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TOPOLOGICAL COMPLETENESS OF FUNCTION SPACES ARISING IN THE HAUSDORFF APPROXIMATION OF FUNCTIONS

(Dedicated to Professor Bl. Sendov)

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ABSTRACT. Let X be a complete metric space. Viewing continuous real functions on X as closed subsets of $X \times R$, equipped with Hausdorff distance, we show that $C(X, R)$ is completely metrizable provided X is complete and sigma compact. Following the Bulgarian school of constructive approximation theory, a bounded discontinuous function may be identified with its completed graph, the set of points between the upper and lower envelopes of the function. We show that the space of completed graphs, too, is completely metrizable, provided X is locally connected as well as sigma compact and complete. In the process, when X is a Polish space, we provide a simple answer to the following foundational question: which subsets of $X \times R$ arise as completed graphs?

1. **Introduction.** The recent publication by Kluwer Publishers of the English edition of *Hausdorff Approximations* by Bl. Sendov [Se] marks roughly thirty years of research by the Bulgarian school on the approximation of bounded real functions, both continuous and discontinuous, with respect to Hausdorff distance. Let $\langle X, d \rangle$ be a metric space and let $C(X, R)$ be the continuous real functions on X. In classical approximation theory, one normally works with uniform distance between continuous functions, where the distance between functions is the "maximal"vertical gap between their graphs. In the Hausdorff approximation of functions, this vertical bias is removed, and functions are viewed as closed sets in $X \times R$, equipped with Hausdorff distance induced by the box metric ρ on $X \times R$. Precisely, for f, g bounded continuous real valued functions on X, the Hausdorff distance $H_\rho(f, g)$ between them is given by the formula

$$H_\rho(f,g) = \max\Big\{ \sup_{x\in X} \inf_{z\in X} \max\{d(x,z), |f(x)-g(z)|\}, \sup_{x\in X} \inf_{z\in X} \max\{d(x,z), |g(x)-f(z)|\}\Big\}.$$

For bounded possibly discontinuous functions f and g, the distance between them is computed in terms of the Hausdorff distance between two auxiliary sets $\bar{f}$ and $\bar{g}$, called

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the completed graphs of the respective functions. The completed graph of a function is simply the set of points in $X \times R$ lying between its upper and lower envelopes.

One characteristic of the Hausdorff distance, even in the nicest spaces, is that a sequence of bounded continuous functions can converge in Hausdorff distance to a set that is not only not the graph of a continuous function, but is also not even the completed graph of a bounded function. For example, for $n > 3$, let $f_n \in C([0,1], R)$ be the piecewise linear function whose graph connects the following points in succession:

$$(0,0),\ (1/n,1),\ (2/n,-1),\ (3/n,0),\ \text{and}\ (1,0)$$

Then $\langle f_n \rangle$ converges in Hausdorff distance in $[0,1] \times R$ to the T-shaped set

$$A = \{(x,0) : 0 \le x \le 1\} \cup \{(0,y) : -1 \le y \le 1\}.$$

Evidently, A cannot be the completed graph of any function $f : [0,1] \longrightarrow R$ because we would have to have $f(x) = 0$ for each $x > 0$.

The above example shows that neither the space of bounded continuous functions on $[0,1]$ nor the space of completed graphs, equipped with Hausdorff distance, is a complete metric space. In this paper we show, much more generally, that these spaces are, however, completely metrizable. In the process we are forced to address the following basic question: exactly what subsets of $X \times R$ are completed graphs of bounded functions? When X is a Polish space, we provide a simple but nontrivial characterization.

2. **Preliminaries.** Let $\langle X, d \rangle$ be a metric space. If A is a closed subset of X and $x \in X$, we write $d(x, A)$ for $\inf\{d(x,a) : a \in A\}$.

