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On the equivalence of invariant integrals and minimal ideals in semigroups

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Let S be a Hausdorff topological semigroup and $C_b(S)$, $C_c(S)$, the spaces of real valued continuous functions on S which are respectively bounded and have compact support. A regular measure m on S is r^*-invarient if $m(B) = m(t_x^{-1}(B))$ for every Borel $B \subset S$ and every $x \in S$, where $t_x : s \to sx$ is the right translation by x . The following theorem is proved: Let S be locally compact metric with the t_x's closed. Then the following statements are equivalent: (i) S admits a right invariant integral on $C_c(S)$. (ii) S admits an r^*-invariant measure. (iii) S has a unique minimal left ideal. The above equivalence is considered also for normal semigroups and analogous results are obtained for finitely additive r^*-invariant measures. Also in the case when S is a complete separable metric semigroup with the t_x's closed, the following statements are equivalent: (i) S admits a right invariant integral I on $C_b(S)$ such that $I(1) = 1$ and satisfying Daniel's condition. (ii) S admits an r^*-invariant probability measure. (iii) S has a right ideal which is a compact group and which is contained in a unique minimal left ideal. Finally, in order that a locally compact S admit a right invariant measure, it suffices that S contain a right ideal F which is a left group such that $(B \cap F)x = BX \cap Fx$ for all Borel $B \subset S$.

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1. Introduction

In what follows S will be at least a Hausdorff topological semigroup and $C_b(S)$, $C_c(S)$, the spaces of all real valued continuous functions on S which are bounded, have compact support respectively. For $x \in S$, t_x denotes the right translation $s \to sx$ and for $B \subset S$, Bx^{-1} denotes the set $t_x^{-1}(B) = [s ; sx \in B]$. By a right (left) ideal of S is meant a nonempty subset $A \subset S$ such that $AS \subset A$ $(SA \subset A)$. S is called a left group if S is left simple (i.e., $Sx = S$ for every $x \in S$) and contains an idempotent element; equivalently S is a left group if S is homeomorphic to a direct product $E \times G$ where $G = qS$, q being any (fixed) idempotent of S , is a group, and E is the set of idempotents of S . For example, consider the complex numbers $\neq 0$ with multiplication $x*y = x|y|$.

In case S is locally compact then $C_c(S) \neq [0]$ and by an integral on S we mean a nontrivial positive linear functional I on $C_c(S)$. Given such a functional I on $C_c(S)$, an integral can be defined on $C_b(S)$ by $\overline{I}(\phi) = \sup[I(\psi) ; \psi \in C_c(S) , \psi \leqq \phi]$ for $\phi \in C_b^+(S)$, that is, $\overline{I}$ is the usual "canonical extension" of an integral from $C_c(S)$ to $C_b(S)$. [See 6, p. 121]. In the non-locally compact case by an integral on S we will always mean a nontrivial non-negative linear functional I on $C_b(S)$. In general I would have to satisfy additional conditions in order to have a representation with respect to a measure. The support of an integral I on C_b is defined as the set of all $x \in S$ such that for every open $U \ni x$ there is $f \in C_b$, $0 \leqq f \leqq 1$, $f = 0$ on $S - U$, and $I(f) \neq 0$. We denote by $B(S)$ [resp. $\mathcal{B}(S)$] the Borel algebra [resp. σ-algebra] generated by all open sets of S . Unless otherwise specified, m will denote a measure, i.e., m is countably additive on $\mathcal{B}(S)$. m is right invariant if $m(B) = \overline{m}(Bx)$ for every $B \in \mathcal{B}(S)$ and every $x \in S$, where $\overline{m}$ is the completion of m relative to $\mathcal{B}(S)$. (Note that the definition includes that Bx be completion measurable.) m is r^*-invariant if $m(B) = m(Bx^{-1})$ for every $B \in \mathcal{B}$ and every $x \in S$. The support of m , always denoted by F , is defined as the set of all $x \in S$, every open

