

THE NON-LOCAL CAUCHY PROBLEM FOR SEMILINEAR INTEGRODIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

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Abstract The aim of this paper is to prove the existence and uniqueness of mild and classical solutions of the non-local Cauchy problem for a semilinear integrodifferential equation with deviating argument. The results are established by using the method of semigroups and the contraction mapping principle. The paper generalizes certain results of Lin and Liu.

Keywords: existence of solutions; resolvent operator; integrodifferential equation; deviating argument

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1. Introduction

Using the method of semigroups and the Banach fixed-point theorem, Byszewski [9] proved the existence and uniqueness of mild, classical and strong solutions of the following non-local Cauchy problem,

$$u'(t) + Au(t) = f(t, u(t)), \quad t \in (t_0, t_0 + a], \quad (1.1)$$

$$u(t_0) + g(t_1, \dots, t_p, u(\cdot)) = u_0, \quad (1.2)$$

where $-A$ is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t > 0$, on a Banach space X , $0 \leq t_0 < t_1 < \dots < t_p \leq t_0 + a$, $a > 0$, $u_0 \in X$, $f : [t_0, t_0 + a] \times X \rightarrow X$ and $g : [t_0, t_0 + a]^p \times X \rightarrow X$ are given functions. Subsequently, Byszewski investigated the same type of problem for a different class of evolution equations in a Banach space [4–11]. Moreover, Corduneanu [12] and Gripenberg *et al.* [15] studied the problem for Volterra integral equations of various types using a semigroup approach. The solution of (1.1) with $u(t_0) = u_0$ can be written as [18]:

$$u(t) = T(t - t_0)u_0 + \int_{t_0}^t T(t - s)f(s, u(s)) ds, \quad t_0 \leq t \leq t_0 + a. \quad (1.3)$$

Byszewski [9] proved the existence and uniqueness of the solution of (1.1) and (1.2) by using a fixed-point argument. He also proved that the mild solutions are classical solutions when $f \in C^1([t_0, t_0 + a] \times X, X)$.

Now, consider the classical heat equation for a material with memory [15]:

$$\left. \begin{aligned} q(t, x) &= -E u_x(t, x) - \int_0^t b(t-s) u_x(s, x) ds, \\ u_t(t, x) &= -\frac{\partial}{\partial x} q(t, x) + f(t, x), \\ u(x, 0) &= u_0(x). \end{aligned} \right\} \quad (1.4)$$

The first equation gives the heat flux and the second is the balance equation. Equation (1.4) can be written as

$$u_t(t, x) = \frac{\partial^2}{\partial x^2} \left[u(t, x) + \int_0^t b(t-s) u(s, x) ds \right] + f(t, x), \quad u(0, x) = u_0(x). \quad (1.5)$$

It is clear that if non-local condition (1.2) is introduced to (1.5), then it will have a better effect than the classical condition $u(0, x) = u_0(x)$, since condition (1.2) is usually more precise for physical measurements than classical ones [2–8, 10, 11, 16].

Lin and Liu [17] have studied semilinear integrodifferential equations with the non-local Cauchy problem

$$\begin{aligned} u'(t) &= A \left[u(t) + \int_0^t F(t-s) u(s) ds \right] + f(t, u(t)), \quad 0 \leq t \leq T, \\ u(0) + g(t_1, \dots, t_p, u(t_1), \dots, u(t_p)) &= u_0, \end{aligned}$$

in a Banach space X with A the generator of a strongly continuous semigroup and $F(t)$ a bounded operator for $t \in [0, T]$. $f : [0, T] \times X \rightarrow X$ and $g : [0, T]^p \times X^p \rightarrow X$ are given functions.

