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TWISTING OPERATORS, TWISTED TENSOR PRODUCTS AND SMASH PRODUCTS FOR HOM-ASSOCIATIVE ALGEBRAS

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Abstract. The purpose of this paper is to provide new constructions of Hom-associative algebras using Hom-analogues of certain operators called twistors and pseudotwistors, by deforming a given Hom-associative multiplication into a new Hom-associative multiplication. As examples, we introduce Hom-analogues of the twisted tensor product and smash product. Furthermore, we show that the construction by the *twisting principle* introduced by Yau and the twisting of associative algebras using pseudotwistors admit a common generalization.

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1. Introduction. The motivation to introduce Hom-type algebras comes for examples related to q-deformations of Witt and Virasoro algebras, which play an important role in Physics, mainly in conformal field theory. A q-deformation of an algebra of vector fields is obtained when the derivation is replaced by a σ-derivation. It was observed in the pioneering works [**3, 8–11, 15, 20**] that q-deformations of Witt and Virasoro algebras are no longer Lie algebras, but satisfy a twisted Jacobi condition. Motivated by these examples and their generalization, Hartwig, Larsson and Silvestrov in [**14, 17–19**] introduced the notion of Hom–Lie algebra as a deformation of Lie algebras in which the Jacobi identity is twisted by a homomorphism. The associative-type objects corresponding to Hom–Lie algebras, called Hom-associative algebras, have been introduced in [**23**]. Usual functors between the categories of Lie algebras and associative algebras have been extended to the Hom-setting. It was shown that a commutator of a Hom-associative algebra gives rise to a Hom–Lie algebra; the construction of the free Hom-associative algebra and the enveloping algebra of a Hom–Lie algebra have been provided in [**29**]. Since then, Hom-analogues of various classical structures and results have been introduced and discussed by many authors. For instance, representation theory, cohomology and deformation theory for Hom-associative algebras and Hom–Lie algebras have been developed in [**4, 28**]; see also [**13, 22**] for other properties of Hom-associative algebras. All these generalizations coincide with the usual definitions when the structure map equals the identity.

The dual concept of Hom-associative algebras, called Hom-coassociative coalgebras, as well as Hom-bialgebras and Hom–Hopf algebras, have been introduced in [**24, 25**] and also studied in [**6, 31**]. As expected, the enveloping Hom-associative algebra of a Hom–Lie algebra is naturally a Hom-bialgebra. A twisted version of module algebras called module Hom-algebras has been studied in [**30**], where q-deformations of the $\mathfrak{sl}(2)$-action on the affine plane were provided. Objects admitting coactions by Hom-bialgebras have been studied first in [**31**]. In [**36–38**], various generalizations of Yang–Baxter equations and related algebraic structures have been studied. D. Yau provided solutions of HYBE, a twisted version of the Yang–Baxter equation called the Hom–Yang–Baxter equation, from Hom–Lie algebras, quantum enveloping algebra of $\mathfrak{sl}(2)$, the Jones–Conway polynomial, Drinfeld's (co)quasitriangular bialgebras and Yetter–Drinfeld modules (over bialgebras). Yetter–Drinfeld modules over Hom-bialgebras and their category have been studied in [**26**]. For further results about generalizations of quantum groups and related structures, see [**5, 33–35**]. In [**12**], Hom-quasi-bialgebras have been introduced and concepts like gauge transformation and Drinfeld twist generalized. Moreover, an example of a twisted quantum double was provided.

One of the main tools to construct examples of Hom-type algebras is the *twisting principle* (called sometimes *composition method*). It was introduced by D. Yau for Hom-associative algebras and since then extended to various Hom-type algebras. It allows to construct a Hom-type algebra starting from a classical-type algebra and an algebra homomorphism.

The twisted tensor product $A \otimes_R B$ of two associative algebras A and B is a certain algebra structure on the vector space $A \otimes B$, defined in terms of a so-called twisting map $R : B \otimes A \to A \otimes B$, having the property that it coincides with the usual tensor product algebra $A \otimes B$ if R is the usual flip map. This construction was introduced in [**7, 27**] and it may be regarded as a representative for the Cartesian product of noncommutative spaces. An important example of a twisted tensor product of associative algebras is a smash product $A\#H$, where H is a bialgebra and A is a left H-module algebra. Motivated by the desire to express the multiplication of $A \otimes_R B$ as a *deformation* of the multiplication of $A \otimes B$, in [**21**] was introduced the concept of *pseudotwistor* (with a particular case called *twistor*) for an associative algebra D, with multiplication $\mu : D \otimes D \to D$, as a linear map $T : D \otimes D \to D \otimes D$ satisfying some axioms that imply that the new multiplication $\mu \circ T$ on D is also associative. It turns out that many other *deformed multiplications* that appear in the literature (such as twisted bialgebras and Fedosov products) are afforded by such pseudotwistors.

The aim of this paper is to introduce Hom-analogues of twistors, pseudotwistors and twisted tensor products and to use them to obtain new Hom-associative algebras starting with one or more given Hom-associative algebras.

The paper is organized as follows. In Section 2, we review the main definitions and results about twisting associative algebras by means of twistors and pseudotwistors and the basics on Hom-associative algebras, Hom-bialgebras and related structures. In Section 3, we introduce the concepts of Hom-twistor, Hom-pseudotwistor, Hom-twisting map and Hom-twisted tensor product of Hom-associative algebras; we prove that these concepts are compatible with the twisting principle and that the Hom-twisted tensor product can be iterated. Section 4 deals with smash products in the Hom-setting. Given a Hom-bialgebra H and a left (respectively right) H-module Hom-algebra A (respectively C) such that all structure maps α_H, α_A (respectively α_C) are bijective, we define in a natural way a Hom-twisting map R between A and

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H (respectively between H and C) and a Hom-associative algebra $A\#H := A \otimes_R H$ (respectively $H\#C := H \otimes_R C$), called the left (respectively right) Hom-smash product. Given both A and C as above, we define also the so-called two-sided Hom-smash product $A\#H\#C$.

In the last section, we show that Yau's procedure of obtaining a Hom-associative algebra from an associative algebra via the twisting principle and the procedure of obtaining a new associative algebra from a given associative algebra via a pseudotwistor admit a common generalization, by means of a new concept called α-pseudotwistor, where α is an algebra endomorphism of an associative algebra.

2. Preliminaries. We work over a base field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. For a comultiplication $\Delta : C \to C \otimes C$ on a vector space C, we use a Sweedler-type notation $\Delta(c) = c_1 \otimes c_2$, for $c \in C$. Unless otherwise specified, the (co)algebras ((co)associative or not) that will appear in what follows are *not* supposed to be (co)unital, and a multiplication $\mu : V \otimes V \to V$ on a linear space V is denoted by juxtaposition: $\mu(v \otimes v') = vv'$.

We recall some concepts and results, fixing the terminology to be used throughout the paper.

Definition 2.1 ([**7, 27**]). Let (A, μ_A), (B, μ_B) be two associative algebras. A *twisting map* between A and B is a linear map $R : B \otimes A \to A \otimes B$ satisfying the conditions:

$$R \circ (id_B \otimes \mu_A) = (\mu_A \otimes id_B) \circ (id_A \otimes R) \circ (R \otimes id_A), \tag{2.1}$$
$$R \circ (\mu_B \otimes id_A) = (id_A \otimes \mu_B) \circ (R \otimes id_B) \circ (id_B \otimes R). \tag{2.2}$$

If this is the case, the map $\mu_R = (\mu_A \otimes \mu_B) \circ (id_A \otimes R \otimes id_B)$ is an associative product on $A \otimes B$; the associative algebra $(A \otimes B, \mu_R)$ is denoted by $A \otimes_R B$ and called the *twisted tensor product* of A and B afforded by R.

If we use a Sweedler-type notation $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$, for $a \in A, b \in B$, then (2.1) and (2.2) may be rewritten as

$$(aa')_R \otimes b_R = a_R a'_r \otimes (b_R)_r, \tag{2.3}$$
$$a_R \otimes (bb')_R = (a_R)_r \otimes b_r b'_R, \tag{2.4}$$

and the multiplication of $A \otimes_R B$ may be written as $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$.

Example 2.2. We construct a twisted tensor product of $k^2 \otimes k^2$. The multiplication of k^2 with respect to $\{e_1, e_2\}$ is defined as $e_i e_j = \delta_{i,j} e_i$ for $i, j = 1, 2$, where δ is the Kronecker symbol. We provide a one-parameter family of twisting maps (λ is a parameter in k):

$$R(e_1 \otimes e_1) = \lambda e_1 \otimes e_1 + \lambda e_1 \otimes e_2 + \lambda e_2 \otimes e_1 + (\lambda - 1) e_2 \otimes e_2,$$
$$R(e_1 \otimes e_2) = (1 - \lambda) e_1 \otimes e_1 - \lambda e_1 \otimes e_2 + (1 - \lambda) e_2 \otimes e_1 + (1 - \lambda) e_2 \otimes e_2,$$
$$R(e_2 \otimes e_1) = (1 - \lambda) e_1 \otimes e_1 + (1 - \lambda) e_1 \otimes e_2 - \lambda e_2 \otimes e_1 + (1 - \lambda) e_2 \otimes e_2,$$
$$R(e_2 \otimes e_2) = (\lambda - 1) e_1 \otimes e_1 + \lambda e_1 \otimes e_2 + \lambda e_2 \otimes e_1 + \lambda e_2 \otimes e_2.$$

Therefore, we obtain the following new multiplication on $k^2 \otimes k^2$:

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	$e_1 \otimes e_1$	$e_1 \otimes e_2$	$e_2 \otimes e_1$	$e_2 \otimes e_2$
$e_1 \otimes e_1$	$\lambda e_1 \otimes e_1$	$\lambda e_1 \otimes e_2$	$(1-\lambda)e_1 \otimes e_1$	$-\lambda e_1 \otimes e_2$
$e_1 \otimes e_2$	$(1-\lambda)e_1 \otimes e_1$	$(1-\lambda)e_1 \otimes e_2$	$(\lambda-1)e_1 \otimes e_1$	$\lambda e_1 \otimes e_2$
$e_2 \otimes e_1$	$\lambda e_2 \otimes e_1$	$(\lambda-1)e_2 \otimes e_2$	$(1-\lambda)e_2 \otimes e_1$	$(1-\lambda)e_2 \otimes e_2$
$e_2 \otimes e_2$	$-\lambda e_2 \otimes e_1$	$(1-\lambda)e_2 \otimes e_2$	$\lambda e_2 \otimes e_1$	$\lambda e_2 \otimes e_2$

The following two concepts are versions for nonunital algebras of the ones introduced in [**21**]:

DEFINITION 2.3. Let (D, μ) be an associative algebra and $T : D \otimes D \to D \otimes D$ a linear map. Assume that there exist two linear maps $\tilde{T}_1, \tilde{T}_2 : D \otimes D \otimes D \to D \otimes D \otimes D$ such that the following conditions are satisfied:

$$T \circ (id_D \otimes \mu) = (id_D \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_D), \tag{2.5}$$

$$T \circ (\mu \otimes id_D) = (\mu \otimes id_D) \circ \tilde{T}_2 \circ (id_D \otimes T), \tag{2.6}$$

$$\tilde{T}_1 \circ (T \otimes id_D) \circ (id_D \otimes T) = \tilde{T}_2 \circ (id_D \otimes T) \circ (T \otimes id_D). \tag{2.7}$$

Then, $D^T := (D, \mu \circ T)$ is also an associative algebra. The map T is called a *pseudotwistor* and the two maps $\tilde{T}_1, \tilde{T}_2$ are called the *companions* of T.

DEFINITION 2.4. Let (D, μ) be an associative algebra and $T : D \otimes D \to D \otimes D$ a linear map, with Sweedler-type notation $T(d \otimes d') = d^T \otimes d'_T$, for $d, d' \in D$, satisfying the following conditions:

$$T \circ (id_D \otimes \mu) = (id_D \otimes \mu) \circ T_{13} \circ T_{12}, \tag{2.8}$$

$$T \circ (\mu \otimes id_D) = (\mu \otimes id_D) \circ T_{13} \circ T_{23}, \tag{2.9}$$

$$T_{12} \circ T_{23} = T_{23} \circ T_{12}, \tag{2.10}$$

where we used a standard notation for the operators T_{ij}, namely $T_{12} = T \otimes id_D$, $T_{23} = id_D \otimes T$ and $T_{13}(d \otimes d' \otimes d'') = d^T \otimes d' \otimes d''_T$. Then, $D^T := (D, \mu \circ T)$ is also an associative algebra, and the map T is called a *twistor* for D.

Obviously, any twistor T is a pseudotwistor with companions $\tilde{T}_1 = \tilde{T}_2 = T_{13}$.

