Bull. Aust. Math. Soc. **88** (2013), 123–127
doi:10.1017/S000497271200072X

CRITERIA FOR THE SEQUENCE OF DIFFERENCES OF A BOUNDED SEQUENCE TO BE NULL

DAVID BORWEIN

(Received 25 June 2012; accepted 10 July 2012; first published online 17 September 2012)

Abstract

Conditions are established for the sequence of differences $\{a_n - a_{n-1}\}$ of a bounded sequence $\{a_n\}$ of complex terms to converge to zero when a certain linear nonhomogeneous difference expression of the form $k_0 a_n + k_1 a_{n-1} + \cdots + k_n a_0$ tends to zero as $n \to \infty$.

2010 *Mathematics subject classification*: primary 40A05.

Keywords and phrases: sequence, bounded, linear nonhomogeneous difference expression.

1. Introduction and the main results

Suppose throughout that $K(z) := \sum_{n=0}^{\infty} K_n z^n$, where K_n is complex, and that $k_n = K_n - K_{n-1}$ with $K_{-1} := 0$. Let D be the open unit disc $\{z : |z| < 1\}$, let $\bar{D}$ be its closure, and let $\partial D := \bar{D} \setminus D$.

The object of this paper is to prove Theorems 1.1 and 1.2 stated below.

THEOREM 1.1. *If*

$$\sum_{n=0}^{\infty} |K_n| < \infty, \tag{1.1}$$

$$K(z) \neq 0 \quad \text{on } \partial D, \tag{1.2}$$

and if

$$\{a_n\} \text{ is a bounded complex sequence} \tag{1.3}$$

such that

$$\lim_{n\to\infty} \sum_{r=0}^{n} k_r a_{n-r} = 0, \tag{1.4}$$

then $\lim_{n\to\infty}(a_n - a_{n-1}) = 0$.

The next theorem shows that condition (1.2) is necessary in a sense for the validity of Theorem 1.1.

THEOREM 1.2. *If* $K(z) = p(z)q(z)$ *where* $p(z)$ *is a polynomial and* $q(z) = \sum_{n=0}^{\infty} q_n z^n$, *and if*

$$\sum_{n=0}^{\infty} |q_n| < \infty, \tag{1.5}$$

$$q(z) \neq 0 \quad \text{on } \bar{D}, \tag{1.6}$$

$$K(\zeta) = 0 \quad \text{for } \zeta \neq 1,\ |\zeta| = 1, \tag{1.7}$$

then there exist a bounded sequence $\{a_n\}$ *and a positive integer* N *such that*

$$\sum_{r=0}^{n} k_r a_{n-r} = 0 \quad \text{for all } n \geq N, \tag{1.8}$$

but $\{a_n - a_{n-1}\}$ *does not converge.*

Note that (1.5) in fact implies (1.1).

Theorem 1.1 generalises the following theorem proved by Stević [4].

THEOREM S. *If* $k_0 = -1$, $\sum_{n=1}^{N} k_n = 1$ *with* k_n *real,* $\sum_{n=0}^{N} k_n z^n \neq 0$ *on* $\partial D \setminus \{1\}$, *and if* $\{a_n\}$ *is a bounded real sequence such that* $\lim_{n\to\infty} \sum_{r=0}^{N} k_r a_{n-r} = 0$, *then*

$$\lim_{n\to\infty} (a_n - a_{n-1}) = 0.$$

That Theorem S is a special case of Theorem 1.1 can be seen by taking $K_0 = -1$, $K_n = 0$ for $n > N$, and observing that $\sum_{n=0}^{N} k_n z^n = (1 - z)K(z)$. In [4] Stević cites many examples from mathematical biology which use results of this type, and also produces an extensive list of related results. A companion to Theorem 1.1 is the following result proved in [1].

THEOREM B. *If (1.1) and (1.2) hold, and if* $\{a_n\}$ *is a bounded real sequence such that* $\sum_{r=0}^{n} k_r a_{n-r} \geq 0$ *for all* n *larger than some positive integer* N, *then* $\{a_n\}$ *is convergent.*

Theorem B generalises a theorem of Copson's [2] which in turn generalises the result that a bounded monotonic real sequence converges. Incidentally, Stević in [3] also proved a slight generalisation of Copson's theorem, but failed to observe that his result was in fact a special case of the earlier Theorem B.

2. An auxiliary result

Our proof of Theorem 1.1 is largely modelled on the proof of Theorem B [1, Theorem 1]. We require the following lemma.

LEMMA 2.1. *Suppose that (1.1)–(1.4) hold, and that* $K(\alpha) = 0$ *with* $0 < |\alpha| < 1$. *Then*

$$\frac{1}{\alpha - z} K(z) = \sum_{n=0}^{\infty} P_n z^n \quad \text{where} \quad \sum_{n=0}^{\infty} |P_n| < \infty,$$

https://doi.org/10.1017/S000497271200072X Published online by Cambridge University Press

and

$$\lim_{n\to\infty}\sum_{r=0}^{n} p_r a_{n-r} = 0 \quad \textit{with } p_r := P_r - P_{r-1},\ P_{-1} := 0.$$

Proof. Since $K(\alpha) = 0$,

$$\alpha P_n = \sum_{r=0}^{n} \alpha^{r-n} K_r = -\sum_{r=n+1}^{\infty} \alpha^{r-n} K_r,$$

and so, by (1.1),

$$\sum_{n=0}^{\infty} |P_n| \le \sum_{r=1}^{\infty} |K_r| \sum_{n=0}^{r-1} |\alpha|^{r-1-n} \le \frac{1}{1-|\alpha|} \sum_{r=1}^{\infty} |K_r| < \infty.$$