The purpose of this paper is to extend the technique of Lin and Liu [17] to the following integrodifferential equation with a deviating argument and non-local condition of the form

$$u'(t) = A \left[u(t) + \int_0^t Z(t-s) u(s) ds \right] + f(t, u(\sigma(t))), \quad 0 \leq t \leq T, \quad (1.6)$$

$$u(0) + g(u(t_1), \dots, u(t_p)) = u_0, \quad (1.7)$$

where A generates a strongly continuous semigroup in a Banach space X , $Z(t)$ is a bounded operator for $t \in [0, T]$, $\sigma : [0, T] \rightarrow [0, T]$ is a continuous function such that $\sigma(t) \leq t$ for all t , $f : [0, T] \times X \rightarrow X$ and $g : X^p \rightarrow X$ are given functions. Let $J = [0, T]$. Since equation (1.3) played a very important role in the study of (1.1) with $u(t_0) = u_0$, the corresponding formula for (1.6) and (1.7) is given by [17],

$$u(t) = R(t)[u_0 - g(u(t_1), \dots, u(t_p))] + \int_0^t R(t-s) f(s, u(\sigma(s))) ds, \quad (1.8)$$

where the semigroup $T(\cdot)$ in (1.3) is replaced by the resolvent operator $R(\cdot)$, the counterpart of $T(\cdot)$ for integrodifferential equations.

The existence and uniqueness of solutions via variations of constants formula and other properties of resolvent operators have been studied in [13, 14]. We are able to use the techniques developed by Pazy [18], Byszewski [9] and Lin and Liu [17] to study the problem (1.6) and (1.7). The results generalize Theorems 3.2 and 3.3 in [17].

2. Preliminaries

We make the following assumptions:

- (i) A generates a strongly continuous semigroup in a Banach space X ; and
- (ii) $Z(t) \in B(X)$, $0 \leq t \leq T$, $Z(t) : Y \rightarrow Y$ for $x(\cdot)$ continuous in Y , $AZ(\cdot)x(\cdot) \in L^1([0, T], X)$. For $x \in X$, $Z'(t)x$ is continuous in $t \in [0, T]$, where $B(X)$ is the space of all linear and bounded operators on X , and Y is the Banach space formed from $D(A)$, the domain of A , endowed with the graph norm. We give the following definitions.

Definition 2.1 (see [17]). $R(\cdot)$ is a resolvent operator of (1.6) with $f \equiv 0$ if $R(t) \in B(X)$ for $0 \leq t \leq T$ and satisfies

- (1) $R(0) = I$ (the identity operator on X);
- (2) for all $u \in X$, $R(t)u$ is continuous for $0 \leq t \leq T$; and
- (3) $R(t) \in B(Y)$, $0 \leq t \leq T$. For $y \in Y$, $R(\cdot)y \in C^1([0, T], X) \cap C([0, T], Y)$ and

$$\begin{aligned} \frac{dR(t)}{dt}y &= A \left[R(t)y + \int_0^t Z(t-s)R(s)y \, ds \right] \\ &= R(t)Ay + \int_0^t R(t-s)AZ(s)y \, ds, \quad 0 \leq t \leq T. \end{aligned}$$

Definition 2.2. $u(\cdot, u_0) \in C([0, T], X)$ is a mild solution of (1.6) and (1.7) if it satisfies

$$u(t) = R(t)[u_0 - g(u(t_1), \dots, u(t_p))] + \int_0^t R(t-s)f(s, u(\sigma(s))) \, ds.$$

Definition 2.3. A classical solution of (1.6) and (1.7) is a function $u(\cdot) \in C([0, T], Y) \cap C^1([0, T], X)$ which satisfies (1.6) and (1.7) on $[0, T]$ and we denote it by $u(\cdot, u_0)$.

The existence and uniqueness of resolvent operators is guaranteed by the following theorem, whose proof can be found in [14].

