Canad. Math. Bull. Vol. **50** (1), 2007 pp. 85–96

Classification of Finite Group-Frames and Super-Frames

Deguang Han

Abstract. Given a finite group G, we examine the classification of all frame representations of G and the classification of all G-frames, *i.e.,* frames induced by group representations of G. We show that the exact number of equivalence classes of G-frames and the exact number of frame representations can be explicitly calculated. We also discuss how to calculate the largest number L such that there exists an L-tuple of strongly disjoint G-frames.

1 Introduction

A *finite frame* for a Hilbert space H is a set of vectors (not necessarily linearly independent) that spans H. Unlike linearly independent bases, frames have redundancy but still provide stable expansions for signals. In particular, some special finite frames (*e.g.,* tight, normalized tight, uniform, harmonic frames) could play an important role in many applications, including internet coding and wireless communications [8, 14, 17, 19]. We refer to [4–9, 14, 17, 19] for more information and recent developments on finite frames. Since many finite frames are associated with group representations, in this paper we plan to investigate the classification problem for finite group-frames and their associated frame representations.

The redundancy property of frames also allows us to design *disjoint* bases for super-spaces to deal with multiplexing problems in signal transmissions. The concept of disjoint frames (also called orthogonal frames) was first formally introduced and systematically studied by R. Balan [2], and D. Han and D. Larson [16], and was used in the investigation of orthogonal Weyl–Heisenberg frames, super wavelets and sampling [1, 3, 10–13, 16]. Another purpose of this paper is to examine the "disjointness" properties for finite group-frames.

Recall that a *frame* for a separable Hilbert space H is a sequence $\{\varphi_n\}_{n\in\mathcal{J}}$ of H such that there exist $A, B > 0$ with the property that

$$A\|x\|^2 \leq \sum_{n\in\mathcal{J}} |\langle \phi_n\rangle|^2 \leq B\|x\|^2$$

holds for all $x \in H$. The optimal constants (maximal for A and minimal for B) are called frame bounds. A *tight frame* is a frame with frame bound $A = B$. A frame is called *normalized tight* if $A = B = 1$, and *uniform* if all the elements in the frame sequence have the same norm.

Received by the editors February 26, 2004.
AMS subject classification: 42C15, 46C05, 47B10.
Keywords: frames, group-frames, frame representations, disjoint frames.

Let $\{\varphi_n\}_{n\in\mathcal{J}}$ be a frame for H. The *analysis operator* is the mapping $L\colon H \to \ell^2(\mathcal{J})$ defined by:

$$Lx = \sum_{n\in\mathcal{J}} \langle x, \varphi_n\rangle e_n,$$

where $\{e_n\}$ is the standard orthonormal basis for $\ell^2(\mathcal{J})$. Set $S = L^*L$. Then we have

$$Sx = \sum_{n\in\mathcal{J}} \langle x, \varphi_n\rangle \varphi_n, \quad x \in H.$$

Thus S is a positive invertible bounded linear operator on H, which is called the *frame operator* for $\{\varphi_n\}$.

A direct calculation yields

$$x = \sum_n \langle x, S^{-1/2}\varphi_n\rangle S^{-1/2}\varphi_n = \sum_n \langle x, S^{-1}\varphi_n\rangle \varphi_n \quad x \in H,$$

which implies that $\{S^{-1/2}\varphi_n\}$ is a normalized tight frame and $\{S^{-1}\varphi_n\}$ is also a frame. The frame $\{S^{-1}\varphi_n\}$ is called the *canonical* (or *standard*) *dual* of $\{x_n\}$.