If $A \otimes_R B$ is a twisted tensor product of associative algebras, the map $T : (A \otimes B) \otimes (A \otimes B) \to (A \otimes B) \otimes (A \otimes B)$, $T((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_R) \otimes (a'_R \otimes b')$, is a twistor for the ordinary tensor product algebra $A \otimes B$ and $A \otimes_R B = (A \otimes B)^T$ as associative algebras, cf. [**21**].

We recall now several things about Hom-structures. Since various authors use different terminology, some caution is necessary. In what follows, we use terminology as in our previous paper [**26**].

DEFINITION 2.5.

(i) A *Hom-associative algebra* is a triple (A, μ, α), in which A is a linear space, $\alpha : A \to A$ and $\mu : A \otimes A \to A$ are linear maps, with notation $\mu(a \otimes a') = aa'$, satisfying the following conditions, for all $a, a', a'' \in A$:

$$\alpha(aa') = \alpha(a)\alpha(a'), \qquad (multiplicativity)$$
$$\alpha(a)(a'a'') = (aa')\alpha(a''). \qquad (Hom-associativity)$$

We call α the *structure map* of A.

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A morphism $f : (A, \mu_A, \alpha_A) \to (B, \mu_B, \alpha_B)$ of Hom-associative algebras is a linear map $f : A \to B$ such that $\alpha_B \circ f = f \circ \alpha_A$ and $f \circ \mu_A = \mu_B \circ (f \otimes f)$.

(ii) A *Hom-coassociative coalgebra* is a triple (C, Δ, α), in which C is a linear space, $\alpha : C \to C$ and $\Delta : C \to C \otimes C$ are linear maps, satisfying the following conditions:

$$(\alpha \otimes \alpha) \circ \Delta = \Delta \circ \alpha, \qquad (comultiplicativity)$$
$$(\Delta \otimes \alpha) \circ \Delta = (\alpha \otimes \Delta) \circ \Delta. \qquad (Hom\text{-}coassociativity)$$

A morphism $g : (C, \Delta_C, \alpha_C) \to (D, \Delta_D, \alpha_D)$ of Hom-coassociative coalgebras is a linear map $g : C \to D$ such that $\alpha_D \circ g = g \circ \alpha_C$ and $(g \otimes g) \circ \Delta_C = \Delta_D \circ g$.

REMARK 2.6. Assume that (A, μ_A, α_A) and (B, μ_B, α_B) are two Hom-associative algebras; then $(A \otimes B, \mu_{A\otimes B}, \alpha_A \otimes \alpha_B)$ is a Hom-associative algebra (called the tensor product of A and B), where $\mu_{A\otimes B}$ is the usual multiplication: $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$.

DEFINITION 2.7 ([**30, 35**]).

(i) Let (A, μ_A, α_A) be a Hom-associative algebra, M a linear space and $\alpha_M : M \to M$ a linear map. A *left A-module* structure on (M, α_M) consists of a linear map $A \otimes M \to M$, $a \otimes m \mapsto a \cdot m$, satisfying the conditions:

$$\alpha_M(a \cdot m) = \alpha_A(a) \cdot \alpha_M(m), \tag{2.11}$$
$$\alpha_A(a) \cdot (a' \cdot m) = (aa') \cdot \alpha_M(m), \tag{2.12}$$

for all $a, a' \in A$ and $m \in M$. If (M, α_M) and (N, α_N) are left A-modules (both A-actions denoted by $\cdot$), a morphism of left A-modules $f : M \to N$ is a linear map satisfying the conditions $\alpha_N \circ f = f \circ \alpha_M$ and $f(a \cdot m) = a \cdot f(m)$, for all $a \in A$ and $m \in M$.

(ii) Let (C, Δ_C, α_C) be a Hom-coassociative coalgebra, M a linear space and $\alpha_M : M \to M$ a linear map. A *left C-comodule* structure on (M, α_M) consists of a linear map $\lambda : M \to C \otimes M$ satisfying the following conditions:

$$(\alpha_C \otimes \alpha_M) \circ \lambda = \lambda \circ \alpha_M, \tag{2.13}$$
$$(\Delta_C \otimes \alpha_M) \circ \lambda = (\alpha_C \otimes \lambda) \circ \lambda. \tag{2.14}$$

If (M, α_M) and (N, α_N) are left C-comodules, with structures $\lambda_M : M \to C \otimes M$ and $\lambda_N : N \to C \otimes N$, a morphism of left C-comodules $g : M \to N$ is a linear map satisfying the conditions $\alpha_N \circ g = g \circ \alpha_M$ and $(id_C \otimes g) \circ \lambda_M = \lambda_N \circ g$.

DEFINITION 2.8. ([**24, 25**]) A *Hom-bialgebra* is a quadruple (H, μ, Δ, α), in which (H, μ, α) is a Hom-associative algebra, (H, Δ, α) is a Hom-coassociative coalgebra and moreover Δ is a morphism of Hom-associative algebras.

In other words, a Hom-bialgebra is a Hom-associative algebra (H, μ, α) endowed with a linear map $\Delta : H \to H \otimes H$, with notation $\Delta(h) = h_1 \otimes h_2$, such that the

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following conditions are satisfied, for all $h, h' \in H$:

$$\Delta(h_1) \otimes \alpha(h_2) = \alpha(h_1) \otimes \Delta(h_2), \tag{2.15}$$
$$\Delta(hh') = h_1h'_1 \otimes h_2h'_2, \tag{2.16}$$
$$\Delta(\alpha(h)) = \alpha(h_1) \otimes \alpha(h_2). \tag{2.17}$$

PROPOSITION 2.9 ([**25**, **32**]).
(*i*) *Let (A, μ) be an associative algebra and $\alpha : A \to A$ an algebra endomorphism. Define a new multiplication $\mu_\alpha := \alpha \circ \mu : A \otimes A \to A$. Then, (A, μ_α, α) is a Hom-associative algebra, denoted by A_α.*
(*ii*) *Let (C, Δ) be a coassociative coalgebra and $\alpha : C \to C$ a coalgebra endomorphism. Define a new comultiplication $\Delta_\alpha := \Delta \circ \alpha : C \to C \otimes C$. Then, $(C, \Delta_\alpha, \alpha)$ is a Hom-coassociative coalgebra, denoted by C_α.*
(*iii*) *Let (H, μ, Δ) be a bialgebra and $\alpha : H \to H$ a bialgebra endomorphism. If we define μ_α and Δ_α as in (i) and (ii), then $H_\alpha = (H, \mu_\alpha, \Delta_\alpha, \alpha)$ is a Hom-bialgebra.*

PROPOSITION 2.10 ([**35**]). *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra and (M, α_M) and (N, α_N) two left H-modules. Then, $(M \otimes N, \alpha_M \otimes \alpha_N)$ is also a left H-module, with H-action defined by $H \otimes (M \otimes N) \to M \otimes N$, $h \otimes (m \otimes n) \mapsto h \cdot (m \otimes n) := h_1 \cdot m \otimes h_2 \cdot n$.*

DEFINITION 2.11 ([**31**]). Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra. A *left H-comodule Hom-algebra* is a Hom-associative algebra (D, μ_D, α_D) endowed with a left H-comodule structure $\lambda_D : D \to H \otimes D$ such that λ_D is a morphism of Hom-associative algebras.

DEFINITION 2.12 ([**30**]). Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra. A Hom-associative algebra (A, μ_A, α_A) is called a *left H-module Hom-algebra* if (A, α_A) is a left H-module, with action denoted by $H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$, such that the following condition is satisfied:

$$\alpha_H^2(h) \cdot (aa') = (h_1 \cdot a)(h_2 \cdot a'), \quad \forall\, h \in H,\ a, a' \in A. \tag{2.18}$$

One may wonder why it was chosen α_H^2 in the above formula (and not, for instance, α_H). The answer is provided by the following result:

PROPOSITION 2.13 ([**30**]). *Let (H, μ_H, Δ_H) be a bialgebra and (A, μ_A) a left H-module algebra in the usual sense, with action denoted by $H \otimes A \to A, h \otimes a \mapsto h \cdot a$. Let $\alpha_H : H \to H$ be a bialgebra endomorphism and $\alpha_A : A \to A$ an algebra endomorphism, such that $\alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a)$, for all $h \in H$ and $a \in A$. If we consider the Hom-bialgebra $H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H)$ and the Hom-associative algebra $A_{\alpha_A} = (A, \alpha_A \circ \mu_A, \alpha_A)$, then A_{α_A} is a left H_{α_H}-module Hom-algebra in the above sense, with action $H_{\alpha_H} \otimes A_{\alpha_A} \to A_{\alpha_A}$, $h \otimes a \mapsto h \triangleright a := \alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a)$.*

3. Hom-pseudotwistors and Hom-twisted tensor products.

We begin by introducing the Hom-analogues of twistors and pseudotwistors.

PROPOSITION 3.1. *Let (D, μ, α) be a Hom-associative algebra and $T : D \otimes D \to D \otimes D$ a linear map. Assume that there exist two linear maps $\tilde{T}_1, \tilde{T}_2 : D \otimes D \otimes D \to$*

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$D \otimes D \otimes D$ such that the following relations hold:

$$(\alpha \otimes \alpha) \circ T = T \circ (\alpha \otimes \alpha), \tag{3.1}$$
$$T \circ (\alpha \otimes \mu) = (\alpha \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_D), \tag{3.2}$$
$$T \circ (\mu \otimes \alpha) = (\mu \otimes \alpha) \circ \tilde{T}_2 \circ (id_D \otimes T), \tag{3.3}$$
$$\tilde{T}_1 \circ (T \otimes id_D) \circ (id_D \otimes T) = \tilde{T}_2 \circ (id_D \otimes T) \circ (T \otimes id_D). \tag{3.4}$$

Then, $D^T := (D, \mu \circ T, \alpha)$ is also a Hom-associative algebra. The map T is called a Hom-pseudotwistor *and the two maps $\tilde{T}_1$, $\tilde{T}_2$ are called the* companions *of T.*

Proof. We record first the obvious relations

$$(\mu \circ T) \otimes \alpha = (\mu \otimes \alpha) \circ (T \otimes id_D), \tag{3.5}$$
$$\alpha \otimes (\mu \circ T) = (\alpha \otimes \mu) \circ (id_D \otimes T). \tag{3.6}$$

The fact that α is multiplicative with respect to $\mu \circ T$ follows immediately from (3.1) and the fact that α is multiplicative with respect to μ. Now, we prove the Hom-associativity of $\mu \circ T$:

$$\begin{aligned}
(\mu \circ T) \circ ((\mu \circ T) \otimes \alpha) &\overset{(3.5)}{=} \mu \circ T \circ (\mu \otimes \alpha) \circ (T \otimes id_D) \\
&\overset{(3.3)}{=} \mu \circ (\mu \otimes \alpha) \circ \tilde{T}_2 \circ (id_D \otimes T) \circ (T \otimes id_D) \\
&\overset{(3.4)}{=} \mu \circ (\mu \otimes \alpha) \circ \tilde{T}_1 \circ (T \otimes id_D) \circ (id_D \otimes T) \\
&\overset{Hom-associativity\ of\ \mu}{=} \mu \circ (\alpha \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_D) \circ (id_D \otimes T) \\
&\overset{(3.2)}{=} \mu \circ T \circ (\alpha \otimes \mu) \circ (id_D \otimes T) \\
&\overset{(3.6)}{=} (\mu \circ T) \circ (\alpha \otimes (\mu \circ T)),
\end{aligned}$$

finishing the proof. □

DEFINITION 3.2. Let (D, μ, α) be a Hom-associative algebra and $T : D \otimes D \to D \otimes D$ be a linear map, satisfying the following conditions:

$$(\alpha \otimes \alpha) \circ T = T \circ (\alpha \otimes \alpha), \tag{3.7}$$
$$T \circ (\alpha \otimes \mu) = (\alpha \otimes \mu) \circ T_{13} \circ T_{12}, \tag{3.8}$$
$$T \circ (\mu \otimes \alpha) = (\mu \otimes \alpha) \circ T_{13} \circ T_{23}, \tag{3.9}$$
$$T_{12} \circ T_{23} = T_{23} \circ T_{12}. \tag{3.10}$$

Such a map T is called a *Hom-twistor*. Obviously, a Hom-twistor T is a Hom-pseudotwistor, with companions $\tilde{T}_1 = \tilde{T}_2 = T_{13}$, so we can consider the Hom-associative algebra $D^T := (D, \mu \circ T, \alpha)$.