Theorem 2.4. *Let assumptions (i) and (ii) be satisfied. Then (1.6) with $f \equiv 0$ has a unique resolvent operator.*

3. Existence of solutions

Theorem 3.1. *Assume that*

(i) *A generates a strongly continuous semigroup in a Banach space X with norm $\|\cdot\|$ and $u_0 \in X$;*

(ii) *$f : J \times X \rightarrow X$ is continuous in t on J and there exists a constant $L > 0$ such that*

$$\|f(t, u_1) - f(t, u_2)\|_X \leq L\|u_1 - u_2\|_X, \quad t \in J, \quad u_1, u_2 \in X;$$

(iii) *$\sigma : J \rightarrow J$ is differentiable and $\sigma(t) \leq t$ for all t ;*

(iv) *$g : X^p \rightarrow X$ and there exists a constant $K > 0$ such that*

$$\|g(u(t_1), \dots, u(t_p)) - g(v(t_1), \dots, v(t_p))\|_X \leq K \sup_{t \in J} \|u(t) - v(t)\|_X;$$

(v) *$M = \max_{t \in J} \|R(t)\|_{B(X)}$, $q = (MK + MLT) < 1$;*

then problem (1.6) and (1.7) has a unique mild solution on J .

Proof. Let us take $E = C(J, X)$.

Then define an operator $F : E \rightarrow E$ by

$$(Fv)(t) = R(t)[u_0 - g(v(t_1), \dots, v(t_p))] + \int_0^t R(t-s)f(s, v(\sigma(s))) \, ds, \quad t \in J.$$

Now, for every $v_1, v_2 \in E$ and $t \in J$, we have

$$\begin{aligned} & \| (Fv_1)(t) - (Fv_2)(t) \|_X \\ & \leq \| R(t) \|_{B(X)} \| g(v_1(t_1), \dots, v_1(t_p)) - g(v_2(t_1), \dots, v_2(t_p)) \|_X \\ & \quad + \int_0^t \| R(t-s) \|_{B(X)} \| f(s, v_1(\sigma(s))) - f(s, v_2(\sigma(s))) \|_X \, ds \\ & \leq MK \sup_{t \in J} \| v_1(t) - v_2(t) \|_X + ML \int_0^t \| v_1(\sigma(s)) - v_2(\sigma(s)) \|_X \, ds \\ & \leq MK \sup_{t \in J} \| v_1(t) - v_2(t) \|_X + MLT \sup_{t \in J} \| v_1(t) - v_2(t) \|_X \\ & \leq (MK + MLT) \sup_{t \in J} \| v_1(t) - v_2(t) \|_X, \quad v_1, v_2 \in X, \quad t \in J \\ & \leq q \sup_{t \in J} \| v_1(t) - v_2(t) \|_X, \quad 0 < q < 1. \end{aligned}$$

This shows that operator F is a contraction on E . Applying Banach's fixed-point theorem we get a unique fixed-point for F and this point is the mild solution of problem (1.6) and (1.7) on J . \square

Remark 3.2. If the inequality in assumption (ii) holds only for u, v in a small ball in X , as shown in [9], then similar conditions as those in [9] can be obtained here.

Next, we prove that the mild solutions are classical solutions when $f \in C^1(J \times X, X)$.

Theorem 3.3. *Let assumptions (i)–(v) in Theorem 3.1 be satisfied and let $u(\cdot)$ be the unique mild solution of (1.6) and (1.7). Assume further that $u_0 \in D(A)$, $f \in C^1(J \times X, X)$ and $g : X^p \rightarrow D(A)$. Then $u(\cdot)$ is a unique classical solutions of (1.6) and (1.7).*

Proof. Since all the assumptions of Theorem 3.1 are satisfied, then problem (1.6) and (1.7) possesses a unique mild solution which is denoted by $u(\cdot)$. We will show that $u(\cdot) \in C^1(J, X)$.

Next, we shall show that this mild solution is a classical solution of problem (1.6) and (1.7) on J .