When $H = \mathbb{C}^N$ (the N-dimensional complex Hilbert space), the analysis operator L for a frame $\{\varphi_j\}_{j=1}^M$ with M-elements is an $M \times N$ matrix with row vector $\varphi_j = (\varphi_{j1}, \dots, \varphi_{jN})$ on its j-th row position. Let η_k ($k = 1, \dots, N$) denote its column vectors. Then $\operatorname{span}\{\eta_k : k = 1, \dots, N\}$ is the range space of L. The following is evident:

Proposition 1 *Let L be the analysis operator associated with a sequence $\{\varphi_1, \dots, \varphi_M\}$ in $\mathbb{C}^N$. Then*

(i) $\{\varphi_1, \dots, \varphi_M\}$ *is a frame if and only if the set of column vectors of L is linearly independent.*
(ii) $\{\varphi_1, \dots, \varphi_M\}$ *is a normalized tight frame if and only if the set of column vectors of L is an orthonormal set.*
(iii) $\{\varphi_1, \dots, \varphi_M\}$ *is a uniform frame if and only if the set of column vectors of L are linearly independent and the row vectors have the same ℓ^2-norm.*

There is a natural way to define equivalent frames: two frames $\{\varphi_n\}$ (for H) and $\{\psi_n\}$ (for $\mathcal{K}$) are said to be *equivalent* (or, *similar*) if there exists a bounded invertible operator T from H onto $\mathcal{K}$ such that $T\phi_n = \psi_n$ for all n. From the discussion prior to Proposition 1, we know that every frame is similar to a normalized tight one. Thus, when we discuss the classification of frames, we can focus only on the normalized tight ones. The reader will have no difficulty in phrasing the results in more general settings.

Let $\{\varphi_j^{(\ell)}\}_{j\in\mathcal{J}}$ be a normalized frame for Hilbert spaces H_ℓ ($\ell = 1, \dots, k$). We say that $\big(\{\varphi_j^{(1)}\}, \{\varphi_j^{(2)}\}, \dots, \{\varphi_j^{(k)}\}\big)$ is a *disjoint k-tuple* if $\{\varphi_j^{(1)} \oplus \cdots \oplus \varphi_j^{(k)}\}_{j\in\mathcal{J}}$ is a frame for the orthogonal direct sum space $H_1 \oplus \cdots \oplus H_k$, and is a *strongly disjoint k-tuple* if it is a normalized tight frame for the direct sum space. A strongly disjoint k-tuple is also called a *superframe* [2]. The following was proved in [16]:

Proposition 2 *Let L_ℓ be the analysis operator for $\{\varphi_j^{(\ell)}\}$. Then*

(i) $\big(\{\varphi_j^{(1)}\}, \{\varphi_j^{(2)}\}, \dots, \{\varphi_j^{(k)}\}\big)$ *is a disjoint k-tuple if and only if* $\mathrm{Range}(L_1) + \cdots + \mathrm{Range}(L_k)$ *is a direct sum in* $\ell^2(\mathcal{J})$*;*

(ii) $\big(\{\varphi_j^{(1)}\}, \{\varphi_j^{(2)}\}, \dots, \{\varphi_j^{(k)}\}\big)$ *is a strongly disjoint k-tuple if and only if* $\mathrm{Range}(L_1), \dots, \mathrm{Range}(L_k)$ *are mutually orthogonal subspaces of* $\ell^2(\mathcal{J})$.

Strongly disjoint frames have potential applications to data transmission involving multiplexing: suppose that we have a k-tuple strongly disjoint normalized tight frames $\{\varphi_j^{(\ell)}\}(\ell = 1, \dots, k)$ and are planning to transmit k-signals $(f_1, \dots, f_k)$. By using the strong disjointness property of frame bases, instead of transmitting all the k-groups of data $\{\langle f_\ell, \varphi_j^{(\ell)}\rangle\}$, it suffices to transform *only one* group of data $\{c_j\}$, where $c_j = \sum_{\ell=1}^{k}\langle f_\ell, \varphi_j^{(\ell)}\rangle$. For more discussion about disjoint frames we refer to [16, Ch. 2].

The following is a simple consequence of Proposition 1.

