J. Aust. Math. Soc. **96** (2014), 36–43
doi:10.1017/S144678871300061X

ON THE GROUP INVERSE FOR THE SUM OF MATRICES

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(Received 5 February 2013; accepted 14 August 2013; first published online 8 November 2013)

Communicated by M. Jackson

Abstract

Let $\mathbb{K}^{m\times n}$ denote the set of all $m \times n$ matrices over a skew field $\mathbb{K}$. In this paper, we give a necessary and sufficient condition for the existence of the group inverse of $P + Q$ and its representation under the condition $PQ = 0$, where $P, Q \in \mathbb{K}^{n\times n}$. In addition, in view of the natural characters of block matrices, we give the existence and representation for the group inverse of $P + Q$ and $P + Q + R$ under some conditions, where $P, Q, R \in \mathbb{K}^{n\times n}$.

2010 *Mathematics subject classification*: primary 15A09.

Keywords and phrases: group inverse, group invertibility, skew field.

1. Introduction

Let $\mathbb{K}^{m\times n}$ and $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over a skew field $\mathbb{K}$ and complex field $\mathbb{C}$, respectively. For $A \in \mathbb{K}^{n\times n}$, the smallest nonnegative integer k such that $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$ is called the index of A and denoted by $\mathrm{ind}(A)$. Let $A \in \mathbb{K}^{n\times n}$ with $\mathrm{ind}(A) = k$. The matrix $X \in \mathbb{K}^{n\times n}$ satisfies $XAX = X, AX = XA; A^{k+1}X = A^k$ is called the Drazin inverse of A and denoted by A^D. The Drazin inverse of a square matrix always exists and is unique (see [3, 18]). If $\mathrm{ind}(A) = 1$, then A^D is called the group inverse of A and denoted by $A^\sharp$. If $A^\sharp$ exists, A is called group invertible. In this paper, we let $A^\pi = I - AA^\sharp$ if A is group invertible.

In [14], Hartwig *et al.* gave a representation for the Drazin inverse of $P + Q$ under the condition $PQ = 0$, and there are some results on the representation for the Drazin inverse of $P + Q$, for example [4, 12, 13, 16]. In [1], Benítez *et al.* studied the invertibility of $c_1P + c_2Q$ when $P, Q \in \mathbb{C}^{n\times n}$ are two k-potent matrices and $PQ = 0$, where $c_1, c_2 \in \mathbb{C}$. The representation of the group inverse of $c_1P + c_2Q$ was also obtained in [1] when $P, Q \in \mathbb{C}^{n\times n}$ are two k-potent matrices and $PQ = 0$. Benítez *et al.* also [2] gave a representation of the group inverse of $P + Q$ when $PQ = 0$

This work is supported by the National Science Foundation of China under grant 11371109 and the Fundamental Research Funds for the Central Universities.

and $P, Q \in \mathcal{A}$ are group invertible, where $\mathcal{A}$ is an algebra. In this paper, we give a necessary and sufficient condition for the existence of the group inverse of $P + Q$ and its representation under the condition $PQ = 0$, where $P, Q \in \mathbb{K}^{n\times n}$.

In 1979, Campbell and Meyer proposed an open problem to find an explicit representation of the Drazin inverse for a 2×2 block matrix $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A and D are square (see [8]). Hitherto, this problem has not been solved completely. However, there are some results on the group inverse for the block matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ under certain conditions (see [6, 7, 10, 11, 15, 17]). Notice that $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & B \\ C & D \end{smallmatrix}\right) := P + Q$, then $PQP = 0$; and that $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & B \\ 0 & D \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right) := P + Q + R$, then $PR = 0$ and $QP = 0$.

In this paper, we give the existence and representation for the group inverse of $P + Q$ under $PQP = 0$ and other conditions, where $P, Q \in \mathbb{K}^{n\times n}$. We also give the existence and representation for the group inverse of $P + Q + R$ under $PR = 0$, $QP = 0$ and other conditions, where $P, Q, R \in \mathbb{K}^{n\times n}$.

2. Lemmas

In order to obtain our main results, we give the following two lemmas which play an important role throughout this paper.

