, and the one-dimensional slab equilibrium closely related to the galactic disk.

The former type has been rather extensively investigated (Antonov, 1960. 1962, 1969; Lynden Bell and Sanitt, 1969; Ispser and Thorne, 1968), but as yet no instability has been found. The nonrotating slab, although not so realistic since it is only a true equilibrium in one direction, has been investigated with more success. In this paper we wish to discuss an instability in this equilibrium.

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Astrophysics and Space Science **14** (1971) 52–55. *All Rights Reserved*

https://doi.org/10.1017/S0252921100028499 Published online by Cambridge University Press

2. The Slab Equilibrium

By a slab equilibrium we mean a one-dimensional, self-consistent equilibrium with the distribution function $F=F(\varepsilon, v_y, v_z)$, $\varepsilon\equiv v_x^2/2+\Phi$, and potential $\Phi=\Phi(x)$. (We restrict ourselves to those equilibria for which $\partial F/\partial\varepsilon\equiv F_\varepsilon<0$.) One-dimensional perturbations, depending only on x, are stable (Milder, 1966). However, perturbations ϕ in Φ, f in F (which depend on y as $\phi\sim\exp iky$, and which are symmetric in x) are unstable for sufficiently small k (Kulsrud and Mark, 1970) – see Figure 1a. These are of the same character as normal Jeans instabilities for fluid systems, and have the same growth

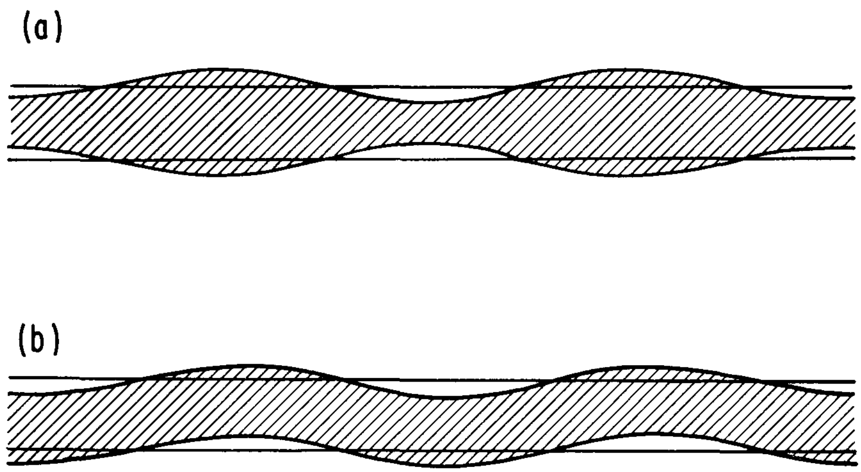

Fig. 1. (a) Jeans instability; (b) Hose-pipe instability.

rates and critical wave number. However, before one can be sure that such instabilities are realistic, one must worry about rotational forces or whatever gives the slab equilibrium in the y and z directions. If one agrees that rotational forces are negligible if the growth rate ω exceeds the rotation rate Ω, one can conclude that these instabilities are real.

In this paper we wish to report on the other possibility for instability; namely, the buckling instability of the slab for which ϕ is not symmetric in x (Figure 1b). This instability occurs when $\sigma_y>\sigma_x$, where $\sigma_y\equiv\langle v_y^2\rangle$ and $\sigma_x\equiv\langle v_x^2\rangle$, where $\langle v_x^2\rangle$ and $\langle v_y^2\rangle$ are the dispersions of the velocity in the x and y directions, respectively. It will grow when the centrifugal force arising from stars passing over the buckling exceeds the gravitational restoring force. This instability is analogous to the hose-pipe instability in plasma physics, but there are differences. For long horizontal wavelengths the gravitational restoring force dominates and the buckling is stable. Thus, the instability occurs for *shorter wavelengths* only, but not, however, for wavelengths short enough to be comparable to the thickness of the slab. On the other hand, the plasma instability

https://doi.org/10.1017/S0252921100028499 Published online by Cambridge University Press

exists for all wave numbers. This instability is thus complementary to the Jeans instability, in which the longer wavelengths are unstable.

There are two reasons for discussing this instability. First, we admit that the model is not quite realistic, since the horizontal forces are not self-consistent because the slab extends infinitely in the y and z directions. Still, it is possible to give a precise treatment of the Vlasov-Poisson equations for the linearized small-amplitude instabilities. By employing one small parameter kx_0, where $2x_0$ is the slab thickness, a simple closed solution can be obtained to lowest significant order. This solution would ordinarily be that for simple gravitational waves of the slab, but if we order $\sigma_y/\sigma_x \sim (kx_0)^{-1}$ the wave is converted to an instability, while the perturbation theory remains correct.