Now let

$$v_n := \sum_{r=0}^{n} K_r a_{n-r}, \quad u_n := \sum_{r=0}^{n} P_r a_{n-r},$$

$$a(z) := \sum_{n=0}^{\infty} a_n z^n, \quad v(z) := \sum_{n=0}^{\infty} v_n z^n \quad \text{and} \quad u(z) := \sum_{n=0}^{\infty} u_n z^n.$$

Then, by (1.4),

$$v_n - v_{n-1} = \sum_{r=0}^{n} k_r a_{n-r} \to 0 \quad \text{as } n \to \infty.$$

Also, since $v(z) = K(z)a(z)$, so that $v(\alpha) = 0$ and $u(z) = (\alpha - z)^{-1} v(z)$,

$$u_n = -\sum_{r=n+1}^{\infty} \alpha^{r-n-1} v_r = -\sum_{r=0}^{\infty} \alpha^r v_{n+1+r},$$

and hence, by the series version of Lebesgue's dominated convergence theorem,

$$\sum_{r=0}^{n} p_r a_{n-r} = u_n - u_{n-1} = -\sum_{r=0}^{\infty} \alpha^r (v_{n+1+r} - v_{n+r}) \to 0 \quad \text{as } n \to \infty.$$

3. Proofs of the theorems

Proof of Theorem 1.1. *Case* 1. $K(0) \ne 0$. By (1.1), $K(z)$ is holomorphic on D and continuous on $\bar{D}$. Hence, by (1.2), $K(z)$ can have at most a finite number of zeros in D. We can use Lemma 2.1 to remove the zeros, and thus we may assume without loss of generality that $K(z)$ has no zeros on $\bar{D}$. Then, by the Wiener–Lévy theorem [5, p. 246],

$$\frac{1}{K(z)} = \sum_{n=0}^{\infty} c_n z^n \quad \text{for } z \in \bar{D} \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

https://doi.org/10.1017/S000497271200072X Published online by Cambridge University Press

Using the notation introduced in the previous section, we have $a(z) = v(z)c(z)$, so that $a_n = \sum_{r=0}^{n} c_r v_{n-r}$. Hence, for $w_n := \sum_{r=0}^{n} k_r a_{n-r}$, by (1.4),

$$a_n - a_{n-1} = \sum_{r=0}^{n} c_r w_{n-r} \to 0 \quad \text{as } n \to \infty.$$

Case 2. $z = 0$ is a zero of order m of $K(z)$. Since $K_m \neq 0$ and

$$z^{-m} K(z) = \sum_{n=0}^{\infty} K_{n+m} z^n,$$

it follows easily from Case 1 that $a_{n+m} - a_{n+m-1} \to 0$ as $n \to \infty$. □

Proof of Theorem 1.2. As in the proof of its companion theorem [1, Theorem 2], we define a sequence $\{a_n\}$ and a function $a(z)$ by

$$a(z) := \sum_{n=0}^{\infty} a_n z^n := \frac{1}{q(z)(\zeta - z)} \quad \text{for } z \in D. \tag{3.1}$$

Let

$$w(z) := \sum_{n=0}^{\infty} w_n z^n \quad \text{with } w_n := \sum_{r=0}^{n} k_r a_{n-r}.$$

Then

$$w(z) = (1 - z)K(z)a(z) = \frac{(1 - z)p(z)}{\zeta - z}$$

and, by (1.6) and (1.7), $\zeta - z$ is a factor of the polynomial $p(z)$. Consequently $w(z)$ is a polynomial of degree $N - 1$ say, and (1.8) follows.

Further, by the Wiener–Lévy theorem, hypotheses (1.5) and (1.6) imply that there is a sequence $\{c_n\}$ such that, for $z \in \bar{D}$,

$$\frac{1}{q(z)} = \sum_{n=0}^{\infty} c_n z^n \quad \text{and} \quad \sum_{n=0}^{\infty} |c_n| < \infty.$$

It follows, on equating coefficients in (3.1), that

$$\zeta^{n+1} a_n = \zeta^n \sum_{r=0}^{n} c_r \zeta^{r-n} \to \frac{1}{q(\zeta)} \quad \text{as } n \to \infty.$$

The sequence $\{a_n\}$ is bounded, and

$$\zeta^{n+1} a_n - \zeta^n a_{n-1} \to 0 \Rightarrow \zeta a_n - a_{n-1} \to 0 \Rightarrow a_n - a_{n-1} + (\zeta - 1)a_n \to 0.$$

Since $\{a_n\}$ does not converge, it follows that neither can $\{a_n - a_{n-1}\}$. □

https://doi.org/10.1017/S000497271200072X Published online by Cambridge University Press

References

[1] D. Borwein, 'Convergence criteria for bounded sequences', *Proc. Edinb. Math. Soc.* **18** (1972), 99–103.

[2] E. T. Copson, 'On a generalisation of monotonic sequences', *Proc. Edinb. Math. Soc.* **17** (1970), 159–164.

[3] S. Stević, 'A note on bounded sequences satisfying linear inequality', *Indian J. Math.* **43** (2001), 223–230.

[4] S. Stević, 'A note on bounded sequences satisfying linear nonhomogeneous difference equation', *Indian J. Math.* **45** (2003), 357–367.

[5] A. Zygmund, *Trigonometric Series, 1* (Cambridge University Press, Cambridge, 1959).

DAVID BORWEIN, Department of Mathematics, Western University,
London ON, Canada N6A 5B7
e-mail: dborwein@uwo.ca

https://doi.org/10.1017/S000497271200072X Published online by Cambridge University Press