Lemma 2.1 [5]. *Let $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathbb{K}^{n\times n}$, where $A \in \mathbb{K}^{r\times r}$ is invertible, and the group inverse of $S = D - CA^{-1}B$ exists. Then $M^\sharp$ exists if and only if $G = A^2 + BS^\pi C$ is invertible. If $M^\sharp$ exists, then*

$$M^\sharp = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where

$$\begin{aligned}
X &= AG^{-1}(A + BS^\sharp C)G^{-1}A, \\
Y &= AG^{-1}(A + BS^\sharp C)G^{-1}BS^\pi - AG^{-1}BS^\sharp, \\
Z &= S^\pi CG^{-1}(A + BS^\sharp C)G^{-1}A - S^\sharp CG^{-1}A, \\
W &= S^\pi CG^{-1}(A + BS^\sharp C)G^{-1}BS^\pi - S^\sharp CG^{-1}BS^\pi - S^\pi CG^{-1}BS^\sharp + S^\sharp.
\end{aligned}$$

Lemma 2.2 [9]. *Let $M = \left(\begin{smallmatrix} A & B \\ 0 & C \end{smallmatrix}\right) \in \mathbb{K}^{n\times n}$, where $A \in \mathbb{K}^{r\times r}$. Then $M^\sharp$ exists if and only if $A^\sharp$, $C^\sharp$ exist and* $\mathrm{rank}(M) = \mathrm{rank}(A) + \mathrm{rank}(C)$. *If $M^\sharp$ exists, then*

$$M^\sharp = \begin{pmatrix} A^\sharp & (A^\sharp)^2BC^\pi + A^\pi B(C^\sharp)^2 - A^\sharp BC^\sharp \\ 0 & C^\sharp \end{pmatrix}.$$

3. Main results

In this section, we give our main results.

Theorem 3.1. *Let $P, Q \in \mathbb{K}^{n\times n}$ and $PQ = 0$. Then $(P + Q)^\sharp$ exists if and only if P^2, Q^2 are group invertible and* $\mathrm{rank}(P + Q) = \mathrm{rank}(P^2) + \mathrm{rank}(Q^2)$. *If $(P + Q)^\sharp$ exists, then*

$$(P + Q)^\sharp = (P^2)^\sharp P + Q(Q^2)^\sharp + QXP,$$

https://doi.org/10.1017/S144678871300061X Published online by Cambridge University Press

where

$$X = ((Q^2)^\sharp)^2(P+Q)(P^2)^\pi + (Q^2)^\pi(P+Q)((P^2)^\sharp)^2 - (Q^2)^\sharp(P+Q)(P^2)^\sharp.$$

Proof. It is well known [3] that there exists an invertible matrix U such that

$$P = U\begin{pmatrix}\Delta & 0\\ 0 & N\end{pmatrix}U^{-1}, \tag{3.1}$$

where Δ is invertible and N is a nilpotent matrix. Let $Q = U\left(\begin{smallmatrix}Q_1 & Q_2\\ Q_3 & Q_4\end{smallmatrix}\right)U^{-1}$, where Q_1 has the same order as Δ. From $PQ = 0$,

$$Q = U\begin{pmatrix}0 & 0\\ Q_3 & Q_4\end{pmatrix}U^{-1}, \quad NQ_3 = 0, \quad NQ_4 = 0. \tag{3.2}$$

By Lemma 2.2, the group inverse of $P + Q = U\left(\begin{smallmatrix}\Delta & 0\\ Q_3 & N+Q_4\end{smallmatrix}\right)U^{-1}$ exists if and only if $(N + Q_4)^\sharp$ exists. Similarly, there exists an invertible matrix V such that

$$Q_4 = V\begin{pmatrix}R & 0\\ 0 & S\end{pmatrix}V^{-1}, \tag{3.3}$$

where R is invertible and S is a nilpotent matrix. Let $N = V\left(\begin{smallmatrix}N_1 & N_2\\ N_3 & N_4\end{smallmatrix}\right)V^{-1}$, where N_1 has the same order as R. From $NQ_4 = 0$,

$$N = V\begin{pmatrix}0 & N_2\\ 0 & N_4\end{pmatrix}V^{-1}, \quad N_2S = 0, \quad N_4S = 0. \tag{3.4}$$

Applying Lemma 2.2, the group inverse of $N + Q_4 = V\left(\begin{smallmatrix}R & N_2\\ 0 & N_4+S\end{smallmatrix}\right)V^{-1}$ exists if and only if $(N_4 + S)^\sharp$ exists. Since N_4, S are nilpotent matrices and $N_4S = 0$, $N_4 + S$ is a nilpotent matrix. Since $(P + Q)^\sharp$ exists if and only if $(N + Q_4)^\sharp$ exists, $(P + Q)^\sharp$ exists if and only if $N_4 + S = 0$.