To this end, let

$$B(s) = \frac{\partial}{\partial u(\sigma(s))} f(s, u(\sigma(s))) \sigma'(s), \quad s \in J \quad (3.1)$$

and

$$\begin{aligned} k(t) = R(t)f(0, u(\sigma(0))) + A \left[R(t)u(0) + \int_0^t Z(t-s)R(s)u(0) ds \right] \\ + \int_0^t R(t-s) \frac{\partial}{\partial s} f(s, u(\sigma(s))) ds, \quad t \in J. \end{aligned} \quad (3.2)$$

Note that $u(0) \in Y$, from Definition 2.1 and our assumptions, $k(\cdot) \in E$. Thus, the method used in Pazy [18, pp. 184–187] or in the proof of the Theorem 3.3 above can be applied here to show that the integral equation

$$w(t) = k(t) + \int_0^t R(t-s)B(s)w(\sigma(s)) ds, \quad t \in J, \quad (3.3)$$

has a unique solution $w(\cdot) \in E$ (see also [1]). Moreover, from the assumptions we have

$$f(s, u(\sigma(s+h))) - f(s, u(\sigma(s))) = B(s)[u(\sigma(s+h)) - u(\sigma(s))] + w_1(s, h) \quad (3.4)$$

and

$$f(s+h, u(\sigma(s+h))) - f(s, u(\sigma(s+h))) = \frac{\partial}{\partial s} f(s, u(\sigma(s+h)))h + w_2(s, h), \quad (3.5)$$

where

$$h^{-1} \|w_i(s, h)\| \rightarrow 0, \quad \text{as } h \rightarrow 0, \quad (3.6)$$

uniformly on $s \in J$ for $i = 1, 2$.

Define

$$w_h(t) = \frac{u(t+h) - u(t)}{h} - w(t). \quad (3.7)$$

Then, from (3.1), (3.2), (3.3), (3.7) and the fact that $u(\cdot)$ is a mild solution, we obtain

$$\begin{aligned} w_h(t) &= \left(h^{-1} [R(t+h)u(0) - R(t)u(0)] - A \left[R(t)u(0) + \int_0^t Z(t-s)R(s)u(0) \, ds \right] \right) \\ &\quad + \left(h^{-1} \left[\int_0^{t+h} R(t+h-s)f(s, u(\sigma(s))) \, ds - \int_0^t R(t-s)f(s, u(\sigma(s))) \, ds \right] \right) \\ &\quad - \left[R(t)f(0, u(\sigma(0))) + \int_0^t R(t-s) \frac{\partial}{\partial s} f(s, u(\sigma(s))) \, ds \right] \\ &\quad - \int_0^t R(t-s)B(s)w(\sigma(s)) \, ds \\ &= \left(h^{-1} [R(t+h)u(0) - R(t)u(0)] - A \left[R(t)u(0) + \int_0^t Z(t-s)R(s)u(0) \, ds \right] \right) \\ &\quad + \left[h^{-1} \int_0^h R(t+h-s)f(s, u(\sigma(s))) \, ds - R(t)f(0, u(\sigma(0))) \right] \\ &\quad + h^{-1} \left[\int_0^t R(t-s) \left[\frac{\partial}{\partial s} f(s, u(\sigma(s+h)))h + w_2(s, h) \right] \, ds \right. \\ &\quad \quad \left. + \int_0^t R(t-s) \left[\frac{\partial}{\partial u(\sigma(s))} f(s, u(\sigma(s)))\sigma'(s) \right. \right. \\ &\quad \quad \quad \left. \left. \times [u(\sigma(s+h)) - u(\sigma(s))] + w_1(s, h) \right] \, ds \right] \\ &\quad - \int_0^t R(t-s) \frac{\partial}{\partial s} f(s, u(\sigma(s))) \, ds - \int_0^t R(t-s)B(s)w(\sigma(s)) \, ds \\ &= \left(h^{-1} [R(t+h)u(0) - R(t)u(0)] - A \left[R(t)u(0) + \int_0^t Z(t-s)R(s)u(0) \, ds \right] \right) \\ &\quad + \left[h^{-1} \int_0^h R(t+h-s)f(s, u(\sigma(s))) \, ds - R(t)f(0, u(\sigma(0))) \right] \\ &\quad + h^{-1} \left[\int_0^t R(t-s)[w_1(s, h) + w_2(s, h)] \, ds \right. \\ &\quad \quad \left. + \int_0^t R(t-s) \frac{\partial}{\partial s} [f(s, u(\sigma(s+h))) - f(s, u(\sigma(s)))] \, ds \right] \\ &\quad + \int_0^t R(t-s) \frac{\partial}{\partial u(\sigma(s))} f(s, u(\sigma(s)))\sigma'(s) \\ &\quad \quad \times \left[\frac{u(\sigma(s+h)) - u(\sigma(s))}{h} - w(\sigma(s)) \right] \, ds \end{aligned}$$