neighborhood of x has positive measure. m is regular if m is finite on compact sets, m is outer regular [i.e., $m(B)$, $B \in \mathcal{B}$, is the inf of the measures of its open supersets], and
$m(U) = \sup[m(C)\ ;\ C = \text{compact} \subset U]$ for all open $U \subset S$. m is weakly regular if m is outer regular and $m(B) = \sup[m(K)\ ;\ K = \text{closed} \subset B]$ for every $B \in \mathcal{B}$. m is a probability measure if $m(S) = 1$. An integral I on a locally compact S is right invariant if $I(f) = \overline{I}(f \circ t_x) \equiv \overline{I}(f_x)$, for every $f \in C_c(S)$ and every $x \in S$. In the non-locally compact case an integral I on $C_b(S)$ is right invariant if $I(f) = I(f_x)$ for every $f \in C_b(S)$ and every $x \in S$. In many of our theorems we will require that S be complete, i.e., that S admit a uniformity compatible with its topology with respect to which S is a complete uniform space. Metric, paracompact, realcompact [3b, pp. 114, 226; called Q-spaces by Hewitt] spaces are complete in the above sense.

Argabright [2] proved the following result:

THEOREM 1.1. *Let S be locally compact and satisfy*

(#) $\quad Cx^{-1}$ *is compact for every $x \in S$ and every compact $C \subset S$.*

Then the following statements are equivalent:

(i) There is a right invariant integral I on $C_c(S)$.

(ii) There is an r^*-invariant regular measure m on S.

(iii) S has a unique minimal left ideal.

In this paper we consider the above equivalence for normal or locally compact metric semigroups without the compactness condition (#). It turns out that if S is metric then (#) can be replaced by a weaker condition, i.e., by

(C) $\quad$ *All right translations t_x, $x \in S$, are closed maps.*

Substantial use of (C) in connection with invariant integrals was made in [7, p. 186] and it was implied by the conditions used in a related work in [8]. Using a characterization of r^*-invariant measures on left groups given in [2, p. 378] we give a sufficient condition for the existence of a regular right invariant measure (integral) on a locally compact semigroup.

2. Some useful lemmata

LEMMA 2.1. ([2]) *Each right invariant integral* I *(resp. each* r^**-invariant measure* m*) on a locally compact left group* $S = E \times qS$ *is given by* $I(f) = \int_S f(x)\, d(m_1 \times m_2)$ *, where* m_1 *= a Borel measure on* E *and* m_2 *= a right Haar measure on the locally compact group* qS *(resp.* $m = m_1 \times m_2$ *, which is also right invariant on* S *, the mappings* t_x *on left groups being homeomorphisms). Moreover if* m *is* r^**-invariant on a Hausdorff semigroup* S *and* $m(S-F) = 0$ *, then* F *is a right ideal of* S *and* $\overline{Fx} = \overline{F}$ *for all* $x \in S$ *.* [$\overline{Fx}$ = *the closure of* Fx *in* S].

LEMMA 2.2. *The closure of a left subgroup* F *of* S *which is also a right ideal of* S *is also a left subgroup.*

Proof. By continuity, every right identity for F is also a right identity for $\overline{F}$; for if $x_\beta \to z \in \overline{F}$, $x_\beta \in F$, and e is an idempotent in F (e is also a right identity), then $x_\beta e \to ze$, so that $ze = z$. Hence for $a \in F$, there is $c \in F$ such that $ca = e$ and so $\overline{F}a \supset \overline{F}ca = \overline{F}$. Next for $z \in \overline{F}$, $\overline{F}z \supset \overline{F}fz$, f being any element of F , and $fz \in F$, so that $\overline{F}z \supset \overline{F}$.

That condition (#) implies condition (C) is shown by the following

LEMMA 2.3. *Let* $f : X \to Y$ *be any map,* X, Y *Hausdorff spaces; then*

(i) f *closed and* $f^{-1}(y)$ = compact, *for all* $y \in Y$ *, implies*

(ii) $f^{-1}(C)$ = compact, *for all compact* $C \subset Y$.

The proof of this lemma (formulated in different terminology) is essentially contained in N. Bourbaki [3*a*, pp. 101, 104 and p. 37, Prop. 3]. As an example, consider $S = [0, 1/2)$ with ordinary multiplication and usual topology; clearly the t_x are closed but $[0]0^{-1} = S$ = not compact.