We prove the 'only if' part. If $(P + Q)^\sharp$ exists, then $N_4 + S = 0$, $N_4 = -S$. By $N_4S = 0$ we get $N_4^2 = 0$ and $S^2 = 0$. From $N_4 = -S$ and (3.4) we have $N^2 = 0$. From (3.1) we know that P^2 is group invertible. By $S^2 = 0$ and (3.3) we know that Q_4^2 is group invertible. By $Q^2 = U\left(\begin{smallmatrix}0 & 0\\ Q_4Q_3 & Q_4^2\end{smallmatrix}\right)U^{-1}$, we have $\mathrm{rank}(Q^2) = \mathrm{rank}\left(\begin{smallmatrix}0 & 0\\ Q_4Q_3 - Q_4^2Q_4^\sharp Q_3 & Q_4^2\end{smallmatrix}\right) = \mathrm{rank}(Q_4^2)$. By Lemma 2.2, we know that Q^2 is group invertible. So

$$\begin{aligned}\mathrm{rank}(P+Q) &= \mathrm{rank}(\Delta) + \mathrm{rank}(N+Q_4) = \mathrm{rank}(\Delta) + \mathrm{rank}(Q_4^2)\\ &= \mathrm{rank}(P^2) + \mathrm{rank}(Q^2).\end{aligned}$$

We now turn to the 'if' part. Since P^2 and Q^2 are group invertible, by (3.1) and (3.2), we know that $N^2 = 0$, Q_4^2 is group invertible and $\mathrm{rank}(Q^2) = \mathrm{rank}(Q_4^2) = \mathrm{rank}(R)$. Since $\mathrm{rank}(P + Q) = \mathrm{rank}(\Delta) + \mathrm{rank}(N + Q_4) = \mathrm{rank}(P^2) + \mathrm{rank}(N + Q_4) = \mathrm{rank}(P^2) + \mathrm{rank}(Q^2) = \mathrm{rank}(P^2) + \mathrm{rank}(R)$, we have $\mathrm{rank}(N + Q_4) = \mathrm{rank}(R)$. By $N + Q_4 = V\left(\begin{smallmatrix}R & N_2\\ 0 & N_4+S\end{smallmatrix}\right)V^{-1}$, we have $N_4 + S = 0$. Hence $(P + Q)^\sharp$ exists.

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If $(P+Q)^\sharp$ exists,

$$\begin{aligned}(P+Q)^\sharp &= U\begin{pmatrix}\Delta & 0\\ Q_3 & N+Q_4\end{pmatrix}^\sharp U^{-1}\\ &= U\begin{pmatrix}\Delta^{-1} & 0\\ (N+Q_4)^\pi Q_3\Delta^{-2}-(N+Q_4)^\sharp Q_3\Delta^{-1} & (N+Q_4)^\sharp\end{pmatrix}U^{-1}\\ &= (P^2)^\sharp P + Q(Q^2)^\sharp + QXP,\end{aligned}$$

where $X = ((Q^2)^\sharp)^2(P+Q)(P^2)^\pi + (Q^2)^\pi(P+Q)((P^2)^\sharp)^2 - (Q^2)^\sharp(P+Q)(P^2)^\sharp$. □

THEOREM 3.2. *Let $P, Q \in \mathbb{K}^{n\times n}$ and $PQ = 0$. Then $P+Q$ is invertible if and only if P, Q are group invertible and* $\operatorname{rank}(P) + \operatorname{rank}(Q) = n$. *If $P+Q$ is invertible, then*

$$(P+Q)^{-1} = Q^\pi P^\sharp + Q^\sharp P^\pi.$$

PROOF. We begin with the 'only if' part. If $P+Q$ is invertible, according to Theorem 3.1, we have $\operatorname{rank}(P+Q) = \operatorname{rank}(P^2) + \operatorname{rank}(Q^2) = n$. By $PQ = 0$, we get $\operatorname{rank}(P) + \operatorname{rank}(Q) \le n = \operatorname{rank}(P^2) + \operatorname{rank}(Q^2)$, which implies that P and Q are group invertible.