EXAMPLE 3.3. We consider the two-dimensional Hom-associative algebra (D, μ, α) defined with respect to a basis $\{e_1, e_2\}$ by

$$\begin{aligned}
&\mu(e_1, e_1) = ae_1,\ \ \mu(e_1, e_2) = \mu(e_2, e_1) = \lambda_1 ae_1 + \lambda_2 ae_2, \\
&\mu(e_2, e_2) = \frac{\lambda_1^2(1 - 2\lambda_2)a}{(1 - \lambda_2)^2} e_1 + \frac{2\lambda_1\lambda_2 a}{1 - \lambda_2} e_2, \\
&\alpha(e_1) = e_1,\ \ \alpha(e_2) = \lambda_1 e_1 + \lambda_2 e_2,
\end{aligned}$$

where a, λ_1, λ_2 are parameters in k, with $\lambda_2 \neq 1$ and $a \neq 0$. It is easy to see that D is an associative algebra if and only if $\lambda_2 = 0$.

We provide an example of a Hom-twistor T for D; it is defined with respect to the basis by

$$T(e_1 \otimes e_1) = e_1 \otimes e_1, \quad T(e_1 \otimes e_2) = \frac{\lambda_1}{1-\lambda_2} e_1 \otimes e_1,$$
$$T(e_2 \otimes e_1) = e_2 \otimes e_1, \quad T(e_2 \otimes e_2) = \frac{\lambda_1}{1-\lambda_2} e_2 \otimes e_1.$$

By Proposition 3.1, we have the new Hom-associative algebra $D^T = (D, \mu_T = \mu \circ T, \alpha)$ whose multiplication is defined on the basis by

$$\mu_T(e_1, e_1) = ae_1, \quad \mu_T(e_1, e_2) = \frac{\lambda_1 a}{1-\lambda_2} e_1,$$
$$\mu_T(e_2, e_1) = \lambda_1 a e_1 + \lambda_2 a e_2, \quad \mu_T(e_2, e_2) = \frac{\lambda_1 a}{1-\lambda_2}(\lambda_1 e_1 + \lambda_2 e_2).$$

Notice that the new multiplication is no longer commutative.

Proposition 3.4. *Let (D, μ) be an associative algebra, $\alpha : D \to D$ an algebra endomorphism and $T : D \otimes D \to D \otimes D$ a pseudotwistor with companions $\tilde{T}_1$, $\tilde{T}_2$. Consider the associative algebra $D^T = (D, \mu \circ T)$ and the Hom-associative algebra $D_\alpha = (D, \alpha \circ \mu, \alpha)$. Assume that moreover we have $(\alpha \otimes \alpha) \circ T = T \circ (\alpha \otimes \alpha)$. Then, T is a Hom-pseudotwistor for D_α with companions $\tilde{T}_1$, $\tilde{T}_2$, the map α is an algebra endomorphism of the associative algebra D^T and the Hom-associative algebras $(D_\alpha)^T$ and $(D^T)_\alpha$ coincide. In particular, if T is a twistor for D, then T is a Hom-twistor for D_α.*

Proof. The only nontrivial things to prove are the relations (3.2) and (3.3) with μ there replaced by the multiplication of D_α, that is $\alpha \circ \mu$. We compute:

$$\begin{aligned}
(\alpha \otimes (\alpha \circ \mu)) \circ \tilde{T}_1 \circ (T \otimes id_D) &= (\alpha \otimes \alpha) \circ (id_D \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_D) \\
&\overset{(2.5)}{=} (\alpha \otimes \alpha) \circ T \circ (id_D \otimes \mu) \\
&= T \circ (\alpha \otimes \alpha) \circ (id_D \otimes \mu) \\
&= T \circ (\alpha \otimes (\alpha \circ \mu)),
\end{aligned}$$

so (3.2) holds; similarly one can prove (3.3). □

We introduce now the Hom-analogue of twisted tensor products of algebras.

Definition 3.5. Let (A, μ_A, α_A) and (B, μ_B, α_B) be two Hom-associative algebras. A linear map $R : B \otimes A \to A \otimes B$ is called a *Hom-twisting map* between A and B if the following conditions are satisfied:

$$(\alpha_A \otimes \alpha_B) \circ R = R \circ (\alpha_B \otimes \alpha_A), \tag{3.11}$$
$$R \circ (\alpha_B \otimes \mu_A) = (\mu_A \otimes \alpha_B) \circ (id_A \otimes R) \circ (R \otimes id_A), \tag{3.12}$$
$$R \circ (\mu_B \otimes \alpha_A) = (\alpha_A \otimes \mu_B) \circ (R \otimes id_B) \circ (id_B \otimes R). \tag{3.13}$$

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If we use the standard Sweedler-type notation $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$, for $a \in A$, $b \in B$, then the above conditions may be rewritten as

$$\alpha_A(a_R) \otimes \alpha_B(b_R) = \alpha_A(a)_R \otimes \alpha_B(b)_R, \tag{3.14}$$

$$(aa')_R \otimes \alpha_B(b)_R = a_R a'_r \otimes \alpha_B((b_R)_r), \tag{3.15}$$

$$\alpha_A(a)_R \otimes (bb')_R = \alpha_A((a_R)_r) \otimes b_r b'_R, \tag{3.16}$$

for all $a, a' \in A$ and $b, b' \in B$.

PROPOSITION 3.6. *Let* (A, μ_A, α_A) *and* (B, μ_B, α_B) *be two Hom-associative algebras and* $R : B \otimes A \to A \otimes B$ *a Hom-twisting map. Define the linear map* $T : (A \otimes B) \otimes (A \otimes B) \to (A \otimes B) \otimes (A \otimes B)$, $T((a \otimes b) \otimes (a' \otimes b')) = (a \otimes b_R) \otimes (a'_R \otimes b')$. *Then,* T *is a Hom-twistor for the Hom-associative algebra* $(A \otimes B, \mu_{A\otimes B}, \alpha_A \otimes \alpha_B)$*, the tensor product of* A *and* B*. The Hom-associative algebra* $(A \otimes B)^T$ *is denoted by* $A \otimes_R B$ *and is called the* Hom-twisted tensor product *of* A *and* B*; its multiplication is defined by* $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$*, and the structure map is* $\alpha_A \otimes \alpha_B$.

Proof. We need to prove that T satisfies the conditions (3.7)–(3.10) for $A \otimes B$. The condition (3.10) is trivially satisfied, (3.7) follows immediately from (3.14), while (3.8) and (3.9) follow after some easy computations by using (3.15) and respectively (3.16). □

REMARK 3.7. Let (A, μ_A, α_A) and (B, μ_B, α_B) be two Hom-associative algebras. Then obviously the linear map $R : B \otimes A \to A \otimes B$, $R(b \otimes a) = a \otimes b$, is a Hom-twisting map and the Hom-twisted tensor product $A \otimes_R B$ coincides with the ordinary tensor product $A \otimes B$.

EXAMPLE 3.8. We assume that the characteristic of k is zero and consider the algebra D defined in Example 3.3 with $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Recall that this D is associative, but we regard it as a Hom-associative algebra with the same multiplication but with structure map as defined in Example 3.3, that is $\alpha(e_1) = e_1$, $\alpha(e_2) = \lambda_1 e_1$. One can see that this Hom-associative algebra D is a twisting, in the sense of Proposition 2.9, of the associative algebra D, via the map α. We introduce two families of examples of Hom-twisting maps, denoted by R_1 and R_2, between this Hom-associative algebra and itself. They are defined with respect to the given basis by

$$\begin{aligned}
R_1(e_1 \otimes e_1) &= 0,\\
R_1(e_1 \otimes e_2) &= a_1 e_1 \otimes e_1 + a_2 e_1 \otimes e_2 - \left(a_2 + \frac{a_1}{\lambda_1}\right) e_2 \otimes e_1,\\
R_1(e_2 \otimes e_1) &= a_3 e_1 \otimes e_1 - \frac{1}{2\lambda_1}(a_1 + a_3 - a_4 + a_5 + 2a_2\lambda_1) e_1 \otimes e_2\\
&\quad + \frac{1}{2\lambda_1}(a_1 - a_3 - a_4 + a_5 + 2a_2\lambda_1) e_2 \otimes e_1,\\
R_1(e_2 \otimes e_2) &= \frac{\lambda_1}{2}(a_1 + a_3 - a_4 - a_5) e_1 \otimes e_1 + a_4 e_1 \otimes e_2 + a_5 e_2 \otimes e_1\\
&\quad - \frac{1}{2\lambda_1}(a_1 + a_3 + a_4 + a_5) e_2 \otimes e_2
\end{aligned}$$

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and

$$\begin{aligned}
R_2(e_1 \otimes e_1) &= e_1 \otimes e_1,\\
R_2(e_1 \otimes e_2) &= a_1 e_1 \otimes e_1 + a_2 e_1 \otimes e_2 + \left(1 - a_2 - \frac{a_1}{\lambda_1}\right) e_2 \otimes e_1,\\
R_2(e_2 \otimes e_1) &= a_3 e_1 \otimes e_1 - \frac{1}{2\lambda_1}(a_1 + a_3 - a_4 + a_5 + 2a_2\lambda_1 - 2\lambda_1) e_1 \otimes e_2\\
&\quad + \frac{1}{2\lambda_1}(a_1 - a_3 - a_4 + a_5 + 2a_2\lambda_1) e_2 \otimes e_1,\\
R_2(e_2 \otimes e_2) &= \frac{\lambda_1}{2}(a_1 + a_3 - a_4 - a_5) e_1 \otimes e_1 + a_4 e_1 \otimes e_2 + a_5 e_2 \otimes e_1\\
&\quad - \frac{1}{2\lambda_1}(a_1 + a_3 + a_4 + a_5 - 2\lambda_1) e_2 \otimes e_2,
\end{aligned}$$

where $a_1, \ldots, a_5$ are parameters in k. It is worth mentioning that in general R_1 and R_2 are *not* twisting maps for the associative algebra D.

EXAMPLE 3.9. We present now a family of examples of Hom-twisting maps between the Hom-associative algebra D defined in Example 3.3, for which we choose again $\lambda_1 \neq 0$, $\lambda_2 = 0$, and the associative algebra k^2 (with a basis $\{f_1, f_2\}$ with multiplication $f_i \cdot f_j = \delta_{ij} f_i$, for $i, j \in \{1, 2\}$, where δ_{ij} is the Kronecker symbol), considered as a Hom-associative algebra with structure map equal to the identity. Namely, the twisting maps are defined with respect to the bases by the following formulae:

$$\begin{aligned}
R(f_1 \otimes e_1) &= 0,\\
R(f_1 \otimes e_2) &= a_1 e_1 \otimes f_1 + a_2 e_1 \otimes f_2 - \frac{a_1}{\lambda_1} e_2 \otimes f_1 - \frac{a_2}{\lambda_1} e_2 \otimes f_2,\\
R(f_2 \otimes e_1) &= 0,\\
R(f_2 \otimes e_2) &= a_1 \lambda_1 e_1 \otimes f_1 + a_2 \lambda_1 e_1 \otimes f_2 - a_1 e_2 \otimes f_1 - a_2 e_2 \otimes f_2,
\end{aligned}$$

where a_1, a_2 are parameters in k. Note also that in general R is *not* a twisting map between the associative algebras D and k^2, therefore the obtained algebra is no longer associative.

PROPOSITION 3.10. *Let (A, μ_A) and (B, μ_B) be two associative algebras, $\alpha_A : A \to A$ and $\alpha_B : B \to B$ algebra maps and $R : B \otimes A \to A \otimes B$ a twisting map satisfying the condition $(\alpha_A \otimes \alpha_B) \circ R = R \circ (\alpha_B \otimes \alpha_A)$. Then, R is a Hom-twisting map between the Hom-associative algebras A_{α_A} and B_{α_B} and the Hom-associative algebras $A_{\alpha_A} \otimes_R B_{\alpha_B}$ and $(A \otimes_R B)_{\alpha_A \otimes \alpha_B}$ coincide.*

Proof. Note first that $\alpha_A \otimes \alpha_B$ is an algebra endomorphism of $A \otimes_R B$ because of the relation $(\alpha_A \otimes \alpha_B) \circ R = R \circ (\alpha_B \otimes \alpha_A)$. We need to prove (3.12) and (3.13) with those μ_A and μ_B replaced by $\alpha_A \circ \mu_A$ and respectively $\alpha_B \circ \mu_B$. We prove only (3.12),

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while (3.13) is similar and left to the reader:

$$\begin{aligned}((\alpha_A\circ\mu_A)\otimes\alpha_B)\circ(id_A\otimes R)\circ(R\otimes id_A) &= (\alpha_A\otimes\alpha_B)\circ(\mu_A\otimes id_B)\\ &\quad\circ(id_A\otimes R)\circ(R\otimes id_A)\\ &\overset{(2.1)}{=} (\alpha_A\otimes\alpha_B)\circ R\circ(id_B\otimes\mu_A)\\ &= R\circ(\alpha_B\otimes\alpha_A)\circ(id_B\otimes\mu_A)\\ &= R\circ(\alpha_B\otimes(\alpha_A\circ\mu_A)),\quad q.e.d.\end{aligned}$$

The fact that the multiplications of $A_{\alpha_A}\otimes_R B_{\alpha_B}$ and $(A\otimes_R B)_{\alpha_A\otimes\alpha_B}$ coincide is an immediate consequence of the relation $(\alpha_A\otimes\alpha_B)\circ R = R\circ(\alpha_B\otimes\alpha_A)$. □

Let (A,μ_A) be a (not necessarily associative) algebra over k, let $q\in k$ be a nonzero fixed element and $\sigma: A\to A$ an involutive (i.e. $\sigma^2=id_A$) algebra automorphism. We denote by $C(k,q)$ the two-dimensional associative algebra $k[v]/(v^2=q)$. Define the linear map

$$R: C(k,q)\otimes A\to A\otimes C(k,q),\quad R(1\otimes a)=a\otimes 1,\quad R(v\otimes a)=\sigma(a)\otimes v, \tag{3.17}$$

for all $a\in A$. The *Clifford process*, as introduced in [**1**], [**39**], associates to the pair (A,σ) a (not necessarily associative) algebra structure on $A\otimes C(k,q)$, with multiplication

$$(a\otimes 1+b\otimes v)(c\otimes 1+d\otimes v)=(ac+qb\sigma(d))\otimes 1+(ad+b\sigma(c))\otimes v, \tag{3.18}$$

for all $a,b,c,d\in A$. This algebra structure is denoted by $\overline{A}$. As noted in [**1**], if A is associative then so is $\overline{A}$, and in this case R is a twisting map and $\overline{A}$ is the twisted tensor product of associative algebras $\overline{A}=A\otimes_R C(k,q)$. If A is a quasialgebra, i.e. A is a left H-module algebra over a quasi-bialgebra H and moreover σ is H-linear, then $\overline{A}$ is also a quasialgebra, cf. [**2**].