We denote the nonempty closed subsets of $\langle X, d \rangle$ by CL(X). For $A \in \mathrm{CL}(X)$ and $\varepsilon > 0$, we denote by $S_\varepsilon[A]$ the *ε-parallel body of* A, $\{x \in X : d(x,A) < \varepsilon\}$. We may now define *Hausdorff distance* in CL(X) as follows (see, e.g., [Ha, CV, KT, Se]):

$$\begin{aligned} H_d(A,B) &= \max\{\sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A)\} \\ &= \inf\{\varepsilon > 0 : A \subset S_\varepsilon[B] \text{ and } B \subset S_\varepsilon[A]\}. \end{aligned}$$

As is well-known, if a sequence $\langle A_n \rangle$ is convergent in Hausdorff distance to A, then $A = \mathrm{Li}\, A_n = \mathrm{Ls}\, A_n$, where $\mathrm{Li}\, A_n$ (resp. $\mathrm{Ls}\, A_n$) consists of all points $x \in X$ each neighborhood of which meets all but finitely many (resp. meets infinitely many) of the sets A_n. Conversely, when X is compact, then $A = \mathrm{Li}\, A_n = \mathrm{Ls}\, A_n$ implies $\lim_{n \to \infty} H_d(A_n, A) = 0$ [Ha, §28].

Hausdorff distance so defined yields an infinite valued metric on CL(X), which is complete provided $\langle X, d \rangle$ is complete [CV, KT]. Uniformly equivalent metrics give rise to uniformly equivalent Hausdorff metrics; thus, for the purposes of completeness questions, if the reader is bothered by infinite values, the distance $H_d(A,B)$ may be replaced by the finite distance $\min\{H_d(A,B), 1\}$. We denote the topology on CL(X) induced by Hausdorff distance by τ_{H_d} in the sequel.

If $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ are metric spaces, the *box metric* ρ on $X \times Y$ will be understood:

$$\rho[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), d_Y(y_1, y_2)].$$

In view of the remarks just made, using any other standard matrix on $X \times Y$ gives rise to a uniformly equivalent Hausdorff metric on $\mathrm{CL}(X \times Y)$. We denote the continuous functions from X to Y by $C(X, Y)$. Since elements of $C(X, Y)$ are closed subsets of $X \times Y$, we may speak of the Hausdorff distance between them with respect to the box metric. In particular, if $Y = R$ with the usual metric, the Hausdorff distance between two real continuous functions f and g is given by the formula of §1. As is well known, the topology of uniform convergence is finer than the H_ρ-topology on $C(X, Y)$, and provided Y has some nontrivial path, the topologies agree if and only if each element of $C(X, R)$ is uniformly continuous [Be3]. Spaces on which real continuous functions are uniformly continuous properly contain the compact spaces; they are called UC-*spaces* or *Atsuji spaces* in the literature, and admit a number of beautiful characterizations [At, To, Ra, Wa, Be3, Be5, RZ]. As noted in the introduction, if $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ are complete metric spaces, then $\langle C(X, Y), \tau_{H_\rho}\rangle$ need not be a complete metric space, unlike $C(X, Y)$ equipped with the uniform metric. Completeness criteria for subfamilies of $\langle C(X, Y), \tau_{H_\rho}\rangle$ are explored in [Be6]. For other work on continuous functions and Hausdorff distance in the context of metric spaces, the reader may consult [Bo, HN, Ho, Na].

If $A \in \mathrm{CL}(X \times Y)$, we write $A(x)$ for the *slice* $\{y \in Y : (x, y) \in A\}$, and $A^*(x)$ for the *stalk* $\{(x, y) : y \in A(x)\}$. Evidently, $A(x)$ is a closed, but possibly empty, subset of Y. We write $\mathrm{CL}_0(X \times Y)$ for $\{A \in \mathrm{CL}(X \times Y) : \forall x \in X, A(x) \neq \emptyset\}$.