Corollary 3 *Let $M \ge N$ and $\{\varphi_j^{(1)}\}_{j=1}^M$ be a normalized tight frame for $\mathbb{C}^N$. Then there exist normalized tight frames $\{\varphi_j^{(\ell)}\}_{j=1}^M$ for H $(\ell = 2, \dots, [M/N])$ such that*

$$\big(\{\varphi_j^{(1)}\}, \{\varphi_j^{(2)}\}, \dots, \{\varphi_j^{([M/N])}\}\big)$$

is a strongly disjoint $[M/N]$-tuple, where $[M/N]$ is the integer part of M/N.

Proof Let L be the analysis operator for the frame $\{\varphi_j^{(1)}\}_{j=1}^M$. Then, by Proposition 1, the set of the N-column vectors of L is orthonormal. Let $k = [M/N]N$. Then we can complete the $M \times N$ matrix L to a $M \times k$ matrix such that all the column vectors of the new matrix are orthonormal. Thus, by to Proposition 1(ii) and Proposition 2(ii), we can use the added columns to construct the other $k - 1$ normalized tight frames for $\mathbb{C}^N$. ■

One of our goals is to investigate whether the above result still holds if we restrict ourselves to a special class of finite frames. In particular, we will examine the case when the frames are induced by group representations. More precisely, for a given finite group G and a unitary representation, we want to know how to explicitly calculate the largest number k such that there exists a k-tuple of strongly disjoint G-frames (frames induced by the group representation of G).

Let G be a group. Recall that a *unitary representation* π of G is a group homomorphism from G into the group $U(H_\pi)$ of unitary operators on some nonzero Hilbert space H_π. Then H_π is called the representation space and $\dim H_\pi$ is called the *dimension* of the representation. Two unitary representations π and Δ are said to be *equivalent* if there is a unitary operator $U\colon H_\pi \to H_\Delta$ such that $U\pi(g)U^* = \Delta(g)$ holds for every $g \in G$. An *invariant subspace* for a unitary representation π is a closed subspace M such that $\pi(g)M \subseteq M$ for all $g \in G$, or equivalently, the orthogonal projection P onto M commutes with every unitary operator $\pi(g)$. Apparently, the restriction π_P of π to M is also a unitary representation of G, and is called a *subrepresentation*.

A unitary representation π for a group G is called a *frame representation* if there is a vector $\varphi \in H$ such that $\{\pi(g)\varphi : g \in G\}$ is a normalized tight frame for H, and in this case we say that $\{\pi(g)\varphi : g \in G\}$ is a *G-frame*. We emphasize that from the definition of frames, we treat a G-frame as a sequence indexed by the elements of G. We also remark that the existence of a normalized tight frame $\{\pi(g)\varphi : g \in G\}$ is equivalent to the existence of an arbitrary frame $\{\pi(g)\psi : g \in G\}$. In fact, let S be the analysis operator for an arbitrary frame $\{\pi(g)\psi : g \in G\}$. Then S commutes with every $\pi(g)$, and thus $\{\pi(g)S^{-1/2}\psi : g \in G\}$ is a normalized tight one. Frame unitary representations were introduced in [16, Ch. 6], and were investigated more systematically and more generally in [10]. In particular, frame representations play an interesting role in studying Gabor frames and semi-orthogonal wavelet frames [10–12, 15]). Some recent results on frame representation of abelian groups can be found in [1].

Any G-frame is clearly a uniform frame. When G is a cyclic group, a G-frame is called a *general harmonic frame* in [8], and it is called a *geometrically uniform* frame in [9] when $\mathcal{G}$ is abelian. In Section 2 we will give a complete classification for G-frames and frame representations. In particular, we discuss how to calculate the cardinality of the set of all the equivalence classes of G-frames (resp., frame representations). Section 3 is devoted to studying strongly disjoint G-frames.