Assume now that (A,μ_A,α_A) is Hom-associative and we have $\alpha_A\circ\sigma=\sigma\circ\alpha_A$. We can regard $B=C(k,q)$ as a Hom-associative algebra with $\alpha_B=id$.

Proposition 3.11. *The map R defined by (3.17) is a Hom-twisting map and the Hom-twisted tensor product $A\otimes_R C(k,q)$ and $\overline{A}$ are isomorphic as algebras. Consequently, $\overline{A}$ is a Hom-associative algebra.*

Proof. We begin with (3.11), which is enough to be checked on elements of the type $1\otimes a$ and $v\otimes a$, with $a\in A$:

$$\begin{aligned}((\alpha_A\otimes id)\circ R)(1\otimes a) &= (\alpha_A\otimes id)(a\otimes 1)=\alpha_A(a)\otimes 1\\ &= R(1\otimes\alpha_A(a))=(R\circ(id\otimes\alpha_A))(1\otimes a),\\ ((\alpha_A\otimes id)\circ R)(v\otimes a) &= (\alpha_A\otimes id)(\sigma(a)\otimes v)=\alpha_A(\sigma(a))\otimes v=\sigma(\alpha_A(a))\otimes v\\ &= R(v\otimes\alpha_A(a))=(R\circ(id\otimes\alpha_A))(v\otimes a).\end{aligned}$$

Similarly, one has to check (3.12) on elements of the type $1\otimes a\otimes a'$ and $v\otimes a\otimes a'$ and (3.13) on elements of the type $1\otimes 1\otimes a$, $1\otimes v\otimes a$, $v\otimes 1\otimes a$ and $v\otimes v\otimes a$, with

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$a, a' \in A$. Let us only check (3.13) on $v \otimes v \otimes a$:

$$(R \circ (\mu \otimes \alpha_A))(v \otimes v \otimes a) = R(v^2 \otimes \alpha_A(a)) = R(q1 \otimes \alpha_A(a)) = q\alpha_A(a) \otimes 1,$$
$$((\alpha_A \otimes \mu) \circ (R \otimes id) \circ (id \otimes R))(v \otimes v \otimes a) = ((\alpha_A \otimes \mu) \circ (R \otimes id))(v \otimes \sigma(a) \otimes v)$$
$$= (\alpha_A \otimes \mu)(\sigma^2(a) \otimes v \otimes v)$$
$$= \alpha_A(a) \otimes v^2 = \alpha_A(a) \otimes q1, \quad q.e.d.$$

The fact that $\overline{A}$ is exactly the Hom-twisted tensor product $A \otimes_R C(k, q)$ is obvious. □

REMARK 3.12. In particular, if (A, μ_A, α_A) is a Hom-associative algebra such that $\alpha_A^2 = id_A$, we can perform the Clifford process with $\sigma := \alpha_A$ to obtain the new Hom-associative algebra $\overline{A}$.

We prove that, under certain circumstances, Hom-twisted tensor products can be iterated, generalizing thus the corresponding result obtained for associative algebras in [**16**].

THEOREM 3.13. *Let (A, μ_A, α_A), (B, μ_B, α_B) and (C, μ_C, α_C) be three Hom-associative algebras and $R_1 : B \otimes A \to A \otimes B$, $R_2 : C \otimes B \to B \otimes C$, $R_3 : C \otimes A \to A \otimes C$ three Hom-twisting maps, satisfying the braid condition*

$$(id_A \otimes R_2) \circ (R_3 \otimes id_B) \circ (id_C \otimes R_1) = (R_1 \otimes id_C) \circ (id_B \otimes R_3) \circ (R_2 \otimes id_A). \quad (3.19)$$

Define the maps

$$P_1 : C \otimes (A \otimes_{R_1} B) \to (A \otimes_{R_1} B) \otimes C, \quad P_1 = (id_A \otimes R_2) \circ (R_3 \otimes id_B),$$
$$P_2 : (B \otimes_{R_2} C) \otimes A \to A \otimes (B \otimes_{R_2} C), \quad P_2 = (R_1 \otimes id_C) \circ (id_B \otimes R_3).$$

Then, P_1 is a Hom-twisting map between $A \otimes_{R_1} B$ and C, P_2 is a Hom-twisting map between A and $B \otimes_{R_2} C$, and the Hom-associative algebras $(A \otimes_{R_1} B) \otimes_{P_1} C$ and $A \otimes_{P_2} (B \otimes_{R_2} C)$ coincide; this Hom-associative algebra will be denoted by $A \otimes_{R_1} B \otimes_{R_2} C$ and will be called the iterated Hom-twisted tensor product *of A, B, C.*

Proof. We prove that P_1 is a Hom-twisting map (the proof for P_2 is similar and left to the reader). We will use the Sweedler-type notation introduced before. With this notation, P_1 is defined by $P_1(c \otimes a \otimes b) = a_{R_3} \otimes b_{R_2} \otimes (c_{R_3})_{R_2}$, and (3.19) may be written as

$$(a_{R_1})_{R_3} \otimes (b_{R_1})_{R_2} \otimes (c_{R_3})_{R_2} = (a_{R_3})_{R_1} \otimes (b_{R_2})_{R_1} \otimes (c_{R_2})_{R_3}, \quad (3.20)$$

for all $a \in A$, $b \in B$, $c \in C$. We prove (3.11) for P_1:

$$((\alpha_A \otimes \alpha_B \otimes \alpha_C) \circ P_1)(c \otimes a \otimes b) = \alpha_A(a_{R_3}) \otimes \alpha_B(b_{R_2}) \otimes \alpha_C((c_{R_3})_{R_2})$$
$$\overset{(3.14)}{=} \alpha_A(a_{R_3}) \otimes \alpha_B(b)_{R_2} \otimes \alpha_C(c_{R_3})_{R_2}$$
$$\overset{(3.14)}{=} \alpha_A(a)_{R_3} \otimes \alpha_B(b)_{R_2} \otimes (\alpha_C(c)_{R_3})_{R_2}$$
$$= (P_1 \circ (\alpha_C \otimes \alpha_A \otimes \alpha_B))(c \otimes a \otimes b), \quad q.e.d.$$

Now, we prove (3.12) for P_1:

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$$(P_1\circ(\alpha_C\otimes\mu_{A\otimes_{R_1}B}))(c\otimes a\otimes b\otimes a'\otimes b')$$
$$\begin{aligned}&= P_1(\alpha_C(c)\otimes aa'_{R_1}\otimes b_{R_1}b')\\ &= (aa'_{R_1})_{R_3}\otimes(b_{R_1}b')_{R_2}\otimes(\alpha_C(c)_{R_3})_{R_2}\\ &\overset{(3.15)}{=} a_{R_3}(a'_{R_1})_{r_3}\otimes(b_{R_1}b')_{R_2}\otimes\alpha_C((c_{R_3})_{r_3})_{R_2}\\ &\overset{(3.15)}{=} a_{R_3}(a'_{R_1})_{r_3}\otimes(b_{R_1})_{R_2}b'_{r_2}\otimes\alpha_C((((c_{R_3})_{r_3})_{R_2})_{r_2}),\end{aligned}$$

$$((\mu_{A\otimes_{R_1}B}\otimes\alpha_C)\circ(id_A\otimes id_B\otimes P_1)\circ(P_1\otimes id_A\otimes id_B))(c\otimes a\otimes b\otimes a'\otimes b')$$
$$\begin{aligned}&= ((\mu_{A\otimes_{R_1}B}\otimes\alpha_C)\circ(id_A\otimes id_B\otimes P_1))(a_{R_3}\otimes b_{R_2}\otimes(c_{R_3})_{R_2}\otimes a'\otimes b')\\ &= (\mu_{A\otimes_{R_1}B}\otimes\alpha_C)(a_{R_3}\otimes b_{R_2}\otimes a'_{r_3}\otimes b'_{r_2}\otimes(((c_{R_3})_{R_2})_{r_3})_{r_2})\\ &= a_{R_3}(a'_{r_3})_{R_1}\otimes(b_{R_2})_{R_1}b'_{r_2}\otimes\alpha_C((((c_{R_3})_{R_2})_{r_3})_{r_2})\\ &\overset{(3.20)}{=} a_{R_3}(a'_{R_1})_{r_3}\otimes(b_{R_1})_{R_2}b'_{r_2}\otimes\alpha_C((((c_{R_3})_{r_3})_{R_2})_{r_2}),\quad q.e.d.\end{aligned}$$

Finally, we prove (3.13) for P_1:

$$\begin{aligned}(P_1\circ(\mu_C\otimes\alpha_A\otimes\alpha_B))(c\otimes c'\otimes a\otimes b) &= P_1(cc'\otimes\alpha_A(a)\otimes\alpha_B(b))\\ &= \alpha_A(a)_{R_3}\otimes\alpha_B(b)_{R_2}\otimes((cc')_{R_3})_{R_2}\\ &\overset{(3.16)}{=} \alpha_A((a_{R_3})_{r_3})\otimes\alpha_B(b)_{R_2}\otimes(c_{r_3}c'_{R_3})_{R_2}\\ &\overset{(3.16)}{=} \alpha_A((a_{R_3})_{r_3})\otimes\alpha_B((b_{R_2})_{r_2})\otimes(c_{r_3})_{r_2}(c'_{R_3})_{R_2},\end{aligned}$$

$$((\alpha_A\otimes\alpha_B\otimes\mu_C)\circ(P_1\otimes id_C)\circ(id_C\otimes P_1))(c\otimes c'\otimes a\otimes b)$$
$$\begin{aligned}&= ((\alpha_A\otimes\alpha_B\otimes\mu_C)\circ(P_1\otimes id_C))(c\otimes a_{R_3}\otimes b_{R_2}\otimes(c'_{R_3})_{R_2})\\ &= (\alpha_A\otimes\alpha_B\otimes\mu_C)((a_{R_3})_{r_3}\otimes(b_{R_2})_{r_2}\otimes(c_{r_3})_{r_2}\otimes(c'_{R_3})_{R_2})\\ &= \alpha_A((a_{R_3})_{r_3})\otimes\alpha_B((b_{R_2})_{r_2})\otimes(c_{r_3})_{r_2}(c'_{R_3})_{R_2}),\quad q.e.d.\end{aligned}$$

The fact that $(A\otimes_{R_1}B)\otimes_{P_1}C$ and $A\otimes_{P_2}(B\otimes_{R_2}C)$ coincide is obvious, because the multiplications in these algebras are both defined by $(a\otimes b\otimes c)(a'\otimes b'\otimes c')=a(a'_{R_3})_{R_1}\otimes b_{R_1}b'_{R_2}\otimes(c_{R_3})_{R_2}c'$. □

4. Hom-smash products.

We introduce now a Hom-analogue of the smash product.