By a *multifunction* or *correspondence* Γ from X to Y, we mean a function that assigns to each $x \in X$ a closed, but perhaps empty, subset $\Gamma(x)$ of Y. By the *graph* of Γ we mean this subset of $X \times Y$:

$$\{(x, y) : x \in X \text{ and } y \in \Gamma(x)\}.$$

Excluding the empty multifunction, there is a one-to-one correspondence between $\mathrm{CL}(X \times Y)$ and the multifunctions with closed graph from X to Y, given by $A \leftrightarrow \Gamma_A$ where $\Gamma_A(x) = A(x)$. We call a multifunction Γ *upper semicontinuous* (u.s.c) [Kt, En] provided for each open subset V of Y, $\{x \in X : \Gamma(x) \subset V\}$ is open in X. Upper semicontinuous multifunctions have closed graphs, and a multifunction with closed graph with values in a compact target space is automatically u.s.c. [KT, p. 78]. As a counterexample in R^2,

$$A = \{(x, y) : x \neq 0 \text{ and } y = 1/x\} \cup \{(0, 0)\}$$

is closed, but $x \longrightarrow A(x)$ is not an upper semicontinuous multifunction. We call a multifunction Γ *bounded* provided $\bigcup_{x\in X} \Gamma(x)$ is a bounded subset of Y.

In the sequel, we will freely identify a multifunction with closed graph with its graph.

3. **Topological completeness of $C(X, Y)$ with the Hausdorff metric topology.** Let $\langle X, d_X\rangle$ and $\langle Y, d_Y\rangle$ be complete metric spaces. In this section, we obtain sufficient conditions for the complete metrizability of $\langle C(X,Y), \tau_{H_\rho}\rangle$ where, as usual, ρ is the box matrix on $X \times Y$. Since $\langle X \times Y, \rho\rangle$ is a complete metric space, so is the hyperspace $\langle \mathrm{CL}(X \times Y), \tau_{H_\rho}\rangle$ [KT, Theorem 4.3.8]. Thus, by the celebrated theorem of Alexandroff (see, e.g., [En, p. 342]), complete metrizability of $\langle C(X,Y), \tau_{H_\rho}\rangle$ holds provided we can show that $C(X, Y)$ is a G_δ-subset of $\langle \mathrm{CL}(X \times Y), \tau_{H_\rho}\rangle$. This will be accomplished for X σ-compact, and Y either compact or a Euclidean space. We obtain these results through a series of lemmas, some of which will be used in the next section where we consider completed graphs.

LEMMA 3.1. *Let $\langle X, d_X\rangle$ be a metric space and let $\langle Y, d_Y\rangle$ be a metric space in which closed and bounded sets are compact. Let $\langle A_n\rangle$ be a sequence of bounded multifunctions in $\mathrm{CL}_0(X \times Y)$ H_ρ-convergent to $A \in \mathrm{CL}(X \times Y)$. Then $A \in \mathrm{CL}_0(X \times Y)$.*

PROOF. Evidently, the multifunctions are uniformly bounded. Fix $x \in X$. For each $n \in Z^+$, choose $y_n \in A_n(x)$. By the compactness of closed and bounded subsets of Y, some subsequence of $\langle y_n\rangle$ is convergent to a point $y \in Y$. This means that $(x, y) \in \mathrm{Ls}\, A_n$, and by Hausdorff metric convergence, $\mathrm{Ls}\, A_n = A$. Thus, $y \in A(x)$ and $A \in \mathrm{CL}_0(X \times Y)$. ∎

EXAMPLE. In the plane, $A = \{(x, y) : y = 1/x \text{ and } x \neq 0\}$ is the Hausdorff metric limit of $\langle A_n\rangle$, where $A_n = \{(x, y) : y = 1/x \text{ and } x \neq 0\} \cup \{(0, n)\}$.

COROLLARY 3.2. *Let $\langle X, d_X\rangle$ be a metric space and let $\langle Y, d_Y\rangle$ be a compact metric space. Then $\mathrm{CL}_0(X \times Y)$ is a closed subset of $\langle \mathrm{CL}(X \times Y), \tau_{H_\rho}\rangle$.*

A standard proof [En, p. 344] of the Lebesgue covering lemma for a compact metric space yields something a little stronger, that we require in Lemma 3.3 below.