2 Classification of Group-Frames and Frame Representations

It is known [16] that two frames (indexed by the same set $\mathcal{J}$) are equivalent if and only if their analysis operators have the same range space in $\ell^2(\mathcal{J})$. This implies that there exists a one-to-one correspondence between all the equivalence classes of finite frames with M-elements for $\mathbb{C}^N$ and all the N-dimensional orthogonal projections in $\mathbb{C}^M$. Similarly, there is a one-to-one correspondence between all the equivalence classes of uniform normalized tight frames with M-elements in $\mathbb{C}^N$ and all the N-dimensional orthogonal projections P in $\mathbb{C}^M$ such that $\|Pe_j\|$ $(j = 1, \dots, M)$ are equal, where $\{e_j\}$ is the standard orthonormal basis of $\mathbb{C}^M$. Therefore, we have the following:

Proposition 4 *If $M = N$, then there is only one equivalence class of frames for $\mathbb{C}^N$ with M elements. If $M > N$, then there are infinitely many equivalence classes of frames for $\mathbb{C}^N$ (resp., uniform normalized tight frames) with M-elements.*

Proof We only need to point out that when $M > N$, there are infinitely many equivalence classes of uniform normalized tight frames for $\mathbb{C}^N$ with M-elements. In fact, let $\{\varphi_1, \dots, \varphi_M\}$ be a uniform normalized tight frame for $\mathbb{C}^N$ with $M > N$. Then $\{t\varphi_1, \dots, \varphi_M\}$ would be a uniform normalized tight frame when $|t| = 1$, and $\{t\varphi_1, \dots, \varphi_M\}$ is not equivalent to $\{\varphi_1, \dots, \varphi_M\}$ for $t \neq 1$. This can be checked by comparing $L_1L_1^*$ with $L_tL_t^*$, where L_t is the corresponding analysis operator. ■

The above simple fact suggests that if we are looking for finite classifications, we need either to relax the definition of equivalence or to restrict ourselves to more specific frames. In this section we restrict our discussion to group frames.

Let $\mathcal{S}$ be a set of bounded linear operators on a Hilbert space H. We use $\mathcal{S}'$ to denote the *commutant* of $\mathcal{S}$, which is the algebra of all the bounded linear operators on H commuting with every number in $\mathcal{S}$. The algebra generated by a representation π is denoted by $\mathcal{A}_\pi$.

In what follows we always assume that G is a finite group of order M and $H = \mathbb{C}^N$ is a finite-dimensional Hilbert space. Recall that a finite sequence is a frame for $\mathbb{C}^N$ if and only if it spans the space, and that the frame operator for a G-frame $\{\pi(g)\varphi : g \in G\}$ always commutes with every $\pi(g)$. Thus the following is evident:

Proposition 5 *The following are equivalent:*

(i) *π is a frame representation;*
(ii) *there is a vector $\psi \in H$ such that $\{\pi(g)\varphi : g \in G\}$ is a frame for H;*
(iii) *there is a vector $\psi \in H$ such that* $\operatorname{span}\{\pi(g)\varphi : g \in G\} = H$.

Therefore every unitary representation of a finite group G is a finite direct sum of frame representations. This is not true for infinite groups [10].

The most basic model for unitary representations is the so-called *left regular representations* λ of G on $\ell^2(G)$, $\lambda(g)e_h = e_{gh}$, $h \in G$, where $\{e_h : h \in G\}$ is the standard orthonormal basis for $\ell^2(G)$. Similarly, the *right regular representation* ρ is defined by $\rho(g)e_h = e_{hg^{-1}}$, $h \in G$. A very useful fact is that the commutant of $\mathcal{A}_\lambda$ is $\mathcal{A}_\rho$.

We first point out that classifying G-frames is different from classifying frame representations. In fact this can be easily seen from the following result:

Proposition 6 (i) *If $\pi(g)\varphi$ is a G-frame for H, then the range space of its analysis operator is invariant under $\lambda(G)$.*

(ii) *Every frame representation π is equivalent to a subrepresentation λ_P for some orthogonal projection $P \in \lambda(G)'$.*

(iii) *Let λ_P and λ_Q be two subrepresentations of the left regular representation of G. Then they are equivalent if and only if there is an operator $V \in \lambda(\mathcal{G})'$ such that $V^*V = P$ and $VV^* = Q$, where V^* is the adjoint operator of V.*

The operator V in (iii) is an isometry when it is restricted to the range space of P and is zero on the orthogonal complement of P. An operator with such a property is called a *partial isometry* in operator theory. The two projections P and Q satisfying the condition in (iii) are called *equivalent projection* in the algebra $\lambda(G)'$.