Second, the instability has the interesting property that wavelengths *shorter* than a certain critical wavelength are unstable. This critical wavelength is approximately the length that an equilibrium would have if it were self-consistent in the horizontal y and z directions. Thus, the approximation of homogeneity in these directions may not be so bad.

As a possible illustration of this instability we may consider the galactic disk with the y and z directions in the plane of the disk. The large velocity difference between population I and population II gives a large value for σ_y. If we assume $\sigma_y/\sigma_x \approx 10$, one finds instability for $1 < \lambda/2\pi x_0 < 10$ or $1 \ll \lambda < 10$ kpc, where $x_0 = 150$ pc, $\lambda = 2\pi/k$. These values might be compared with the spacing of the spiral arms, although the direction of dispersion is not correct for this interpretation. (Also, it is problematical whether population II stars may be treated by the slab model as they have a much greater thickness.)

3. The Dispersion Relation

The actual calculation of the growth rate of the instability is as follows: the equilibrium equation is

$$(\mathrm{d}^2\Phi/\mathrm{d}x^2) = 4\pi G \int F_0\left[(v_x^2/2) + \Phi, v_y, v_z\right] \mathrm{d}^3 v\,. \tag{1}$$

Assume, for definiteness, that F_0 vanishes for $|x| > x_0$. The linearized equation for f_1 can be solved and substituted into the equation for ϕ_1 to get

$$\begin{aligned}(\mathrm{d}^2\phi/\mathrm{d}x^2) - k^2\phi &= 4\pi G n \\ &= 4\pi G \int \mathrm{d}^3 v\left[(\partial F/\partial\varepsilon)\,\phi - J\left(\varpi\,(\partial F/\partial\varepsilon) + k\,(\partial F/\partial v_y)\,x\right)\right],\end{aligned} \tag{2}$$

where $\varpi = \omega - k v_y$,

$$J(\varepsilon, x) = \int_0^\tau \mathrm{d}\tau'\phi' \cos\varpi(\tau - \tau') - (\sin\varpi\tau/\cos\varpi\tau_0)\int_0^{\tau_0} \mathrm{d}\tau'\phi'\cos\varpi(\tau - \tau'), \tag{3}$$

$$\tau(\varepsilon, x) = \int_0^x \mathrm{d}x/v_x(x, \varepsilon)\,, \tag{4}$$

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and $\tau_0 = \tau(\varepsilon, x_0)$. For ϕ even in x, replace $\sin\bar{\omega}\tau/\cos\bar{\omega}\tau_0$ by $-\cos\bar{\omega}\tau/\sin\bar{\omega}\tau_0$. The boundary condition on ϕ is

$$(\mathrm{d}\phi/\mathrm{d}x) = -|k|\,\phi\,, \qquad x = \pm\, x_0\,. \tag{5}$$

Now, expand in k, but assume $v_y^2/v_x^2 \sim O(k^{-1})$. Then, to lowest order, the solution is $\omega = 0$ and $\phi = \delta\,\mathrm{d}\Phi_0/\mathrm{d}x$, corresponding to a grid displacement δ in the x direction. For this value one finds (if $\langle v_y \rangle = 0$)

$$n = \phi\,(\mathrm{d}N/\mathrm{d}\Phi) + \int \mathrm{d}^3 v F_\varepsilon \delta x\,(\omega^2 + k^2 \langle v_y^2 \rangle)\,. \tag{6}$$

In order to find a solution which in first order satisfies the boundary condition (5), we must choose $\omega \sim O(k^{1/2})$. Then, substituting the result into (2) and demanding a solution in next order which satisfies (5) ,we get [by integrating Equation (2)]

$$\begin{aligned}\omega^2 &= 2\pi k G N - k^2 \langle v_y^2 \rangle \\ &= 4\pi G \langle \rho \rangle \left[k x_0 - \alpha (k x_0)^2 \left(\langle v_y^2 \rangle / \langle v_x^2 \rangle\right)\right].\end{aligned} \tag{7}$$

Here, $\langle v_y^2 \rangle$ is the mean value of v_x^2 averaged over all particles, $N = 2x_0\,\langle \varrho \rangle$. N is the number of particles per unit area and α a constant of order unity defined by

$$\alpha \langle v_x^2 \rangle = 4\pi G \langle \varrho \rangle\, x_0\,. \tag{8}$$

From Equation (7) we see that the system is indeed unstable for

$$1 \gg k x_0 > \langle v_x^2 \rangle / \alpha \langle v_y^2 \rangle\,. \tag{9}$$

This instability was discovered independently by Toomre (1967). He has found in a numerical investigation that the instability only occurs for $\sigma_x/\sigma_y < 0.1$, so it would appear difficult for such an instability to occur.

Acknowledgement

One of the authors (R.M.K.) was supported by U.S. Air Force Office of Scientific Research Contract AF-44620-70-C-0033.

We should like to thank Alar Toomre for many helpful ideas in connection with this work.

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