THEOREM 4.1. *Let $(H,\mu_H,\Delta_H,\alpha_H)$ be a Hom-bialgebra, (A,μ_A,α_A) a left H-module Hom-algebra, with action denoted by $H\otimes A\to A$, $h\otimes a\mapsto h\cdot a$, and assume that the structure maps α_H and α_A are both bijective. Define the linear map*

$$R: H\otimes A\to A\otimes H,\quad R(h\otimes a)=\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a)\otimes\alpha_H^{-1}(h_2). \tag{4.1}$$

Then, R is a Hom-twisting map between A and H. Consequently, we can consider the Hom-associative algebra $A\otimes_R H$, which is denoted by $A\#H$ (we denote $a\otimes h:=a\#h$, for $a\in A$, $h\in H$) and called the Hom-smash product *of A and H. The structure map of $A\#H$ is $\alpha_A\otimes\alpha_H$ and its multiplication is*

$$(a\#h)(a'\#h')=a(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))\#\alpha_H^{-1}(h_2)h'.$$

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Proof. We need to prove that R satisfies the conditions (3.11)–(3.13).

Proof of (3.11):

$$\begin{aligned}((\alpha_A\otimes\alpha_H)\circ R)(h\otimes a) &= \alpha_A(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a))\otimes\alpha_H(\alpha_H^{-1}(h_2))\\ &\overset{(2.11)}{=} \alpha_H^{-1}(h_1)\cdot a\otimes h_2,\end{aligned}$$

$$\begin{aligned}(R\circ(\alpha_H\otimes\alpha_A))(h\otimes a) &= R(\alpha_H(h)\otimes\alpha_A(a))\\ &= \alpha_H^{-2}(\alpha_H(h)_1)\cdot\alpha_A^{-1}(\alpha_A(a))\otimes\alpha_H^{-1}(\alpha_H(h)_2)\\ &\overset{(2.17)}{=} \alpha_H^{-2}(\alpha_H(h_1))\cdot a\otimes\alpha_H^{-1}(\alpha_H(h_2))\\ &= \alpha_H^{-1}(h_1)\cdot a\otimes h_2,\quad q.e.d.\end{aligned}$$

Proof of (3.12):

$$\begin{aligned}(R\circ(\alpha_H\otimes\mu_A))(h\otimes a\otimes a') &= R(\alpha_H(h)\otimes aa')\\ &= \alpha_H^{-2}(\alpha_H(h)_1)\cdot\alpha_A^{-1}(aa')\otimes\alpha_H^{-1}(\alpha_H(h)_2)\\ &\overset{(2.17)}{=} \alpha_H^{-1}(h_1)\cdot\alpha_A^{-1}(aa')\otimes h_2,\end{aligned}$$

$$((\mu_A\otimes\alpha_H)\circ(id_A\otimes R)\circ(R\otimes id_A))(h\otimes a\otimes a')$$

$$\begin{aligned}&= ((\mu_A\otimes\alpha_H)\circ(id_A\otimes R))(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a)\otimes\alpha_H^{-1}(h_2)\otimes a')\\ &= (\mu_A\otimes\alpha_H)(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a)\otimes\alpha_H^{-2}(\alpha_H^{-1}(h_2)_1)\cdot\alpha_A^{-1}(a')\otimes\alpha_H^{-1}(\alpha_H^{-1}(h_2)_2))\\ &\overset{(2.17)}{=} [\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a)][\alpha_H^{-3}((h_2)_1)\cdot\alpha_A^{-1}(a')]\otimes\alpha_H^{-1}((h_2)_2)\\ &\overset{(2.15)}{=} [\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a)][\alpha_H^{-3}((h_1)_2)\cdot\alpha_A^{-1}(a')]\otimes h_2\\ &\overset{(2.17)}{=} [\alpha_H^{-3}(h_1)_1\cdot\alpha_A^{-1}(a)][\alpha_H^{-3}(h_1)_2\cdot\alpha_A^{-1}(a')]\otimes h_2\\ &\overset{(2.18)}{=} \alpha_H^{-1}(h_1)\cdot(\alpha_A^{-1}(a)\alpha_A^{-1}(a'))\otimes h_2\\ &= \alpha_H^{-1}(h_1)\cdot\alpha_A^{-1}(aa')\otimes h_2,\quad q.e.d.\end{aligned}$$

Proof of (3.13):

$$\begin{aligned}(R\circ(\mu_H\otimes\alpha_A))(h\otimes h'\otimes a) &= R(hh'\otimes\alpha_A(a))\\ &= \alpha_H^{-2}((hh')_1)\cdot\alpha_A^{-1}(\alpha_A(a))\otimes\alpha_H^{-1}((hh')_2)\\ &\overset{(2.16)}{=} \alpha_H^{-2}(h_1h'_1)\cdot a\otimes\alpha_H^{-1}(h_2h'_2),\end{aligned}$$

$$((\alpha_A\otimes\mu_H)\circ(R\otimes id_H)\circ(id_H\otimes R))(h\otimes h'\otimes a)$$

$$\begin{aligned}&= ((\alpha_A\otimes\mu_H)\circ(R\otimes id_H))(h\otimes\alpha_H^{-2}(h'_1)\cdot\alpha_A^{-1}(a)\otimes\alpha_H^{-1}(h'_2))\\ &= (\alpha_A\otimes\mu_H)(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(\alpha_H^{-2}(h'_1)\cdot\alpha_A^{-1}(a))\otimes\alpha_H^{-1}(h_2)\otimes\alpha_H^{-1}(h'_2))\\ &\overset{(2.11)}{=} \alpha_A(\alpha_H^{-2}(h_1)\cdot[\alpha_H^{-3}(h'_1)\cdot\alpha_A^{-2}(a)])\otimes\alpha_H^{-1}(h_2h'_2)\\ &\overset{(2.12)}{=} \alpha_A([\alpha_H^{-3}(h_1)\alpha_H^{-3}(h'_1)]\cdot\alpha_A^{-1}(a))\otimes\alpha_H^{-1}(h_2h'_2)\\ &\overset{(2.11)}{=} \alpha_H^{-2}(h_1h'_1)\cdot a\otimes\alpha_H^{-1}(h_2h'_2),\end{aligned}$$

finishing the proof. □

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EXAMPLE 4.2. We consider the class of examples of $U_q(\mathfrak{sl}_2)_\alpha$-module Hom-algebra structures on $\mathbb{A}^{2|0}_{q,\beta}$ given in [35, Example 5.7] (here, we take the base field $k = \mathbb{C}$). The quantum group $U_q(\mathfrak{sl}_2)$ is generated as a unital associative algebra by four generators $\{E, F, K, K^{-1}\}$ with relations

$$KK^{-1} = 1 = K^{-1}K,$$
$$KE = q^2EK,\ KF = q^{-2}FK,$$
$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$$

where $q \in \mathbb{C}$ with $q \neq 0$, $q \neq \pm 1$. The comultiplication is defined by

$$\Delta(E) = 1 \otimes E + E \otimes K,$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K,\ \Delta(K^{-1}) = K^{-1} \otimes K^{-1}.$$

We fix $\lambda \in \mathbb{C}$, $\lambda \neq 0$. The Hom-bialgebra $U_q(\mathfrak{sl}_2)_\alpha = (U_q(\mathfrak{sl}_2), \mu_\alpha, \Delta_\alpha, \alpha)$ is defined by $\mu_\alpha = \alpha \circ \mu$ and $\Delta_\alpha = \Delta \circ \alpha$, where μ and Δ are respectively the multiplication and comultiplication of $U_q(\mathfrak{sl}_2)$ and $\alpha : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ is a bialgebra morphism such that

$$\alpha(E) = \lambda E,\ \alpha(F) = \lambda^{-1}F,\ \alpha(K) = K,\ \alpha(K^{-1}) = K^{-1}.$$

Let $\mathbb{A}^{2|0}_q = k\langle x, y\rangle/(yx - qxy)$ be the quantum plane. We fix also some $\xi \in \mathbb{C}$, $\xi \neq 0$. The Hom-quantum plane $\mathbb{A}^{2|0}_{q,\beta} = (\mathbb{A}^{2|0}_q, \mu_\beta, \beta)$ is defined by $\mu_\beta = \beta \circ \mu_{\mathbb{A}}$, where $\mu_{\mathbb{A}}$ is the multiplication in $\mathbb{A}^{2|0}_q$ and $\beta : \mathbb{A}^{2|0}_q \to \mathbb{A}^{2|0}_q$ is an algebra morphism such that $\beta(x) = \xi x,\ \beta(y) = \xi\lambda^{-1}y$. Then, for any integer $l \geq 0$ there is a $U_q(\mathfrak{sl}_2)_\alpha$-module Hom-algebra structure on $\mathbb{A}^{2|0}_{q,\beta}$ defined by

$$\rho_l(E, x^my^n) = [n]_q\xi^{m+n}\lambda^{l-n+1}x^{m+1}y^{n-1}$$
$$\rho_l(F, x^my^n) = [m]_q\xi^{m+n}\lambda^{-l-n-1}x^{m-1}y^{n+1}$$
$$\rho_l(K^{\pm 1}, P) = P(q^{\pm 1}\xi x, q^{\mp 1}\xi\lambda^{-1}y),$$

where $P = P(x, y) \in \mathbb{A}^{2|0}_q$ and $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$.

Notice that both α and β are bijective for $\lambda \neq 0$ and $\xi \neq 0$. According to Theorem 4.1, the map $R : U_q(\mathfrak{sl}_2)_\alpha \otimes \mathbb{A}^{2|0}_{q,\beta} \to \mathbb{A}^{2|0}_{q,\beta} \otimes U_q(\mathfrak{sl}_2)_\alpha$ defined in (4.1) leads to a smash product $\mathbb{A}^{2|0}_{q,\beta}\#U_q(\mathfrak{sl}_2)_\alpha$ whose multiplication is defined by

$$(a\#h)(a'\#h') = a(\alpha^{-2}(h_1) \cdot \beta^{-1}(a'))\#\alpha^{-1}(h_2)h'.$$

In particular, if we choose $l = 0$, then for any $G \in U_q(\mathfrak{sl}_2)$ and $m, n, r, s \in \mathbb{N}$ we have

$$(x^my^n\#K^{\pm 1})(x^ry^s\#G) = q^{\pm r \mp s + nr}\xi^{m+n+r+s}\lambda^{-n-s}x^{m+r}y^{n+s}\#K^{\pm 1}\alpha(G),$$
$$(x^my^n\#E)(x^ry^s\#G) = q^{nr}\xi^{m+n+r+s}\lambda^{-n-s+1}x^{m+r}y^{n+s}\#E\alpha(G)$$
$$+[s]_q q^{n(r+1)}\xi^{m+n+r+s}\lambda^{-n-s+1}x^{m+r+1}y^{n+s-1}\#K\alpha(G),$$
$$(x^my^n\#F)(x^ry^s\#G) = q^{s-r+nr}\xi^{m+n+r+s}\lambda^{-n-s-1}x^{m+r}y^{n+s}\#F\alpha(G)$$
$$+[r]_q q^{n(r-1)}\xi^{m+n+r+s}\lambda^{-n-s-1}x^{m+r-1}y^{n+s+1}\#\alpha(G),$$

where $K\alpha(G)$, $E\alpha(G)$ and $F\alpha(G)$ are multiplications in $U_q(\mathfrak{sl}_2)$.

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PROPOSITION 4.3. *In the hypotheses of and with notation as in Proposition 2.13, and assuming moreover that the maps α_H and α_A are both bijective, if we denote by $A\#H$ the usual smash product between A and H, then $\alpha_A \otimes \alpha_H$ is an algebra endomorphism of $A\#H$ and the Hom-associative algebras $(A\#H)_{\alpha_A\otimes\alpha_H}$ and $A_{\alpha_A}\#H_{\alpha_H}$ coincide.*

Proof. We recall that the multiplication of $A\#H$ is defined by $(a\#h)(a'\#h') = a(h_1 \cdot a')\#h_2h'$, and the smash product $A\#H$ is the twisted tensor product $A \otimes_P H$, where P is the twisting map $P : H \otimes A \to A \otimes H$, $P(h \otimes a) = h_1 \cdot a \otimes h_2$. By using the condition $\alpha_A(h \cdot a) = \alpha_H(h) \cdot \alpha_A(a)$, one can prove immediately that we have $(\alpha_A \otimes \alpha_H) \circ P = P \circ (\alpha_H \otimes \alpha_A)$. We are thus in the hypotheses of Proposition 3.10, so the Hom-associative algebras $(A\#H)_{\alpha_A\otimes\alpha_H}$ and $A_{\alpha_A} \otimes_P H_{\alpha_H}$ coincide. So, the proof will be finished if we show that the Hom-twisting map R affording the Hom-smash product $A_{\alpha_A}\#H_{\alpha_H}$ and the map P actually coincide. We compute this map R (using the structures of A_{α_A} and H_{α_H}):

$$\begin{aligned} R(h \otimes a) &= \alpha_H^{-2}(\alpha_H(h)_1) \triangleright \alpha_A^{-1}(a) \otimes \alpha_H^{-1}(\alpha_H(h)_2) \\ &= \alpha_H^{-1}(h_1) \triangleright \alpha_A^{-1}(a) \otimes h_2 \\ &= h_1 \cdot a \otimes h_2 = P(h \otimes a), \end{aligned}$$

so indeed we have $R = P$. □

We will need in what follows the right-handed and two-sided analogues of left comodules and comodule algebras over Hom-coassociative coalgebras and Hom-bialgebras.