3. Theorems on invariant integrals and measures

The following theorem has an immediate application to locally compact metric semigroups satisfying (C).

THEOREM 3.1. *Let* S *be locally compact, normal,* 1^{st}*-countable,* *satisfying* (C). *Then the following statements are equivalent:*

(i) There is a right invariant integral I *on* $C_c(S)$.

(ii) There is a regular r^**-invariant Borel measure* m *on* S .

(iii) S *has a closed right ideal* F *which is a left group (and which is necessarily contained in a closed unique minimal left ideal* L *of* S*)* .

Proof. The proof that *(i)* ⇒ *(ii)* is the same as in [2, p. 379]. To prove that *(ii)* ⇒ *(iii)* we only need to show that F has an idempotent, since by (C) and Lemma 2.1, F is left simple and a right ideal. For if F is shown to be a left group, then $L = \bigcap_{x\in S} Sx \supset F$ and by Lemma 3.2, $\overline{L} = L$. [L is the unique minimal left ideal and also a right ideal.] Let $A = aa^{-1} \cap F = [x \in F ; xa=a]$, for $a \in F$; $A \neq \phi$ since the t_a's on F are "onto" and F is left simple. If Int$A$ = Interior$(A) \neq \phi$ then by r^*-invariance, $m[a] > 0$ and every $z \in aF$ is such that $m[z] \geq m[a]$; (note that for K = closed, $Kxx^{-1} \supset K$ and so $m(Kx) \geq m(k)$); hence, if $z \in yy^{-1} \cap aF \neq \phi$ (aF is also left simple), $y \in aF$, then $[z, z^2, z^3, \ldots]$ is finite because $m[yy^{-1}] = m[y] > 0$ and $m[y] < \infty$. Since every finite semigroup has an idempotent, F is a left group. In case Int$A = \phi$, the boundary of A is countably compact by [9, p. 10] and since $\dot{A}$ is complete, A is a compact semigroup and hence it contains at least one idempotent. (See [3b], p. 237.)

Next we show that *(iii)* ⇒ *(i)*; by Lemma 3.1, F admits an r^*-invariant measure $m = m_1 \times m_2$ which can be extended to the whole S as in [2, p. 380].

The following theorem has an immediate application to separable metric semigroups satisfying (C).

THEOREM 3.2. *Let* S *be normal,* 1^{st}*-countable, realcompact, satisfying* (C). *Then (i)* ⟺ *(ii)* ⇒ *(iii), where*

(i) There is a right invariant integral I *on* $C_b(S)$ *such that* $I(1) = 1$ *and* $I(f_n) \searrow 0$ *whenever* $f_n \searrow 0$, $f_n \in C_b(S)$.

(ii) *There is an* r^**-invariant weakly regular probability measure* m *on* S .

(iii) S *has a closed right ideal* F *which is a left group (and which is necessarily contained in a closed unique minimal left ideal* L*)* .

Proof. *(i)* $\Rightarrow$ *(ii):* There is a weakly regular probability m such that $I(f) = \int f(x)m(dx)$, for $f \in C_b(S)$. (see [10, p. 63]). (Actually m is a Baire measure, i.e., m is defined on the σ -field generated by the closed G_δ's and then it is extended to a Borel measure by $m(B) = \sup[m(K) ; K = \text{closed } G_\delta \subset B]$ as in [3, pp. 183, 203, 194].)

We can prove that m is r^*-invariant either by using an argument in [2, p. 379] replacing local compactness and compact sets by normality and closed sets respectively, or, a method of Markov [7] who studied finitely additive outer probability measures on normal spaces.

That *(ii)* $\Rightarrow$ *(iii)* follows as in Theorem 3.1 provided that we can show that such a support F for m exists. Since S is realcompact, the Baire restriction of m has a support $F =$ (the intersection of all closed G_δ's of measure 1) . (See [1, p. 197] and [5, pp. 172-173]). [In particular if S is separable metric and m is a probability measure on S , then $F \neq \phi$ and $m(S-F) = 0$. See [12, p. 27].]

COROLLARY 3.3. *Let* S *be a complete separable metric semigroup with property* (C). *Then each of the statements (i), (ii) of Theorem 3.2 is equivalent to the statement*

(iii) S *has a right ideal* F *which is a compact group (and which is necessarily contained in a closed unique minimal left ideal* L*)* .