EXTENDED LEBESGUE COVERING LEMMA. Let $\langle X, d\rangle$ be a metric space and let K be a compact subset of X. Let $\{V_i : i \in I\}$ be a family open subsets of X with $K \subset \bigcup_{i \in I} V_i$. Then there exists $\delta > 0$ such that for each subset A of X of diameter less that δ that meets K, there exists $i \in I$ with $A \subset V_i$.

LEMMA 3.3. *Let $\langle X, d_X\rangle$ be a sigma compact metric space and let $\langle Y, d_Y\rangle$ be a compact metric space. Then $C(X, Y)$ is a G_δ-subset of $\langle \mathrm{CL}_0(X \times Y), \tau_{H_\rho}\rangle$.*

PROOF. Write $X = \bigcup_{n=1}^\infty K_n$ with each K_n compact. Let Ω_n be the following set of relations:

$$\Omega_n = \{A \in \mathrm{CL}_0(X \times Y) : \ \forall x \in K_n \quad A(x) \text{ is a singleton}\}.$$

The set of functions with closed graph from X to Y is clearly $\bigcap_{n=1}^\infty \Omega_n$, and since each such function determines an upper semicontinuous multifunction [KT, Theorem 7.1.16], each such function is necessarily continuous. Thus, to establish the assertion of the lemma, it suffices to show that each Ω_n is a G_δ-subset of $\mathrm{CL}_0(X \times Y)$.

To this end, fix $n \in Z^+$. Let $\Delta_{nk} = \{A \in \mathrm{CL}_0(X \times Y) : \ \forall x \in K_n, \operatorname{diam} A(x) < 1/k\}$. Now if $A \in \Delta_n$, for each x in K_n there exists an open subset V_x of Y containing $A(x)$ with

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diam $V_x < 1/k$. By the upper semicontinuity of $x \longrightarrow A(x)$, there exists $\{x_1, x_2, \ldots, x_m\} \subset K_n$ and positive numbers $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_m\}$ such that $K_n \subset \bigcup_{i=1}^m S_{\varepsilon_i}[x_i]$, and whenever $x \in S_{\varepsilon_i}[x_i]$, we have $A(x) \subset V_{x_i}$. Choose $\delta > 0$ such that each subset of X of diameter less than δ that meets K_n is contained in $S_{\varepsilon_i}[x_i]$ for some i. Let λ be this positive number

$$\lambda = \frac{1}{2}\left[\frac{1}{k} - \max\{\text{diam}\, V_{x_i} : 1 \leq i \leq m\}\right].$$

Now suppose $B \in CL_0(X \times Y)$ and $H_\rho(A, B) < \min\{\frac{\lambda}{2}, \frac{\delta}{2}\}$. We claim that $B \in \Delta_{nk}$.

To see this, fix $x \in K_n$ and $y \in B(x)$, and choose $i \leq m$ such that $S_{\delta/2}[x] \subset S_{\varepsilon_i}[x_i]$. There exists $(x_A, y_A) \in A$ with $\rho[(x, y), (x_A, y_A)] < \min\{\frac{\lambda}{2}, \frac{\delta}{2}\}$. Since $x_A \in S_{\varepsilon_i}[x_i]$, we have $y_A \in V_{x_i}$. Thus, $y \in S_{\lambda/2}(V_{x_i})$, so that $B(x) \subset S_{\lambda/2}(V_{x_i})$, a set with diameter at most diam $V_{x_i} + \lambda < 1/k$. This proves that $B \in \Delta_{nk}$ so that Δ_{nk} is open in the relative Hausdorff metric topology.

Finally, for each n, we have $\Omega_n = \bigcap_{k=1}^\infty \Delta_{nk}$, completing the proof. ■

Putting together the last two results, we have in view of Alexandroff's Theorem

THEOREM 3.4. *Let $\langle X, d_X \rangle$ be a sigma compact complete metric space and let $\langle Y, d_Y \rangle$ be a compact metric space. Then $\langle C(X, Y), \tau_{H_\rho} \rangle$ is a G_δ-subset of $\langle \mathrm{CL}(X \times Y), \tau_{H_\rho} \rangle$ and is thus completely metrizable.*

We next wish to show the last result is valid with a Euclidean target space in lieu of a compact one. We need one more lemma.