DEFINITION 4.4. Let (C, Δ_C, α_C) be a Hom-coassociative coalgebra, M a linear space and $\alpha_M : M \to M$ a linear map.

(i) A *right C-comodule* structure on (M, α_M) consists of a linear map $\rho : M \to M \otimes C$ satisfying the following conditions:

$$(\alpha_M \otimes \alpha_C) \circ \rho = \rho \circ \alpha_M, \tag{4.2}$$

$$(\alpha_M \otimes \Delta_C) \circ \rho = (\rho \otimes \alpha_C) \circ \rho. \tag{4.3}$$

(ii) If (M, α_M) is both a left C-comodule with structure $\lambda : M \to C \otimes M$ and a right C-comodule with structure $\rho : M \to M \otimes C$, then M is called a *C-bicomodule* if $(\lambda \otimes \alpha_C) \circ \rho = (\alpha_C \otimes \rho) \circ \lambda$.

Obviously, (C, α_C) itself is a C-bicomodule, with $\rho = \lambda = \Delta_C$.

DEFINITION 4.5. Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra.

(i) A *right H-comodule Hom-algebra* is a Hom-associative algebra (D, μ_D, α_D) endowed with a right H-comodule structure $\rho_D : D \to D \otimes H$ such that ρ_D is a morphism of Hom-associative algebras.

(ii) An *H-bicomodule Hom-algebra* is a Hom-associative algebra (D, μ_D, α_D) that is both a left and a right H-comodule Hom-algebra and such that the left and right H-comodule structures form an H-bicomodule.

PROPOSITION 4.6. *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra and (A, μ_A, α_A) a left H-module Hom-algebra, with action denoted by $H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$, such that the structure maps α_H and α_A are both bijective. Then, the Hom-smash product $A\#H$ is a right H-comodule Hom-algebra, via the linear map $\rho_{A\#H} : A\#H \to (A\#H) \otimes H$, $\rho_{A\#H}(a\#h) = (\alpha_A(a)\#h_1) \otimes h_2$.*

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Proof. First, we prove (4.3) for the map $\rho_{A\#H}$:

$$\begin{aligned}((\alpha_A \otimes \alpha_H \otimes \Delta_H)\circ \rho_{A\#H})(a\#h) &= \alpha_A^2(a)\#\alpha_H(h_1)\otimes (h_2)_1\otimes (h_2)_2\\ &\overset{(2.15)}{=} \alpha_A^2(a)\#(h_1)_1\otimes (h_1)_2\otimes \alpha_H(h_2)\\ &= ((\rho_{A\#H}\otimes \alpha_H)\circ \rho_{A\#H})(a\#h), \quad q.e.d.\end{aligned}$$

The relation $(\alpha_A \otimes \alpha_H \otimes \alpha_H)\circ \rho_{A\#H} = \rho_{A\#H}\circ(\alpha_A\otimes\alpha_H)$ follows immediately from (2.17), so the only thing left to prove is the multiplicativity of $\rho_{A\#H}$; we compute:

$$\begin{aligned}\rho_{A\#H}((a\#h)(a'\#h')) &= \rho_{A\#H}(a(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))\#\alpha_H^{-1}(h_2)h')\\ &= \alpha_A(a(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a')))\#(\alpha_H^{-1}(h_2)h')_1\otimes(\alpha_H^{-1}(h_2)h')_2\\ &\overset{(2.11),(2.16)}{=} \alpha_A(a)(\alpha_H^{-1}(h_1)\cdot a')\#\alpha_H^{-1}(h_2)_1h'_1\otimes\alpha_H^{-1}(h_2)_2h'_2\\ &\overset{(2.17)}{=} \alpha_A(a)(\alpha_H^{-1}(h_1)\cdot a')\#\alpha_H^{-1}((h_2)_1)h'_1\otimes\alpha_H^{-1}((h_2)_2)h'_2\\ &\overset{(2.15)}{=} \alpha_A(a)(\alpha_H^{-2}((h_1)_1)\cdot a')\#\alpha_H^{-1}((h_1)_2)h'_1\otimes h_2h'_2,\end{aligned}$$

$$\begin{aligned}\rho_{A\#H}(a\#h)\rho_{A\#H}(a'\#h') &= ((\alpha_A(a)\#h_1)\otimes h_2)((\alpha_A(a')\#h'_1)\otimes h'_2)\\ &= (\alpha_A(a)\#h_1)(\alpha_A(a')\#h'_1)\otimes h_2h'_2\\ &= \alpha_A(a)(\alpha_H^{-2}((h_1)_1)\cdot a')\#\alpha_H^{-1}((h_1)_2)h'_1\otimes h_2h'_2,\end{aligned}$$

and obviously the two terms are equal. □

DEFINITION 4.7 ([**26**]). Let $(H,\mu_H,\Delta_H,\alpha_H)$ be a Hom-bialgebra, M a linear space and $\alpha_M : M\to M$ a linear map such that (M,α_M) is a left H-module with action $H\otimes M\to M$, $h\otimes m\mapsto h\cdot m$ and a left H-comodule with coaction $M\to H\otimes M$, $m\mapsto m_{(-1)}\otimes m_{(0)}$. Then, (M,α_M) is called a (left-left) *Yetter-Drinfeld module* over H if the following relation holds, for all $h\in H$, $m\in M$:

$$(h_1\cdot m)_{(-1)}\alpha_H^2(h_2)\otimes(h_1\cdot m)_{(0)} = \alpha_H^2(h_1)\alpha_H(m_{(-1)})\otimes\alpha_H(h_2)\cdot m_{(0)}. \tag{4.4}$$

LEMMA 4.8. *Let (A,μ,α) be a Hom-associative algebra such that α is bijective and let $a,b,c,d\in A$. Then, the following relation holds:*

$$(ab)(cd) = \alpha(a)(\alpha^{-1}(bc)d). \tag{4.5}$$

Proof. A straightforward computation, using the definition of a Hom-associative algebra. □

PROPOSITION 4.9. *In the hypotheses of and with notation as in Proposition 4.6, assume that moreover (A,α_A) is a left H-comodule with structure $A\to H\otimes A$, $a\mapsto a_{(-1)}\otimes a_{(0)}$, such that (A,μ_A,α_A) is a left H-comodule Hom-algebra and (A,α_A) is a (left–left) Yetter–Drinfeld module over H. Then, $A\#H$ is an H-bicomodule Hom-algebra, via the map $\rho_{A\#H}$ defined in Proposition 4.6 and the linear map $\lambda_{A\#H} : A\#H\to H\otimes(A\#H)$, $\lambda_{A\#H}(a\#h) = a_{(-1)}h_1\otimes(a_{(0)}\#h_2)$.*

Proof. The fact that $\lambda_{A\#H}$ is a left H-comodule structure follows from [**35**], Proposition 5.3 ($\lambda_{A\#H}$ is just the tensor product of the left H-comodules A and H). We check the bicomodule condition:

$$\begin{aligned}
((\lambda_{A\#H}\otimes\alpha_H)\circ\rho_{A\#H})(a\#h) &= (\lambda_{A\#H}\otimes\alpha_H)((\alpha_A(a)\#h_1)\otimes h_2)\\
&= \alpha_A(a)_{(-1)}(h_1)_1\otimes(\alpha_A(a)_{(0)}\#(h_1)_2)\otimes\alpha_H(h_2)\\
&\overset{(2.15),\,(2.13)}{=} \alpha_H(a_{(-1)})\alpha_H(h_1)\otimes(\alpha_A(a_{(0)})\#(h_2)_1)\otimes(h_2)_2\\
&= (\alpha_H\otimes\rho_{A\#H})(a_{(-1)}h_1\otimes(a_{(0)}\#h_2))\\
&= ((\alpha_H\otimes\rho_{A\#H})\circ\lambda_{A\#H})(a\#h), \quad q.e.d.
\end{aligned}$$

The only thing left to prove is that $\lambda_{A\#H}$ is multiplicative. We compute:

$$\begin{aligned}
&\lambda_{A\#H}((a\#h)(a'\#h'))\\
&= \lambda(a(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))\#\alpha_H^{-1}(h_2)h')\\
&= [a_{(-1)}(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))_{(-1)}][\alpha_H^{-1}(h_2)_1h'_1]\otimes a_{(0)}(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))_{(0)}\#\alpha_H^{-1}(h_2)_2h'_2\\
&= [a_{(-1)}(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))_{(-1)}][\alpha_H^{-1}((h_2)_1)h'_1]\otimes a_{(0)}(\alpha_H^{-2}(h_1)\cdot\alpha_A^{-1}(a'))_{(0)}\\
&\quad \#\alpha_H^{-1}((h_2)_2)h'_2\\
&\overset{(2.15)}{=} [a_{(-1)}(\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a'))_{(-1)}][\alpha_H^{-1}((h_1)_2)h'_1]\otimes a_{(0)}(\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a'))_{(0)}\#h_2h'_2\\
&\overset{(4.5)}{=} \alpha_H(a_{(-1)})\{\alpha_H^{-1}[(\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a'))_{(-1)}\alpha_H^{-1}((h_1)_2)]h'_1\}\\
&\quad \otimes a_{(0)}(\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a'))_{(0)}\#h_2h'_2\\
&= \alpha_H(a_{(-1)})\{\alpha_H^{-1}[(\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a'))_{(-1)}\alpha_H^{2}(\alpha_H^{-3}((h_1)_2))]h'_1\}\\
&\quad \otimes a_{(0)}(\alpha_H^{-3}((h_1)_1)\cdot\alpha_A^{-1}(a'))_{(0)}\#h_2h'_2\\
&= \alpha_H(a_{(-1)})\{\alpha_H^{-1}[(\alpha_H^{-3}(h_1)_1\cdot\alpha_A^{-1}(a'))_{(-1)}\alpha_H^{2}(\alpha_H^{-3}(h_1)_2)]h'_1\}\\
&\quad \otimes a_{(0)}(\alpha_H^{-3}(h_1)_1\cdot\alpha_A^{-1}(a'))_{(0)}\#h_2h'_2\\
&\overset{(4.4)}{=} \alpha_H(a_{(-1)})\{\alpha_H^{-1}[\alpha_H^{2}(\alpha_H^{-3}(h_1)_1)\alpha_H(\alpha_A^{-1}(a')_{(-1)})]h'_1\}\\
&\quad \otimes a_{(0)}(\alpha_H(\alpha_H^{-3}(h_1)_2)\cdot\alpha_A^{-1}(a')_{(0)})\#h_2h'_2\\
&\overset{(2.13)}{=} \alpha_H(a_{(-1)})[\alpha_H^{-1}(\alpha_H^{-1}(h_1)_1a'_{(-1)})h'_1]\otimes a_{(0)}(\alpha_H^{-1}(\alpha_H^{-1}(h_1)_2)\cdot\alpha_A^{-1}(a'_{(0)}))\#h_2h'_2\\
&\overset{(4.5)}{=} [a_{(-1)}\alpha_H^{-1}((h_1)_1)][a'_{(-1)}h'_1]\otimes a_{(0)}(\alpha_H^{-2}((h_1)_2)\cdot\alpha_A^{-1}(a'_{(0)}))\#h_2h'_2\\
&\overset{(2.15)}{=} [a_{(-1)}h_1][a'_{(-1)}h'_1]\otimes a_{(0)}(\alpha_H^{-2}((h_2)_1)\cdot\alpha_A^{-1}(a'_{(0)}))\#\alpha_H^{-1}((h_2)_2)h'_2\\
&= [a_{(-1)}h_1][a'_{(-1)}h'_1]\otimes(a_{(0)}\#h_2)(a'_{(0)}\#h'_2)\\
&= \lambda_{A\#H}(a\#h)\lambda_{A\#H}(a'\#h'),
\end{aligned}$$

finishing the proof. □

In order to define two-sided Hom-smash products, we need first the right-handed versions of some concepts and results presented so far; the proofs of these analogues are left to the reader.