Proof For (i) and (ii) see [16]. For the sufficency of (iii), if we also regard V as a unitary from the range space of P onto the range space of Q, then it induces the unitary equivalence between λ_P and λ_Q since

$$\begin{aligned} V\lambda_P(g) &= V\lambda(g)P = \lambda(g)VP \\ &= \lambda(g)V(V^*V) = \lambda(g)(VV^*)V \\ &= \lambda(g)QV = \lambda_Q(g)V. \end{aligned}$$

Secondly, assume that λ_P and λ_Q are equivalent, and let U be a unitary operator from the range of P onto the range of Q such that $U\lambda_P(g) = \lambda_Q(g)U$ holds for every $g \in G$.

We define an operator V on $\ell^2(G)$ by letting it agree with U on $P\ell^2(G)$ and be zero on $P^\perp\ell^2(G)$. Then it is easy to check that V commutes with λg for all g, and thus $V \in \lambda(G)'$. Clearly $VV^* = Q$ and $V^*V = P$. ■

Corollary 7 *The cardinality of the set of all the equivalence classes of frame representations is at most equal to the cardinality of the set of all the equivalence classes of G-frames. The equality holds when G is abelian.*

Proof The first statement follows from Proposition 6(iii). For the second statement, assume that G is abelian. Then $\lambda(G)'$ is commutative. Thus $VV^* = V^*V$ for any $V \in \lambda(G)'$, which implies that two different projections in $\lambda(G)'$ can never be equivalent. ■

By Proposition 6, the number of equivalence classes of G-frames for $\mathbb{C}^N$ is exactly the number of orthogonal projections in the algebra $\mathcal{A}_\rho$ generated by the right regular representation ρ, and the number of equivalence classes of frame representations of G is exactly the number of equivalence classes of the orthogonal projections in $\mathcal{A}_\rho$. To find these numbers we need to recall a few more definitions and results about group representations. A unitary representation of a group G is called *irreducible* if $\pi(G)$ has no nontrivial invariant closed subspaces, which is equivalently to say that $\pi(G)' = \mathbb{C}I$, where I is the identity operator of H. If $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ such that $\pi_1, \dots, \pi_k$ are equivalent irreducible representations, then we call π an *irreducible representation of multiplicity k*. It is well known that for every finite group G, there are only a finite number of equivalence classes of irreducible representations, and this number is equal to the number of conjugacy classes of elements in G. We need the following two lemmas. The second one is obvious and the first one can be found in most standard books, *cf.* [18].

Lemma 8 *If $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ is the direct sum of pairwisely inequivalent irreducible representations π_j of G, then $\mathcal{A}_\pi = \mathcal{A}_{\pi_1} \oplus \cdots \oplus \mathcal{A}_{\pi_k}$.*

Lemma 9 *Let $\pi = \pi_1 \oplus \cdots \oplus \pi_k$ be the direct sum of equivalent representations π_j. Then the number of n-dimensional projections in $\mathcal{A}_\pi$ is the same as the number of kn-dimensional projections in $\mathcal{A}_{\pi_1}$.*

A set $\pi_1, \dots, \pi_k$ of irreducible representations of a group G is called a *complete set of irreducible representations* of G if they are pairwisely inequivalent and every irreducible representation of G is equivalent to one of them. Given a group G of order M, let $\pi_1, \dots, \pi_k$ be a complete set of irreducible representations of G with representation dimensions $m_1, \dots, m_k$. Then, by Burnside's theorem [18], m_j divides M and $m_1^2 + \cdots + m_k^2 = M$.