LEMMA 3.5. *Let $\langle X, d \rangle$ be a sigma compact metric space and let $\langle Y, d \rangle$ be a metric space. Suppose K is a nonempty compact subset of Y and $\Sigma = \{A \in \mathrm{CL}(X \times Y) : A \cap (X \times K) = \emptyset\}$. Then Σ is a G_δ-subset of $\langle \mathrm{CL}(X \times Y), \tau_{H_\rho} \rangle$.*

PROOF. Write $X = \bigcup_{n=1}^\infty K_n$ with each K_n compact. Evidently,

$$\Sigma = \bigcap_{n=1}^{\infty} \{A \in \mathrm{CL}(X \times Y) : A \cap (K_n \times K) = \emptyset\}.$$

Since the set of closed sets that miss a fixed compact set is open in the Hausdorff metric topology and $K_n \times K$ is compact for each n, the result follows. ■

THEOREM 3.6. *Let $\langle X, d_X \rangle$ be a complete sigma compact metric space. Then $\langle C(X, R^n), \tau_{H_\rho} \rangle$ is completely metrizable.*

PROOF. Let U be the closed unit ball of R^n, equipped with the usual metric. For each $A \in \mathrm{CL}(X \times R^n)$, let $\hat{A}$ be this subset of $\mathrm{CL}(X \times U)$:

$$\hat{A} = \mathrm{cl}\Big\{(x, y) : \frac{y}{1 - ||y||} \in A(x)\Big\}.$$

The map $A \longrightarrow \hat{A}$ is nonexpansive (Lipschitz with constant one) with respect to Hausdorff distance and is thus continuous. Moreover, its restriction to $C(X, R^n)$ is 1–1 onto $C(X, \mathrm{int}\, U)$. By Lemma 3.5 and Theorem 3.4, $C(X, \mathrm{int}\, U)$ is a G_δ-subset of $\mathrm{CL}(X \times U)$. Hence, the inverse image of $C(X, \mathrm{int}\, U)$, namely $C(X, R^n)$, is a G_δ-subset of a complete metric space. ■

COROLLARY 3.7. *Let $\langle X, d_X \rangle$ be a complete sigma compact metric space. Then the bounded continuous real functions from X to R^n, equipped with τ_{H_ρ}, are completely metrizable.*

PROOF. The bounded multifunctions form an open and closed subset of $\mathrm{CL}(X \times R^n)$, so that the bounded continuous functions are the intersection of two G_δ-subsets of a completely metrizable space. ∎

4. **Topological completeness of the space of completed graphs of Sendov.** Following Sendov [Se, §1.3], we first introduce the concept of the completed graph of a bounded real function. Let $\langle X, d \rangle$ be a metric space, and let $f: X \longrightarrow R$ be a bounded function with no assumed continuity properties whatsoever. Recall that the *upper envelope* of such a function is the infimum of the upper semicontinuous functions that majorize f, and the *lower envelope* of such a function is the supremum of the lower semicontinuous functions that f majorizes. We denote the upper and lower envelopes of f by $u(f;\cdot)$ and $l(f;\cdot)$, respectively. At each $x \in X$, we have

$$u(f;x) = \inf_{\delta>0} \sup_{w \in S_\delta[x]} f(w) \quad \text{and} \quad l(f;x) = \sup_{\delta>0} \inf_{w \in S_\delta[x]} f(w)$$