DEFINITION 4.10. Let (A,μ_A,α_A) be a Hom-associative algebra, M a linear space and $\alpha_M : M\to M$ a linear map. A *right A-module* structure on (M,α_M) consists of a

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linear map $M \otimes A \to M$, $m \otimes a \mapsto m \cdot a$, satisfying the conditions:

$$\alpha_M(m \cdot a) = \alpha_M(m) \cdot \alpha_A(a), \tag{4.6}$$

$$(m \cdot a) \cdot \alpha_A(a') = \alpha_M(m) \cdot (aa'), \tag{4.7}$$

for all $a, a' \in A$ and $m \in M$. If (M, α_M) and (N, α_N) are right A-modules (both A-actions denoted by $\cdot$), a morphism of right A-modules $f : M \to N$ is a linear map satisfying the conditions $\alpha_N \circ f = f \circ \alpha_M$ and $f(m \cdot a) = f(m) \cdot a$, for all $a \in A$ and $m \in M$.

DEFINITION 4.11. Assume that $(H, \mu_H, \Delta_H, \alpha_H)$ is a Hom-bialgebra. A Hom-associative algebra (C, μ_C, α_C) is called a *right H-module Hom-algebra* if (C, α_C) is a right H-module, with action denoted by $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$, such that the following condition is satisfied:

$$(cc') \cdot \alpha_H^2(h) = (c \cdot h_1)(c' \cdot h_2), \quad \forall\, h \in H,\ c, c' \in C. \tag{4.8}$$

PROPOSITION 4.12. *Let (H, μ_H, Δ_H) be a bialgebra and (C, μ_C) a right H-module algebra in the usual sense, with action denoted by $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$. Let $\alpha_H : H \to H$ be a bialgebra endomorphism and $\alpha_C : C \to C$ an algebra endomorphism, such that $\alpha_C(c \cdot h) = \alpha_C(c) \cdot \alpha_H(h)$, for all $h \in H$ and $c \in C$. Then, the Hom-associative algebra $C_{\alpha_C} = (C, \alpha_C \circ \mu_C, \alpha_C)$ becomes a right module Hom-algebra over the Hom-bialgebra $H_{\alpha_H} = (H, \alpha_H \circ \mu_H, \Delta_H \circ \alpha_H, \alpha_H)$, with action defined by $C_{\alpha_C} \otimes H_{\alpha_H} \to C_{\alpha_C}$, $c \otimes h \mapsto c \triangleleft h := \alpha_C(c \cdot h) = \alpha_C(c) \cdot \alpha_H(h)$.*

THEOREM 4.13. *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra, (C, μ_C, α_C) a right H-module Hom-algebra, with action denoted by $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$, and assume that the structure maps α_H and α_C are both bijective. Define the linear map*

$$R : C \otimes H \to H \otimes C, \quad R(c \otimes h) = \alpha_H^{-1}(h_1) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2). \tag{4.9}$$

Then, R is a Hom-twisting map between H and C. Consequently, we can consider the Hom-associative algebra $H \otimes_R C$, which is denoted by $H\#C$ (we denote $h \otimes c := h\#c$, for $c \in C$, $h \in H$) and called the Hom-smash product *of H and C. The structure map of $H\#C$ is $\alpha_H \otimes \alpha_C$ and its multiplication is*

$$(h\#c)(h'\#c') = h\alpha_H^{-1}(h'_1)\#(\alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h'_2))c'.$$

PROPOSITION 4.14. *In the hypotheses of and with notation as in Proposition 4.12, and assuming moreover that the maps α_H and α_C are both bijective, if we denote by $H\#C$ the usual smash product between H and C (whose multiplication is $(h\#c)(h'\#c') = hh'_1\#(c \cdot h'_2)c'$), then $\alpha_H \otimes \alpha_C$ is an algebra endomorphism of $H\#C$ and the Hom-associative algebras $(H\#C)_{\alpha_H \otimes \alpha_C}$ and $H_{\alpha_H}\#C_{\alpha_C}$ coincide.*

PROPOSITION 4.15. *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra and (C, μ_C, α_C) a right H-module Hom-algebra, with action denoted by $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$, such that the structure maps α_H and α_C are both bijective. Then, the Hom-smash product $H\#C$ is a left H-comodule Hom-algebra, via the map $\lambda_{H\#C} : H\#C \to H \otimes (H\#C)$, $\lambda_{H\#C}(h\#c) = h_1 \otimes (h_2\#\alpha_C(c))$.*

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We are now in the position to define the two-sided Hom-smash product, as a particular case of an iterated Hom-twisted tensor product.

PROPOSITION 4.16. *Let $(H, \mu_H, \Delta_H, \alpha_H)$ be a Hom-bialgebra, (A, μ_A, α_A) a left H-module Hom-algebra and (C, μ_C, α_C) a right H-module Hom-algebra, with actions denoted by $H \otimes A \to A$, $h \otimes a \mapsto h \cdot a$ and $C \otimes H \to C$, $c \otimes h \mapsto c \cdot h$, and assume that the structure maps α_H, α_A, α_C are bijective. Consider the Hom-twisting maps defined by (4.1) and (4.9), namely*

$$R_1 : H \otimes A \to A \otimes H, \quad R_1(h \otimes a) = \alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-1}(h_2),$$
$$R_2 : C \otimes H \to H \otimes C, \quad R_2(c \otimes h) = \alpha_H^{-1}(h_1) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2),$$

as well as the trivial Hom-twisting map $R_3 : C \otimes A \to A \otimes C$, $R_3(c \otimes a) = a \otimes c$. Then, R_1, R_2, R_3 satisfy the braid relation, so, by Theorem 3.13, we can consider the iterated Hom-twisted tensor product $A \otimes_{R_1} H \otimes_{R_2} C$, which will be denoted by $A\#H\#C$ and will be called the two-sided Hom-smash product. *Its multiplication is defined by*

$$(a\#h\#c)(a'\#h'\#c') = a(\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a'))\#\alpha_H^{-1}(h_2h'_1)\#(\alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h'_2))c',$$

and its structure map is $\alpha_A \otimes \alpha_H \otimes \alpha_C$.

Proof. We only need to prove the braid relation. We compute:

$$\begin{aligned}
&((id_A \otimes R_2) \circ (R_3 \otimes id_H) \circ (id_C \otimes R_1))(c \otimes h \otimes a)\\
&\quad= ((id_A \otimes R_2) \circ (R_3 \otimes id_H))(c \otimes \alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-1}(h_2))\\
&\quad= (id_A \otimes R_2)(\alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a) \otimes c \otimes \alpha_H^{-1}(h_2))\\
&\quad= \alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-1}(\alpha_H^{-1}(h_2)_1) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(\alpha_H^{-1}(h_2)_2)\\
&\quad\overset{(2.17)}{=} \alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-2}((h_2)_1) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-3}((h_2)_2),
\end{aligned}$$

$$\begin{aligned}
&((R_1 \otimes id_C) \circ (id_H \otimes R_3) \circ (R_2 \otimes id_A))(c \otimes h \otimes a)\\
&\quad= ((R_1 \otimes id_C) \circ (id_H \otimes R_3))(\alpha_H^{-1}(h_1) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2) \otimes a)\\
&\quad= (R_1 \otimes id_C)(\alpha_H^{-1}(h_1) \otimes a \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2))\\
&\quad= \alpha_H^{-2}(\alpha_H^{-1}(h_1)_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-1}(\alpha_H^{-1}(h_1)_2) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2)\\
&\quad\overset{(2.17)}{=} \alpha_H^{-3}((h_1)_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-2}((h_1)_2) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-2}(h_2)\\
&\quad\overset{(2.15)}{=} \alpha_H^{-2}(h_1) \cdot \alpha_A^{-1}(a) \otimes \alpha_H^{-2}((h_2)_1) \otimes \alpha_C^{-1}(c) \cdot \alpha_H^{-3}((h_2)_2),
\end{aligned}$$

finishing the proof. □

5. Hom-associative algebras obtained from associative algebras. Our aim now is to show that two procedures recalled in the Preliminaries, the one of twisting an associative algebra by a pseudotwistor to obtain another associative algebra and Yau's procedure of twisting an associative algebra by an endomorphism to obtain a Hom-associative algebra admit a common generalization.

THEOREM 5.1. *Let (A, μ) be an associative algebra, $\alpha : A \to A$ an algebra endomorphism, $T : A \otimes A \to A \otimes A$ and $\tilde{T}_1, \tilde{T}_2 : A \otimes A \otimes A \to A \otimes A \otimes A$ linear*

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maps, satisfying the following conditions:

$$(\alpha \otimes \alpha) \circ T = T \circ (\alpha \otimes \alpha), \tag{5.1}$$
$$T \circ (id_A \otimes \mu) = (id_A \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_A), \tag{5.2}$$
$$T \circ (\mu \otimes id_A) = (\mu \otimes id_A) \circ \tilde{T}_2 \circ (id_A \otimes T), \tag{5.3}$$
$$\tilde{T}_1 \circ (T \otimes id_A) \circ (\alpha \otimes T) = \tilde{T}_2 \circ (id_A \otimes T) \circ (T \otimes \alpha). \tag{5.4}$$

Then, $(A, \mu \circ T, \alpha)$ is a Hom-associative algebra, which is denoted by A_α^T. The map T is called an α-pseudotwistor *and the two maps $\tilde{T}_1$, $\tilde{T}_2$ are called the* companions *of T.*

Proof. We record first the obvious relations

$$(\mu \circ T) \otimes \alpha = (\mu \otimes id_A) \circ (T \otimes \alpha), \tag{5.5}$$
$$\alpha \otimes (\mu \circ T) = (id_A \otimes \mu) \circ (\alpha \otimes T). \tag{5.6}$$

The fact that α is multiplicative with respect to $\mu \circ T$ follows immediately from (5.1) and the fact that α is multiplicative with respect to μ, so we only have to prove the Hom-associativity of $\mu \circ T$. We compute:

$$\begin{aligned}
(\mu \circ T) \circ ((\mu \circ T) \otimes \alpha) &\overset{(5.5)}{=} \mu \circ T \circ (\mu \otimes id_A) \circ (T \otimes \alpha) \\
&\overset{(5.3)}{=} \mu \circ (\mu \otimes id_A) \circ \tilde{T}_2 \circ (id_A \otimes T) \circ (T \otimes \alpha) \\
&\overset{(5.4)}{=} \mu \circ (\mu \otimes id_A) \circ \tilde{T}_1 \circ (T \otimes id_A) \circ (\alpha \otimes T) \\
&\overset{\textit{associativity of } \mu}{=} \mu \circ (id_A \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_A) \circ (\alpha \otimes T) \\
&\overset{(5.2)}{=} \mu \circ T \circ (id_A \otimes \mu) \circ (\alpha \otimes T) \\
&\overset{(5.6)}{=} (\mu \circ T) \circ (\alpha \otimes (\mu \circ T)),
\end{aligned}$$

finishing the proof. □

Obviously, if (A, μ) is an associative algebra and we take $\alpha = id_A$, an α-pseudotwistor is the same thing as a pseudotwistor and the Hom-associative algebra A_α^T is actually associative.

We show now that Yau's procedure is a particular case of Theorem 5.1.

PROPOSITION 5.2. *Let (A, μ) be an associative algebra and $\alpha : A \to A$ an algebra endomorphism. Define the maps*

$$T : A \otimes A \to A \otimes A, \quad T = \alpha \otimes \alpha,$$
$$\tilde{T}_1 : A \otimes A \otimes A \to A \otimes A \otimes A, \quad \tilde{T}_1 = id_A \otimes id_A \otimes \alpha,$$
$$\tilde{T}_2 : A \otimes A \otimes A \to A \otimes A \otimes A, \quad \tilde{T}_2 = \alpha \otimes id_A \otimes id_A.$$

Then, T is an α-pseudotwistor with companions $\tilde{T}_1$, $\tilde{T}_2$ and the Hom-associative algebras A_α^T and A_α coincide.