Turning to the 'if' part, suppose that P and Q have the decompositions given in (3.1) and (3.2). Since P is group invertible, $N = 0$. Since Q is group invertible, by Lemma 2.2, Q_4 is group invertible and $\operatorname{rank}(Q) = \operatorname{rank}(Q_4)$. By $\operatorname{rank}(P) + \operatorname{rank}(Q) = \operatorname{rank}(\Delta) + \operatorname{rank}(Q_4) = n$, we have that Q_4 is invertible. Hence $P+Q$ is invertible.

If $P+Q$ is invertible, then

$$\begin{aligned}(P+Q)^{-1} &= U\begin{pmatrix}\Delta & 0\\ Q_3 & Q_4\end{pmatrix}^{-1}U^{-1} = U\begin{pmatrix}\Delta^{-1} & 0\\ -Q_4^{-1}Q_3\Delta^{-1} & Q_4^{-1}\end{pmatrix}U^{-1}\\ &= Q^\pi P^\sharp + Q^\sharp P^\pi.\end{aligned}$$

This concludes the proof. □

THEOREM 3.3. *Let $P, Q \in \mathbb{K}^{n\times n}$, $P^\sharp$ exists, $PQP = 0$ and the group inverse of $V = P^\pi QP^\pi - QP^\sharp Q$ exists. Then $(P+Q)^\sharp$ exists if and only if* $\operatorname{rank}(H) = \operatorname{rank}(P)$, *where $H = P^2 + PP^\sharp QV^\pi QP^\sharp P$. If $(P+Q)^\sharp$ exists, then $H^\sharp$ exists and*

$$(P+Q)^\sharp = (I + V^\pi QP^\sharp)(I - PH^\sharp Q)(PH^\sharp PH^\sharp P + V^\sharp)(I - QH^\sharp P)(I + P^\sharp QV^\pi).$$

PROOF. Since $P^\sharp$ exists, there exist invertible matrices U and Δ such that $P = U\left(\begin{smallmatrix}\Delta & 0\\ 0 & 0\end{smallmatrix}\right)U^{-1}$. Let $Q = U\left(\begin{smallmatrix}Q_1 & Q_2\\ Q_3 & Q_4\end{smallmatrix}\right)U^{-1}$, where Q_1 has the same order as Δ. By $PQP = 0$ we get $Q_1 = 0$. Hence

$$V = P^\pi QP^\pi - QP^\sharp Q = U\begin{pmatrix}0 & 0\\ 0 & Q_4 - Q_3\Delta^{-1}Q_2\end{pmatrix}U^{-1}.$$

Since $V^\sharp$ exists, $S = Q_4 - Q_3\Delta^{-1}Q_2$ is group invertible. Applying Lemma 2.1, the group inverse of $P + Q = U\left(\begin{smallmatrix}\Delta & Q_2\\ Q_3 & Q_4\end{smallmatrix}\right)U^{-1}$ exists if and only if $G = \Delta^2 + Q_2S^\pi Q_3$ is

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invertible. Since $H = P^2 + PP^\sharp QV^\pi QP^\sharp P = U\begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$, G is invertible if and only if rank(H) = rank(P). Hence $(P+Q)^\sharp$ exists if and only if rank(H) = rank(P).

From the above arguments, $(P+Q)^\sharp$ exists if and only if G is invertible. So $H^\sharp$ exists if $(P+Q)^\sharp$ exists. Then we have $H^\sharp = U\begin{pmatrix} G^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$. By Lemma 2.1,

$$(P+Q)^\sharp = U\begin{pmatrix} \Delta & Q_2 \\ Q_3 & Q_4 \end{pmatrix}^\sharp U^{-1} = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}U^{-1},$$

where

$X_1 = \Delta G^{-1}(\Delta + Q_2S^\sharp Q_3)G^{-1}\Delta$,

$X_2 = \Delta G^{-1}(\Delta + Q_2S^\sharp Q_3)G^{-1}Q_2S^\pi - \Delta G^{-1}Q_2S^\sharp$,