Of course, for each x, $l(f;x) \le f(x) \le u(f;x)$, and $u(f;\cdot)$ (resp. $l(f;\cdot)$) is a bounded upper (resp. lower) semicontinuous function. This means that both

$$\operatorname{hypo} u(f;\cdot) = \{(x,y) : x \in X, y \in R, \text{ and } y \le u(f;x)\}$$

and

$$\operatorname{epi} l(f;\cdot) = \{(x,y) : x \in X, y \in R, \text{ and } y \ge l(f;x)\}$$

are closed subsets of $X \times R$ [Au, p. 106]. As a result,

$$\bar{f} = \operatorname{hypo} u(f;\cdot) \cap \operatorname{epi} l(f;\cdot)$$

belongs to $\mathrm{CL}_0(X \times R)$ and moreover, has convex slices. We call $\bar{f}$ the *completed graph* of f. We remark that the completed graph of f is the minimal (with respect to set inclusion) upper semicontinuous convex valued multifunction that contains the graph of f.

We denote by $\mathrm{CG}(X)$ those multifunctions from X to R that arise as the completed graph of some bounded single valued function from X to R. A necessary condition for a nonempty closed subset A of $X \times R$ to be in $\mathrm{CG}(X)$ is that A determines a bounded convex valued multifunction. But this is not sufficient, as we noted in the introduction.

To see what in addition is required, we find it convenient to introduce some more notation. If $A \in \mathrm{CL}_0(X \times R)$ and A determines a convex valued bounded multifunction, we now write $u_A(x)$ for $\max\{y : y \in A(x)\}$ and $l_A(x)$ for $\min\{y : y \in A(x)\}$. By upper semicontinuity of $x \longrightarrow A(x)$ it is clear that u_A (resp. l_A) is an upper (resp. lower) semicontinuous real function.

THEOREM 4.1. *Let $\langle X, d\rangle$ be a Polish space, i.e. X is separable and completely metrizable. Let $A \in \mathrm{CL}_0(X \times R)$. The following are equivalent:*

(i) A is bounded with convex slices and for each $x \in X$ with $u_A(x) \neq l_A(x)$, either $\big(x, u_A(x)\big)$ or $\big(x, l_A(x)\big)$ belongs to $\mathrm{cl}\big(A - A^*(x)\big)$;

(ii) A is the completed graph of a bounded function from X to R.

PROOF. (ii) $\Rightarrow$ (i). We only check the last condition in (i). Suppose $A = \bar{f}$ for some bounded function f, and x is a point at which $u_A(x) \neq l_A(x)$. Now $A = \bar{f}$ implies $u_A = u(f; \cdot)$ and $l_A = l(f; \cdot)$. Suppose that neither $\big(x, u_A(x)\big)$ nor $\big(x, l_A(x)\big)$ belongs to $\mathrm{cl}\big(A - A^*(x)\big)$. In view of the semicontinuity of u_A and v_A, this means that there exists $\delta > 0$ such that

$$u_A(x) > \sup\{u_A(w) : 0 < d(w,x) < \delta\} \text{ and } l_A(x) < \inf\{l_A(w) : 0 < d(w,x) < \delta\},$$

so that

$$u(f;x) > \sup\{f(w) : 0 < d(w,x) < \delta\} \text{ and } l(f;x) < \inf\{f(w) : 0 < d(w,x) < \delta\}.$$

Since $u(f;x) = \inf_{\delta>0} \sup\{f(w) : d(w,x) < \delta\}$, we must have $u(f;x) = f(x)$, and similarly, $l(f;x) = f(x)$. This implies that $u_A(x) = l_A(x)$, a contradiction.

(i) $\Rightarrow$ (ii). Let us call a stalk $A^*(x)$ singular if either $\big(x, u_A(x)\big)$ or $\big(x, l_A(x)\big)$ fails to belong to $\mathrm{cl}\big(A - A^*(x)\big)$. By hypothesis, this cannot be the case for both points if $u_A(x) \neq l_A(x)$. Moreover, since the product topology for $X \times R$ has a countable base, it is easy to see that there can be at most countably many singular stalks.