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Proof. The condition (5.1) is obviously satisfied. We check (5.2), for $a, b, c \in A$:

$$\begin{aligned}((id_A \otimes \mu) \circ \tilde{T}_1 \circ (T \otimes id_A))(a \otimes b \otimes c) &= ((id_A \otimes \mu) \circ \tilde{T}_1)(\alpha(a) \otimes \alpha(b) \otimes c)\\ &= (id_A \otimes \mu)(\alpha(a) \otimes \alpha(b) \otimes \alpha(c))\\ &= \alpha(a) \otimes \alpha(b)\alpha(c)\\ &= \alpha(a) \otimes \alpha(bc)\\ &= T(a \otimes bc)\\ &= (T \circ (id_A \otimes \mu))(a \otimes b \otimes c), \quad q.e.d.\end{aligned}$$

The condition (5.3) is similar, so we check (5.4):

$$\begin{aligned}(\tilde{T}_1 \circ (T \otimes id_A) \circ (\alpha \otimes T))(a \otimes b \otimes c) &= (\tilde{T}_1 \circ (T \otimes id_A))(\alpha(a) \otimes \alpha(b) \otimes \alpha(c))\\ &= \tilde{T}_1(\alpha^2(a) \otimes \alpha^2(b) \otimes \alpha(c))\\ &= \alpha^2(a) \otimes \alpha^2(b) \otimes \alpha^2(c)\\ &= \tilde{T}_2(\alpha(a) \otimes \alpha^2(b) \otimes \alpha^2(c))\\ &= (\tilde{T}_2 \circ (id_A \otimes T))(\alpha(a) \otimes \alpha(b) \otimes \alpha(c))\\ &= (\tilde{T}_2 \circ (id_A \otimes T) \circ (T \otimes \alpha))(a \otimes b \otimes c), \quad q.e.d.\end{aligned}$$

The fact that A^T_α and A_α coincide is obvious. □

DEFINITION 5.3. Let (A, μ_A) and (B, μ_B) be two associative algebras and $\alpha_A : A \to A$ and $\alpha_B : B \to B$ two bijective algebra endomorphisms. A linear map $R : B \otimes A \to A \otimes B$ is called (α_A, α_B)-*twisting map* if the following conditions are satisfied:

$$(\alpha_A \otimes \alpha_B) \circ R = R \circ (\alpha_B \otimes \alpha_A), \tag{5.7}$$
$$R \circ (id_B \otimes \mu_A) = (\mu_A \otimes id_B) \circ (id_A \otimes R) \circ (id_A \otimes \alpha_B^{-1} \otimes id_A) \circ (R \otimes id_A), \tag{5.8}$$
$$R \circ (\mu_B \otimes id_A) = (id_A \otimes \mu_B) \circ (R \otimes id_B) \circ (id_B \otimes \alpha_A^{-1} \otimes id_B) \circ (id_B \otimes R). \tag{5.9}$$

PROPOSITION 5.4. *If R is an (α_A, α_B)-twisting map as above, then the linear map*

$$T : (A \otimes B) \otimes (A \otimes B) \to (A \otimes B) \otimes (A \otimes B),$$
$$T((a \otimes b) \otimes (a' \otimes b')) = (\alpha_A(a) \otimes b_R) \otimes (a'_R \otimes \alpha_B(b')),$$

is an $\alpha_A \otimes \alpha_B$-pseudotwistor for the associative algebra $A \otimes B$, with companions

$$\tilde{T}_1 = T_{13} \circ (\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \otimes id_A \otimes id_B),$$
$$\tilde{T}_2 = T_{13} \circ (id_A \otimes id_B \otimes id_A \otimes id_B \otimes \alpha_A^{-1} \otimes \alpha_B^{-1}).$$

*The Hom-associative algebra $(A \otimes B)^T_{\alpha_A \otimes \alpha_B}$, called the (α_A, α_B)-*twisted tensor product *of A and B, is denoted by $A(\alpha_A) \otimes_R B(\alpha_B)$; its multiplication is $(a \otimes b)(a' \otimes b') = \alpha_A(a)a'_R \otimes b_R\alpha_B(b')$.*

Proof. Note first that (5.8) and (5.9) may be written is Sweedler-type notation as

$$(aa')_R \otimes b_R = a_R a'_r \otimes \alpha_B^{-1}(b_R)_r, \tag{5.10}$$
$$a_R \otimes (bb')_R = \alpha_A^{-1}(a_R)_r \otimes b_r b'_R. \tag{5.11}$$

We need to check the conditions (5.1)–(5.4) for T; (5.1) follows immediately from (5.7).

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Proof of (5.2): We compute:

$$((id_A \otimes id_B \otimes \mu_{A\otimes B}) \circ \tilde{T}_1 \circ (T \otimes id_A \otimes id_B))(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b'')$$

$$\begin{aligned}
&= ((id_A \otimes id_B \otimes \mu_{A\otimes B}) \circ T_{13} \circ (\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \\
&\quad \otimes id_A \otimes id_B))(\alpha_A(a) \otimes b_R \otimes a'_R \otimes \alpha_B(b') \otimes a'' \otimes b'') \\
&= ((id_A \otimes id_B \otimes \mu_{A\otimes B}) \circ T_{13})(a \otimes \alpha_B^{-1}(b_R) \otimes a'_R \otimes \alpha_B(b') \otimes a'' \otimes b'') \\
&= (id_A \otimes id_B \otimes \mu_{A\otimes B})(\alpha_A(a) \otimes \alpha_B^{-1}(b_R)_r \otimes a'_R \otimes \alpha_B(b') \otimes a''_r \otimes \alpha_B(b'')) \\
&= (\alpha_A(a) \otimes \alpha_B^{-1}(b_R)_r \otimes a'_R a''_r \otimes \alpha_B(b'b'') \\
&\overset{(5.10)}{=} \alpha_A(a) \otimes b_R \otimes (a'a'')_R \otimes \alpha_B(b'b'') \\
&= T(a \otimes b \otimes a'a'' \otimes b'b'') \\
&= (T \circ (id_A \otimes id_B \otimes \mu_{A\otimes B}))(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b''), \quad q.e.d.
\end{aligned}$$

Proof of (5.3): We compute:

$$((\mu_{A\otimes B} \otimes id_A \otimes id_B) \circ \tilde{T}_2 \circ (id_A \otimes id_B \otimes T))(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b'')$$

$$\begin{aligned}
&= ((\mu_{A\otimes B} \otimes id_A \otimes id_B) \circ T_{13} \circ (id_A \otimes id_B \otimes id_A \otimes id_B \\
&\quad \otimes \alpha_A^{-1} \otimes \alpha_B^{-1}))(a \otimes b \otimes \alpha_A(a') \otimes b'_R \otimes a''_R \otimes \alpha_B(b'')) \\
&= ((\mu_{A\otimes B} \otimes id_A \otimes id_B) \circ T_{13})(a \otimes b \otimes \alpha_A(a') \otimes b'_R \otimes \alpha_A^{-1}(a''_R) \otimes b'') \\
&= (\mu_{A\otimes B} \otimes id_A \otimes id_B)(\alpha_A(a) \otimes b_r \otimes \alpha_A(a') \otimes b'_R \otimes \alpha_A^{-1}(a''_R)_r \otimes \alpha_B(b'')) \\
&= \alpha_A(aa') \otimes b_r b'_R \otimes \alpha_A^{-1}(a''_R)_r \otimes \alpha_B(b'') \\
&\overset{(5.11)}{=} \alpha_A(aa') \otimes (bb')_R \otimes a''_R \otimes \alpha_B(b'') \\
&= T(aa' \otimes bb' \otimes a'' \otimes b'') \\
&= (T \circ (\mu_{A\otimes B} \otimes id_A \otimes id_B))(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b''), \quad q.e.d.
\end{aligned}$$

Proof of (5.4): Because of how $\tilde{T}_1$ and $\tilde{T}_2$ are defined, it is enough to prove the following relation:

$$\begin{aligned}
&(\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \otimes id_A \otimes id_B) \circ (T \otimes id_A \otimes id_B) \circ (\alpha_A \otimes \alpha_B \otimes T) \\
&= (id_A \otimes id_B \otimes id_A \otimes id_B \otimes \alpha_A^{-1} \otimes \alpha_B^{-1}) \circ (id_A \otimes id_B \otimes T) \circ (T \otimes \alpha_A \otimes \alpha_B).
\end{aligned}$$

We compute:

$((\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \otimes id_A \otimes id_B) \circ (T \otimes id_A \otimes id_B) \circ (\alpha_A \otimes \alpha_B \otimes T))(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b'')$

$$\begin{aligned}
&= ((\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \otimes id_A \otimes id_B) \circ (T \otimes id_A \otimes id_B))(\alpha_A(a) \otimes \alpha_B(b) \\
&\quad \otimes \alpha_A(a') \otimes b'_R \otimes a''_R \otimes \alpha_B(b'')) \\
&= (\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \otimes id_A \otimes id_B)(\alpha_A^2(a) \otimes \alpha_B(b)_r \\
&\quad \otimes \alpha_A(a')_r \otimes \alpha_B(b'_R) \otimes a''_R \otimes \alpha_B(b'')) \\
&\overset{(5.7)}{=} (\alpha_A^{-1} \otimes \alpha_B^{-1} \otimes id_A \otimes id_B \otimes id_A \otimes id_B)(\alpha_A^2(a) \otimes \alpha_B(b_r) \\
&\quad \otimes \alpha_A(a'_r) \otimes \alpha_B(b'_R) \otimes a''_R \otimes \alpha_B(b'')) \\
&= \alpha_A(a) \otimes b_r \otimes \alpha_A(a'_r) \otimes \alpha_B(b'_R) \otimes a''_R \otimes \alpha_B(b''),
\end{aligned}$$

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$((id_A \otimes id_B \otimes id_A \otimes id_B \otimes \alpha_A^{-1} \otimes \alpha_B^{-1}) \circ (id_A \otimes id_B \otimes T) \circ (T \otimes \alpha_A \otimes \alpha_B))(a \otimes b \otimes a' \otimes b' \otimes a'' \otimes b'')$

$$\begin{aligned}
&= ((id_A \otimes id_B \otimes id_A \otimes id_B \otimes \alpha_A^{-1} \otimes \alpha_B^{-1}) \circ (id_A \otimes id_B \otimes T))(\alpha_A(a) \otimes b_r \\
&\quad \otimes a'_r \otimes \alpha_B(b') \otimes \alpha_A(a'') \otimes \alpha_B(b'')) \\
&= (id_A \otimes id_B \otimes id_A \otimes id_B \otimes \alpha_A^{-1} \otimes \alpha_B^{-1})(\alpha_A(a) \otimes b_r \\
&\quad \otimes \alpha_A(a'_r) \otimes \alpha_B(b')_R \otimes \alpha_A(a'')_R \otimes \alpha_B^2(b'')) \\
&\overset{(5.7)}{=} (id_A \otimes id_B \otimes id_A \otimes id_B \otimes \alpha_A^{-1} \otimes \alpha_B^{-1})(\alpha_A(a) \otimes b_r \\
&\quad \otimes \alpha_A(a'_r) \otimes \alpha_B(b'_R) \otimes \alpha_A(a''_R) \otimes \alpha_B^2(b'')) \\
&= \alpha_A(a) \otimes b_r \otimes \alpha_A(a'_r) \otimes \alpha_B(b'_R) \otimes a''_R \otimes \alpha_B(b''),
\end{aligned}$$

and the two terms are obviously equal. □

EXAMPLE 5.5. Let (A, μ_A) and (B, μ_B) be two associative algebras and $\alpha_A : A \to A$ and $\alpha_B : B \to B$ two bijective algebra endomorphisms. Define the map $R : B \otimes A \to A \otimes B$, $R(b \otimes a) = \alpha_A(a) \otimes \alpha_B(b)$. Then, one can easily check that R is an (α_A, α_B)-twisting map, and $A(\alpha_A) \otimes_R B(\alpha_B)$ coincides with $(A \otimes B)_{\alpha_A \otimes \alpha_B}$ as Hom-associative algebras. More generally, assume that $P : B \otimes A \to A \otimes B$ is a twisting map such that $(\alpha_A \otimes \alpha_B) \circ P = P \circ (\alpha_B \otimes \alpha_A)$. Define the map $R : B \otimes A \to A \otimes B$, $R = (\alpha_A \otimes \alpha_B) \circ P$. Then, one can check that R is an (α_A, α_B)-twisting map, and $A(\alpha_A) \otimes_R B(\alpha_B)$ coincides with $(A \otimes_P B)_{\alpha_A \otimes \alpha_B}$ as Hom-associative algebras.

EXAMPLE 5.6. This is a particular case of the previous example, but can also be checked directly. Let (A, μ_A) be an associative algebra, $\sigma : A \to A$ an involutive algebra automorphism of A and $q \in k^*$. Take $B = C(k, q) = k[v]/(v^2 = q)$, take $\alpha_A = \sigma$ and $\alpha_B = id$ and define $R : B \otimes A \to A \otimes B$ a linear map with $R(1 \otimes a) = \sigma(a) \otimes 1$ and $R(v \otimes a) = a \otimes v$, for all $a \in A$. Then, R is an (α_A, α_B)-twisting map. By using the formula of R, one can easily see that the multiplication of the Hom-associative algebra $A(\alpha_A) \otimes_R B(\alpha_B)$ is given by

$$(a \otimes 1 + b \otimes v)(c \otimes 1 + d \otimes v) = (\sigma(ac) + q\sigma(b)d) \otimes 1 + (\sigma(ad) + \sigma(b)c) \otimes v,$$

for all $a, b, c, d \in A$. If we consider again the associative algebra $\overline{A}$ obtained from A by the Clifford process and the algebra automorphism $\overline{\sigma} : \overline{A} \to \overline{A}$, $\overline{\sigma}(a \otimes 1 + b \otimes v) = \sigma(a) \otimes 1 + \sigma(b) \otimes v$, then one can see that $A(\alpha_A) \otimes_R B(\alpha_B) = (\overline{A})_{\overline{\sigma}}$ as Hom-associative algebras.

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