Proof. We show *(ii)* $\Rightarrow$ *(iii)*. As in the proof of Theorem 3.1, F is a left group and $F = E \times qF$, where E is the set of idempotents of F and $q \in F$. Since $qFS = qFqS \subset qFF \subset qF$, qF is also a right ideal of S . We show next that qF is a compact group. Since $F = E \times G$, where G is a group, by using first compact rectangles and the fact that $AB^{-1} = [x ; xb \in A$ for some $b \in B]$ is closed whenever $A,B \subset F$ are compact, one easily verifies that AB^{-1} is compact. Let C be compact $\subset F$ such that $m(x^{-1}C) > 0$ for some $x \in F$. Such a C exists because the measure $\overline{m}(A) = m(x^{-1}A)$ is regular, since m is regular by [12, p. 29]. The proof of this fact given in [4, p. 179] is valid in our case. The function $m(x^{-1}C)$ is upper semi-continuous because $[x \in S ; m(x^{-1}(S-C)) > \alpha]$ is open as in [4, p. 179] whose proof carries over to the present case. Let $\varepsilon > 0$ and K compact $\subset F$ such that $m(K) > 1-\varepsilon$; then for $x \notin CK^{-1}$, $m(x^{-1}C) < \varepsilon$ so that $f(x) = m(x^{-1}C)$ vanishes at infinity. Let $a \in F$ such that $f(a) = \sup_{x\in F} f(x)$. Now $f(a)-f(ax) \geqq 0$ and also $\int_F [f(a)-f(ax)]m(dx) = 0$ because m is an idempotent measure on F. (See [12, p. 67].) Since $[x ; f(a)-f(ax) > 0]$ is open, it must be empty. Hence $f(a) = f(ax)$ for all $x \in F$, so that $\overline{aF} \subset M = [x \in F ; f(x) = \sup_{s\in S} f(s)]$ and so $\overline{aF}$ is compact. Now $\overline{aF}$ being a compact semigroup has an idempotent e and therefore $eF = \overline{eF} \subset M$ and eF is compact group. *(iii)* ⇒ *(i)*. The Haar measure on eF may be extended over the whole S as in [2, pp. 380-381].

THEOREM 3.4. *Let S be a complete metric semigroup satisfying* (C) *and let the set of its idempotents be discrete in the relative topology. Then the following statements are equivalent:*

(i) *There is a continuous r^*-invariant Borel measure m on S with non-empty support. (m is called continuous if $m[x] = 0$ for all $x \in S$.)*

(ii) *S has a perfect right ideal F which is a left group (and which is necessarily contained in a closed unique minimal left ideal).*

Proof. *(i)* ⇒ *(ii)* as in Theorem 3.1. Next we show *(ii)* ⇒ *(i)*: Let $F = E \times qF$; since F has no isolated points, qF is dense-in-itself and a complete metric group. By [11, Th. 1.2], qF admits a right invariant continuous (infinite) Borel measure m_2. By taking m_1 = a probability on E, $m = m_1 \times m_2$ is r^*-invariant on F and it can be extended to the whole S. (Note that r^*-invariance and right invariance coincide on left groups.)

THEOREM 3.5. *(Sufficient condition for existence of right invariant measure.) Let* S *be locally compact. Then, in order that* S *admit a right invariant regular Borel measure it suffices that* S *contain a right ideal* F *which is a left group and such that*

(1) $(B \cap F)x = Bx \cap Fx$ *for every* $x \in S$ *and every Borel* $B \subset S$.

REMARK. Note that (1) is completely algebraic in nature, for if it holds for Borel B , then it holds for any $B \subset S$. In fact, we may consider $\overline{B}$ and then easily prove that it holds for B , since F and $\overline{F}$ are left groups by Lemma 3.2.