We first produce a countable subset of the union of the graphs of u_A and l_A, whose ordered pairs have different first coordinates, that is dense in the union. Let $V_1, V_2, V_3, \ldots$ be those basic open sets in a fixed countable base for the topology of $X \times R$ that meet the union of the graphs. For each n we inductively produce a finite sequence $(x_1, y_1), \ldots, (x_n, y_n)$ of points (not necessarily distinct) in $X \times R$ such that

(a) $(x_i, y_i) \in V_i$;

(b) $(x_i, y_i) \neq (x_k, y_k)$ implies $x_i \neq x_k$;

(c) for each i, $y_i = u_A(x_i)$ or $y_i = l_A(x_i)$;

(d) if $A(x_i)$ is a singular stalk, then $(x_i, y_i) \notin \mathrm{cl}\big(A - A^*(x)\big)$.

To describe this procedure, suppose $(x_1, y_1), \ldots, (x_n, y_n)$ have already been chosen. Choose $x \in X$ such that either $\big(x, u_A(x)\big) \in V_{n+1}$ or $\big(x, l_A(x)\big) \in V_{n+1}$. We consider only the first case; the second is similar and is left to the reader.

If $\big(x, u_A(x)\big) \notin \mathrm{cl}\big(A - A^*(x)\big)$, set $(x_{n+1}, y_{n+1}) = \big(x, u_A(x)\big)$. This choice satisfies (a) for $i = n+1$, (c) and (d), and by (i) it also satisfies (b). Otherwise, choose $\varepsilon > 0$ such that

$$S_{2\varepsilon}[x] \times \big(u_A(x) - 2\varepsilon, u_A(x) + 2\varepsilon\big) \subset V_{n+1}.$$

Choose by upper semicontinuity $\delta < \varepsilon$ such that $u_A(w) < u_A(x) + \varepsilon$ whenever $d(x,w) \leq \delta$. Since we are now assuming that $\big(x, u_A(x)\big) \in \mathrm{cl}\big(A - A^*(x)\big)$, the set

$$E = \Big\{w \in X : 0 < d(x,w) < \delta \text{ and } A(w) \cap \big(u_A(x) - \varepsilon, u_A(x) + \varepsilon\big) \neq \emptyset\Big\}$$

is nonempty. We consider two possibilities for E: E is not dense in itself, or E is dense in itself. In the first case, E has an isolated point w, and since $\big(w, u_A(w)\big) \in S_\delta[x] \times \big(u_A(x) - \varepsilon, u_A(x) + \varepsilon\big)$, we must have $\big(w, u_A(w)\big) \notin \operatorname{cl}\big(A - A^*(w)\big)$. We may safely set $(x_{n+1}, y_{n+1}) = \big(w, u_A(w)\big)$, consistent with the inductive procedure as before. Otherwise, the set $\operatorname{cl} E$ is a perfect subset of the complete metric space X, and is thus uncountable. Since the set of singular stalks is countable, there exists $w \in \operatorname{cl} E$ such that $w \neq x_i$ for $i = 1, 2, 3, \ldots, n$ and $A^*(w)$ is nonsingular. Since A is a closed set, we see that $A(w) \cap [u_A(x) - \varepsilon, u_A(x) + \varepsilon] \neq \emptyset$ and by the choice of δ, we have $u_A(w) < u_a(x) + \varepsilon$. We conclude that

$$\big(w, u_A(w)\big) \in S_{2\delta}[x] \times \big(u_A(x) - 2\varepsilon, u_A(x) + 2\varepsilon\big) \subset V_{n+1}.$$

We now set $(x_{n+1}, y_{n+1}) = \big(w, u_A(w)\big)$, and conditions (a)–(d) also hold for this choice. This verifies the validity of induction procedure as claimed.