$X_3 = S^\pi Q_3G^{-1}(\Delta + Q_2S^\sharp Q_3)G^{-1}\Delta - S^\sharp Q_3G^{-1}\Delta$,

$X_4 = S^\pi Q_3G^{-1}(\Delta + Q_2S^\sharp Q_3)G^{-1}Q_2S^\pi - S^\sharp Q_3G^{-1}Q_2S^\pi - S^\pi Q_3G^{-1}Q_2S^\sharp + S^\sharp$. So

$$\begin{aligned}(P+Q)^\sharp &= U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}U^{-1} \\ &= U\begin{pmatrix} I & 0 \\ S^\pi Q_3\Delta^{-1} & I \end{pmatrix}\begin{pmatrix} I & -\Delta G^{-1}Q_2 \\ 0 & I \end{pmatrix}\begin{pmatrix} G^{-1}\Delta G^{-1}\Delta & 0 \\ 0 & S^\sharp \end{pmatrix} \\ &\quad \times \begin{pmatrix} I & 0 \\ -Q_3G^{-1}\Delta & I \end{pmatrix}\begin{pmatrix} I & \Delta^{-1}Q_2S^\pi \\ 0 & I \end{pmatrix}U^{-1} \\ &= (I + V^\pi QP^\sharp)(I - PH^\sharp Q)(PH^\sharp PH^\sharp P + V^\sharp)(I - QH^\sharp P)(I + P^\sharp QV^\pi).\end{aligned}$$

This concludes the proof. □

Remark 3.1. If $P^\sharp$, $Q^\sharp$ exist and $PQ = 0$, then $P = U\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$, $Q = U\begin{pmatrix} 0 & 0 \\ Q_3 & Q_4 \end{pmatrix}U^{-1}$ and $Q_4^\sharp$ exists. Thus $V = QP^\pi = U\begin{pmatrix} 0 & 0 \\ 0 & Q_4 \end{pmatrix}U^{-1}$ is group invertible and $H = P^2$. By Theorem 3.3, $(P+Q)^\sharp$ exists and

$$(P+Q)^\sharp = Q^\pi P^\sharp + Q^\sharp P^\pi.$$

Theorem 3.4. *Let $P, Q, R \in \mathbb{K}^{n\times n}$, $P^\sharp$ and $Q^\sharp$ exist, $PR = 0$, $QP = 0$, $RP^\pi = 0$ and $RP^\sharp Q = 0$. Then the group inverse of $P + Q + R$ exists and*

$$(P+Q+R)^\sharp = (I + P^\pi Q^\pi RP^\sharp)(I - P^\sharp Q)(P^\sharp + P^\pi Q^\sharp)(I - RP^\sharp).$$

Proof. Since $P^\sharp$ exists, there exist invertible matrices U and Δ such that $P = U\begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}U^{-1}$. Suppose that $Q = U\begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix}U^{-1}$, $R = U\begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}U^{-1}$, where Q_1, R_1 have the same order as Δ. Since $PR = 0$, $QP = 0$ and $RP^\pi = 0$, $Q = U\begin{pmatrix} 0 & Q_2 \\ 0 & Q_4 \end{pmatrix}U^{-1}$, $R = U\begin{pmatrix} 0 & 0 \\ R_3 & 0 \end{pmatrix}U^{-1}$. By $RP^\sharp Q = 0$ we get $R_3\Delta^{-1}Q_2 = 0$. Since $Q^\sharp$ exists, by Lemma 2.2, $Q_4^\sharp$ exists and there exists a matrix X such that $Q_2 = XQ_4$. So we have $Q_2Q_4^\pi = 0$. Since $R_3\Delta^{-1}Q_2 = 0$, by Lemma 2.1, the group inverse of $P + Q + R = U\begin{pmatrix} \Delta & Q_2 \\ R_3 & Q_4 \end{pmatrix}U^{-1}$ exists and

$$(P+Q+R)^\sharp = U\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}U^{-1},$$

https://doi.org/10.1017/S144678871300061X Published online by Cambridge University Press

where

$X_1 = \Delta^{-1}(\Delta + Q_2 Q_4^\sharp R_3)\Delta^{-1}$,

$X_2 = -\Delta^{-1} Q_2 Q_4^\sharp$,

$X_3 = Q_4^\pi R_3 (\Delta^2)^{-1}(\Delta + Q_2 Q_4^\sharp R_3)\Delta^{-1} - Q_4^\sharp R_3 \Delta^{-1}$,