$$\begin{aligned}
\|w_h(t)\| \leq & \left\| h^{-1}[R(t+h)u(0) - R(t)u(0)] - A \left[R(t)u(0) + \int_0^t Z(t-s)R(s)u(0) \, ds \right] \right\| \\
& + \left\| h^{-1} \int_0^h R(t+h-s)f(s, u(\sigma(s))) \, ds - R(t)f(0, u(\sigma(0))) \right\| \\
& + \left\| h^{-1} \int_0^t R(t-s)[w_1(s, h) + w_2(s, h)] \, ds \right\| \\
& + \left\| h^{-1} \int_0^t R(t-s) \frac{\partial}{\partial s} [f(s, u(\sigma(s+h))) - f(s, u(\sigma(s)))] \, ds \right\| \\
& + N \int_0^t \|w_h(\sigma(s))\sigma'(s)\| \, ds, \quad (3.8)
\end{aligned}$$

where

$$N = \max_{t \in J} \left\| R(t-s) \frac{\partial}{\partial u(\sigma(s))} f(s, u(\sigma(s))) \right\|_{B(X)}.$$

From the definition of resolvent operator and our assumptions, it is clear that the norm of each one of the four first terms on the right-hand side of (3.8) tends to zero as $h \rightarrow 0$. Therefore, we have

$$\|w_h(t)\|_X \leq \epsilon(h) + N \int_{\sigma(0)}^{\sigma(t)} \|w_h(s)\|_X \, ds, \quad (3.9)$$

and $\epsilon(h) \rightarrow 0$ as $h \rightarrow 0$.

From (3.9), it follows by Gronwall's inequality that

$$\|w_h(t)\|_X \leq \epsilon(h)e^{TN},$$

and, therefore, $\|w_h(t)\|_X \rightarrow 0$, $h \rightarrow 0$, $t \in J$.

This implies that $u(t)$ is differentiable on J and that $w(t)$ is the derivative of $u(t)$. Since $w \in E$, u is continuously differentiable on J .

Finally, to show that u is the classical solution of problem (1.6) and (1.7). Observe that, from the continuous differentiability of u and $f \in C^1(J \times X, X)$, $t \rightarrow f(t, u(\sigma(t)))$ is continuously differentiable on J . Therefore, the linear Cauchy problem [17, Theorem 2.5]

$$\left. \begin{aligned}
v'(t) &= A \left[v(t) + \int_0^t Z(t-s)v(s) \, ds \right] + f(t, u(\sigma(t))), \quad 0 \leq t \leq T, \\
v(0) &= u_0 - g(u(t_1), \dots, u(t_p)),
\end{aligned} \right\}$$

has a unique classical solution $v(\cdot)$ given by

$$v(t) = R(t)[u_0 - g(u(t_1), \dots, u(t_p))] + \int_0^t R(t-s)f(s, u(\sigma(s))) \, ds. \quad (3.10)$$

The right-hand side of (3.10) is $u(t)$ since $u(\cdot)$ is the mild solution. So we have $v(t) = u(t)$, $t \in J$, and, hence, $u(\cdot)$ is the classical solutions of (1.6) and (1.7). Hence, the theorem is proved. \square

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