Proof. Since $\overline{F}$ is a left group, Lemma 2.1 gives a right invariant measure $m = m_1 \times m_2$ on $\overline{F}$, and by defining $m^*(B) = m(B \cap \overline{F})$ for $B =$ Borel $\subset S$, one easily proves that m^* is r^*-invariant on S [2, p. 380]. Now for $B =$ Borel $\subset S$,

$$m^*(B) = m^*(B\cap\overline{F}) = m[(B\cap\overline{F})qx] = m[(B\cap\overline{F})x] = m(Bx\cap\overline{F}x) = m(Bx\cap\overline{F}) = \overline{m}^*(Bx) ,$$

where $x \in S$ and q is any idempotent (and right identity) of $\overline{F}$. Note that since $(B \cap F)x = Bx \cap Fx = Bx \cap F$, we also have $(B \cap \overline{F})x = Bx \cap \overline{F}x = Bx \cap \overline{F}$, since the mappings t_x are homeomorphisms on F and $\overline{F}$.

4. Finitely additive measures

As it was mentioned earlier an integral I on $C_b(S)$ may not correspond to a Borel measure. However in many cases it is possible to find a finitely additive measure m whose domain of definition includes $B(S)$ and which corresponds to the given functional I in the sense that I can be represented in terms of m as in [1, pp. 180-183]. A finitely additive probability measure m is a non-negative finitely additive set function on $B(S)$ (= the Borel algebra of S) with $m(S) = 1$. The definition of regularity and of the support of m are the same as those given for a Borel measure in Section 1.

THEOREM 4.1. *Let* S *be normal,* 1^{st}*-countable, complete, satisfying* (C). *Then (i)* $\iff$ *(ii)* $\Rightarrow$ *(iii) , where*

(i) There is a right invariant integral I *on* $C_b(S)$ *with* $I(1) = 1$ *and with nonempty support.*

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(ii) *There is an* r^**-invariant weakly regular finitely additive probability measure on* S *with nonempty support.*

(iii) S *has a closed right ideal which is a left group (and which is necessarily contained in a closed unique minimal left ideal of* S*).*

Proof. *(i)* $\Leftrightarrow$ *(ii)* follows as in [7]. *(ii)* $\Rightarrow$ *(iii)* as in Theorem 3.1. It is not known if *(iii)* $\Rightarrow$ *(i)* even when S is a separable metric semigroup.

References

[1] L. Argabright, "Invariant means on topological semigroups", *Pacific J. Math.* 16 (1966), 193-203.

[2] L. Argabright, "A note on invariant integrals on locally compact semigroups", *Proc. Amer. Math. Soc.* 17 (1966), 377-382.

[3] S.K. Berberian, *Measure and Integration*, (MacMillan Co., New York, 1965).

[3a] Nicolas Bourbaki, *General Topology*, Part 1, (Addison-Wesley, Reading, Mento Park, London, Ontario, 1966).

[3b] Leonard Gillman and Meyer Jerison, *Rings of continuous functions*, (Van Nostrand, Princeton, 1960).

[4] M. Heble and M. Rosenblatt, "Idempotent measures on a compact topological semigroup", *Proc. Amer. Math. Soc.* 14 (1963), 177-184.

[5] Edwin Hewitt, "Linear functionals on spaces of continuous functions", *Fund. Math.* 37 (1950), 161-189.

[6] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis* (Academic Press, New York, London, 1963).

[7] A. Markov, "On mean values and exterior densities", *Rec. math. Moscou (Mat. Sb.)* 4 (1938), 165-190.

[8] J.H. Michael, "Right invariant integrals on locally compact semigroups", *J. Austral. Math. Soc.* 4 (1964), 273-286.

[9] Kiiti Morita and Satiro Hanai, "Closed mappings and metric spaces", *Proc. Japan Acad.* 32 (1956), 10-14.

https://doi.org/10.1017/S0004972700041514 Published online by Cambridge University Press

[10] J. Neveu, *Mathematical foundations of the calculus of probabilities* (Holden-Day Inc., San Francisco, 1965).

[11] John C. Oxtoby, "Invariant measures in groups which are not locally compact", *Trans. Amer. Math. Soc.* 60 (1946), 215-237.

[12] K.R. Parthasarathy, *Probability measures on metric spaces* (Academic Press, New York, London, 1967).

[13] G.B. Preston, "Inverse semigroups with minimal right ideals", *J. London Math. Soc.* 29 (1954), 411-419.

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