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NONDIFFERENTIABLE PROGRAMMING AND DUALITY WITH MODIFIED CONVEXITY

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One of the purposes of this thesis is to show that mathematical programming duality theories, which have evolved under the traditional convexity assumptions, can be developed in a more general setting of modified convexity (namely, convexlike or ρ-convexity). The benefit of doing this is not only that results obtained under this setting generalize some well-known classical results for convex programming problems, but also that, in various situations, it provides a unified approach to duality theory. Another purpose is to present optimality conditions of Fritz John and Kuhn-Tucker type with emphasis on derivative conditions, which are not necessarily linear, that is, the directional derivatives are assumed to have some form of convexity, as functions of direction; sometimes non-Lipschitz functions are considered. These questions are approached by first proving theorems of the alternative.

The thesis is divided into five chapters. The first chapter examines the role of convexity, in generalizations of theorems of the alternative, and in a development of optimality conditions for nondifferentiable programming problems. The chapter begins by presenting various generalizations of the Farkas and Motzkin alternative theorems to systems involving sublinear and convex functions in infinite dimensions. These

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theorems are then applied to develop necessary optimality conditions for nondifferentiable programming problems. The results are presented in asymptotic forms and in nonasymptotic forms. This work has been published in Jeyakumar [2].

In the second chapter, it is shown that some basic results, associated with convex programming problems such as the equivalence of saddle points and optima, and duality results of Lagrangian type, hold for a new and a much larger class of nondifferentiable nonconvex problems. These results are derived via theorems of the alternative which are established under much weakened cone convexity hypotheses [see 3]. The basic alternative theorem is also used to generalize a minimax theorem of Ky Fan. This method [see 4] provides a new and different approach to prove minimax theorems avoiding fixed point theorems and von Neumann's minimax theorem. It is also shown that both the basic alternative theorem and the minimax theorem of Ky Fan are equivalent [see 1]. Using these results, saddle-point optimality results and Lagrangian type duality results including fractional programming duality and homogeneous programming duality results, for convex programming problems, are extended to cone convex-like problems.

The next section of this chapter presents, theorems of the alternative involving closed cone systems and convexlike functions, and their applications to vector optimization problems in which the constraints are given by closed convex cones which need not have non-empty interior. In the last section of this chapter, it is shown how the Fritz John and Kuhn-Tucker type (local) optimality conditions for a cone constrained minimization problem can be derived from the theorem of the alternative, under weakened differentiability assumptions. The later results set the stage for the next chapter to study duality results of Wolfe type which are usually proved with the use of a Kuhn-Tucker type theorem.

In the third chapter, various duality theorems of Wolfe type are presented for nondifferentiable cone constrained problems with modified convexity conditions; namely the ρ-convexity conditions, extending the results of Jeyakumar [5]. The ρ-convexity conditions, first introduced by Vial [10] for real valued functions, are now extended to vector valued functions in terms of closed convex cones. The duality results are stated in terms of the local subdifferential, which is a generalization of the subdifferential in the sense of convex analysis. These results include some new subgradient duality theorems, and extensions of some well-known duality theorems. The new duality theorem, which is obtained without the use of a constraint qualification, serves as a strong and converse duality theorem. This is proved under some strengthened cone convexity assumptions, using Fritz John type conditions. The results are also extended to nondifferentiable fractional programming problems.

The fourth chapter is based on the papers [6,7]. The first half of this chapter presents second (and first) order duality theorems for nonlinear programming problems. Our approach, in proving the second order duality theorems, provides unified assumptions that are easily verifiable and are consistent with the corresponding conditions in the first order duality theorems, and paves the way to obtain appropriate first order duality results as special cases. The first section proves second (and first) order duality theorems for cone constrained problems with ρ-convexity conditions and discusses possible computational benefits of second order duality theorems. Then, next two sections present second order symmetric duality theorems, and partial second order duality theorems with generalized convex hypotheses.

The second half of this chapter introduces second order duality theory, for nonlinear fractional programming problems, which does not appear to be available in the mathematical programming literature, and to provide generalizations of some of the known first order duality results. Two different types of second order dual problems for a fractional programming problem are formulated, and the corresponding first order dual problems are obtained by reduction from the second order dual problems. The first order duality relationships are studied with generalized ρ-convexity hypothesis.

In the fifth chapter, a new class of generalized convex functions, called the class of ρ-invex functions [see 8], are introduced. This class includes the classes of (locally Lipschitz) ρ-convex, pseudo convex, and invex functions. Several sufficient conditions for ρ-invexity and concrete examples are given. Using this class of functions and the Clarke generalized subdifferential calculus various basic well-known results such as duality and Kuhn-Tucker sufficiency results for convex problems, are

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generalized to locally Lipschitz programming problems. Hence, the results provide a nonconvex nonsmooth analogue for such results.

Finally, the last section, which is based on the paper [9], presents necessary optimality conditions and sufficient optimality conditions for some more general nonsmooth minimization problems in finite dimensions with inequality constraints. The main result is a Kuhn-Tucker type theorem where necessary conditions are expressed in terms of approximate quasidifferentials which are defined using upper Dini directional derivatives. These results, of course, include the corresponding results, obtained for locally Lipschitz problems and quasi differentiable problems. The sufficient optimality conditions are established in terms of generalized ρ-invex conditions.

References

[1] B.D. Craven and V. Jeyakumar, "Equivalence of a Ky Fan type minimax theorem and a Gordan type alternative theorem", *Operations Research Letters Vol.* 5 *No.*2 (1986) (to appear).

[2] V. Jeyakumar, "Nonlinear alternative theorems and nondifferential programming", *Zeitschrift fur Operations Research*, series A 28 (1984), 175-187.

[3] V. Jeyakumar, "Convexlike alternative theorems and mathematical programming" *Vol.* 16 *No.*2 (1985) 643-652.

[4] V. Jeyakumar, "A generalization of a minimax theorem of Fan via a theorem of the alternative", *Journal of Optimization Theory and Applications Vol.* 48 *No.*3 (1986) 525-533.

[5] V. Jeyakumar, "On subgradient duality with strong and weak convex functions", *Journal of the Australian Mathematical Society, Series A Vol.* 40 *part* 2 (1986) 143-152.

[6] V. Jeyakumar, "ρ-convexity and second order duality", *Utilitas Mathematica, Vol.* 29 (1986) 71-85.

[7] V. Jeyakumar, "First and second order fractional programming duality", *Opsearch Vol.* 22 *No.*1 (1985) 24-41.

[8] V. Jeyakumar, "Strong and weak invexity in mathematical programming", *Methods of Operations Research* 55 (1985) 109-125.

https://doi.org/10.1017/S0004972700013265 Published online by Cambridge University Press

[9] V. Jeyakumar, "On optimality conditions in nonsmooth inequality constrained minimization", *Numerical Functional Analysis and Optimization*, (1986) (to appear).

[10] J.P. Vial, "Strong and weak convexity of sets and functions", *Mathematics of Operations Research* *Vol.* 8 *No.*2 (1983) 231-259.

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