We are now ready to define $f: X \longrightarrow R$ for which $\bar{f} = A$. Let $X_0 = \{x_i : i \in Z^+\}$. For each $i \in Z^+$, let $f(x_i) = y_i$ and for each $x \notin X_0$, let $f(x) = u_A(x)$. By (b) in the induction procedure, f is a function. By (c) in the induction procedure, for each x, we have $l_A(x) \leq f(x) \leq u_A(x)$; so by the lower (resp. upper) semicontinuity of l_A (resp. u_A), we have

$$l_A(x) \leq l(f; x) \text{ and } u(f; x) \leq u_A(x).$$

On the other hand, the density of the graph of $f|X_0$ in the union of the graphs of l_A and u_A as guaranteed by (a) and (c) in the induction procedure yields

$$l(f; x) \leq l_A(x) \text{ and } u_A(x) \leq u(f; x).$$

Together these yield $\bar{f} = A$. ∎

We are able to establish the completeness of the space of completed graphs when X is sigma compact, complete, and locally connected. Local connectedness may seem out of place here, but it has reared its head in a number of function space questions (see, e.g., [Be4, Po, MH]). We will use the following result that is a special case of Theorem 8 of [Be2]. Example 2 of [Be2] and Example 3 of [Be1] show that this result fails if (i) locally connected is replaced by connected; (ii) R is replaced by R^2.

THEOREM. *Let $\langle X, d\rangle$ be a locally connected metric space. Let $\langle A_n\rangle$ be a sequence in* $\mathrm{CL}_0(X \times R)$ *such that for each n, A_n is a convex valued upper-semicontinuous multifunction. If $A = H_\rho - \lim A_n$, then A is also a convex valued upper semicontinuous multifunction.*

Let $\mathrm{CVB}(X) = \{A \in \mathrm{CL}_0(X \times R) : A$ determines a bounded convex valued multifunction from X to $R\}$.

LEMMA 4.2. *Let $\langle X, d\rangle$ be a locally connected metric space. Then* $\mathrm{CVB}(X)$ *is a closed subset of* $\langle \mathrm{CL}(X \times R), \tau_{H_\rho}\rangle$.

PROOF. Evidently, the bounded multifunctions form a closed (and open) subset of $\mathrm{CL}(X \times R)$. Furthermore, by Lemma 3.1 the limit of a sequence of bounded multifunctions

$\langle A_n \rangle$ where for each n, $A_n \in \mathrm{CL}_0(X \times R)$, must also be in $\mathrm{CL}_0(X \times R)$. Since bounded multifunctions into R are automatically upper semicontinuous, the result now follows from the previous cited theorem from [Be2]. ■

THEOREM 4.3. *Let $\langle X, d \rangle$ be sigma compact, complete, and locally connected. Then the completed graph space* $\mathrm{CG}(X)$ *is G_δ-subset of* $\langle \mathrm{CL}(X \times R), \tau_{H_\rho} \rangle$ *and is thus completely metrizable.*

PROOF. Note that X is a Polish space, so that we may use the description of relations that are completed graphs furnished by Theorem 4.1. By Lemma 4.2, it suffices to show that $\mathrm{CG}(X)$ is a G_δ-subset of $\mathrm{CVB}(X)$. We show that the completement is an F_σ-subset. As usual write, $X = \bigcup_{n=1}^{\infty} K_n$ with each K_n compact. For positive integers n, j and k with $k \geq 2j$, define $\Delta(n,j,k) \subset \mathrm{CVB}(X)$ as follows:

$$\Delta(n,j,k) = \Big\{A : \text{there exists } x \in K_n \text{ and points } y_1 \text{ and } y_2 \text{ in } A(x) \text{ with } y_1 + 1/j \leq y_2,$$
$$\big(S_{1/k}[x] \times (-\infty, y_1 + 1/k)\big) \cap \big(A - A^*(x)\big) = \emptyset, \text{ and}$$
$$\big(S_{1/k}[x] \times (y_2 - 1/k, \infty)\big) \cap \big(A - A^*(x)\big) = \emptyset\Big\}.$$

It is routine to show that $\Delta(n,j,k)$ is closed in $\mathrm{CVB}(X)$. Since condition (i) of Theorem 4.1 is satisfied if and only if $A \notin \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{\infty} \bigcup_{k=2j}^{\infty} \Delta(n,j,k)$, the result follows. ■

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