$X_4 = -Q_4^\pi R_3 (\Delta^2)^{-1} Q_2 Q_4^\sharp + Q_4^\sharp$. Hence

$$\begin{aligned}(P+Q+R)^\sharp &= U \begin{pmatrix} I & 0 \\ Q_4^\pi R_3 \Delta^{-1} & I \end{pmatrix} \begin{pmatrix} I & -\Delta^{-1} Q_2 \\ 0 & I \end{pmatrix} \begin{pmatrix} \Delta^{-1} & 0 \\ 0 & Q_4^\sharp \end{pmatrix} \begin{pmatrix} I & 0 \\ -R_3 \Delta^{-1} & I \end{pmatrix} U^{-1} \\ &= (I + P^\pi Q^\pi R P^\sharp)(I - P^\sharp Q)(P^\sharp + P^\pi Q^\sharp)(I - R P^\sharp).\end{aligned}$$

This concludes the proof. □

Next we use $\mathbb{K} = \{a + bi + cj + dk\}$ to denote the real quaternion skew field, where a, b, c, d are real numbers. We give some examples to illustrate the application of the representations given in this paper.

EXAMPLE 3.5. Let $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{K}^{3\times 3}$, $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{K}^{3\times 3}$. By computation,

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and } P + Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So P^2, Q^2 are group invertible and $\operatorname{rank}(P^2) + \operatorname{rank}(Q^2) = \operatorname{rank}(P + Q) = 1$. By Theorem 3.1, $(P + Q)^\sharp$ exists and

$$(P+Q)^\sharp = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

EXAMPLE 3.6. Let

$$P = \begin{pmatrix} i & j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{K}^{4\times 4}, \quad Q = \begin{pmatrix} 0 & 0 & i & j \\ 0 & 0 & 0 & 0 \\ k & k & 2k & 1 \\ k & k & k & 2 \end{pmatrix} \in \mathbb{K}^{4\times 4};$$

then by computation,

$$P^\sharp = \begin{pmatrix} -i & -j & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P^\pi = \begin{pmatrix} 0 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad PQP = 0.$$

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And then

$$V = P^\pi Q P^\pi - Q P^\sharp Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & k-1 & k & 0 \\ 0 & k-1 & k & 1 \end{pmatrix},$$

$$V^\sharp = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1-k & -k & 0 \\ 0 & -k-1 & -1 & 1 \end{pmatrix}, \quad V^\pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1-k & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$H = P^2 + P P^\sharp Q V^\pi Q P^\sharp P = \begin{pmatrix} -1 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So $\operatorname{rank}(H) = \operatorname{rank}(P) = 1$. By Theorem 3.3, $(P+Q)^\sharp$ exists and

$$(P+Q)^\sharp = \begin{pmatrix} i & j & i & j \\ 0 & 0 & 0 & 0 \\ k & k & 2k & 1 \\ k & k & k & 2 \end{pmatrix}^\sharp = \begin{pmatrix} -3i & 6i-9j-2 & k & k \\ 0 & 0 & 0 & 0 \\ i & -2i+3j-k+1 & -k & 0 \\ j & -3i-2j+k-1 & 0 & 1 \end{pmatrix}.$$

Example 3.7. Let

$$P = \begin{pmatrix} i & j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathbb{K}^{3\times 3}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \in \mathbb{K}^{3\times 3}, \quad R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & k & 0 \end{pmatrix} \in \mathbb{K}^{3\times 3};$$

then by computation,

$$P^\sharp = \begin{pmatrix} -i & -j & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P^\pi = \begin{pmatrix} 0 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then $PR = 0$, $QP = 0$, $RP^\pi = 0$ and $RP^\sharp Q = 0$. By Theorem 3.4, we know that $(P + Q + R)^\sharp$ exists and

$$(P+Q+R)^\sharp = \begin{pmatrix} i & j & 0 \\ 0 & 0 & 0 \\ -1 & k & i \end{pmatrix}^\sharp = \begin{pmatrix} -i & -j & 0 \\ 0 & 0 & 0 \\ -1 & k & -i \end{pmatrix}.$$

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