Theorem 10 *Let G be a finite group with $|G| = M$, and let $N \leq M$.*

(i) *If there is a G-frame for $\mathbb{C}^N$, then there exist integers $0 \leq a_j \leq m_j$ such that $N = \sum_{j=1}^k a_j m_j$.*

(ii) *If $N = \sum_{j=1}^k a_j m_j$ for a selection of integers $0 \leq a_j \leq m_j$, and if there exists some j_0 such that $0 < a_{j_0} < m_{j_0}$, then there are infinitely many equivalence classes of G-frames for $\mathbb{C}^N$. Otherwise, there are only finitely many equivalence classes of G-frames, and the number of equivalence classes of G-frames for $\mathbb{C}^N$ is exactly the cardinality of the set of all the possible selections $(a_1, \dots, a_k)$, $a_j \in \{0, m_j\}$, such that $N = \sum_{j=1}^k a_j m_j$.*

Proof We use the standard decomposition theorem for the right regular representation [18]. We can decompose $\ell^2(G)$ into the orthogonal direct sum

$$\ell^2(G) = \sum_{j=1}^{k} \sum_{\ell=1}^{m_j} \oplus M_\ell^j$$

such that each M_ℓ^j is an m_j-dimensional invariant subspace of $\rho(G)$, and the restriction ρ_ℓ^j of ρ to M_ℓ^j is irreducible and equivalent to π_j.

Let $\rho_j = \sum_{\ell=1}^{m_j} \oplus \rho_\ell^j$. Then $\rho = \rho_1 \oplus \cdots \oplus \rho_k$. By Lemma 8, $\mathcal{A}_\rho = \mathcal{A}_{\rho_1} \oplus \cdots \oplus \mathcal{A}_{\rho_k}$. By Lemma 9, every projection in $\mathcal{A}_{\rho_j}$ has dimension am_j for some integers $0 \leq a \leq m_j$. Moreover, the number of am_j-dimensional projections in $\mathcal{A}_{\rho_j}$ is the number of a-dimensional projections in $\mathcal{A}_{\rho_\ell^j}$. Now we prove (i) and (ii).

(i) Assume that $\{\pi(g)\varphi : g \in G\}$ is a frame for $\mathbb{C}^N$. Let P be the orthogonal projection onto the range space of the analysis operator of a G-frame for $\mathbb{C}^N$. Then $P \in \mathcal{A}_\rho$ is an N-dimensional projection. Write $P = \sum_{j=1}^k \oplus P_j$ with $P_j \in \mathcal{A}_{\rho_j}$. So, by the previous discussion, for each j, there exists an integer a_j such that $0 \leq a_j \leq m_j$ and P_j is a projection of dimension $a_j m_j$ Therefore $N = \sum_{j=1}^k a_j m_j$.

(ii) Now assume that $N = \sum_{j=1}^k a_j m_j$ for a selection of integers $0 \leq a_j \leq m_j$. We will see how many N-dimensional projections in $\mathcal{A}_\rho$ we can define. Note that every such projection has the form $P = \sum_{j=1}^k \oplus P_j$ with P_j being a projection in $\mathcal{A}_{\rho_j}$. We let $P_j = 0$ if $a_j = 0$ and $P_j = I_j$ if $a_j = m_j$, where I_j is the identity matrix on the m_j^2-dimensional space $\sum_{\ell=1}^{m_j} \oplus M_\ell^j$. If $0 < a_j < m_j$ for some j, then we have infinitely many $a_j m_j$-dimensional projections in $\mathcal{A}_\rho$. In fact, note that ρ_ℓ^j is an m_j-dimensional irreducible representation. It follows that $\mathcal{A}_{\rho_\ell^j}$ is the algebra of all the $m_j \times m_j$ matrices. Thus it has infinitely many a_j-dimensional projections. Therefore we have infinitely many $a_j m_j$-dimensional projections in $\mathcal{A}_{\rho_j}$. Let P_j be any one of them. Then $P = \sum_{j=1}^k \oplus P_j$ is an N-dimensional projection in $\mathcal{A}_\rho$.

From the above argument, it is clear that there are infinitely many N-dimensional projections in $\mathcal{A}_\rho$ if $0 < a_{j_0} < m_{j_0}$ for some j_0. The second part follows from the above argument and part (i) immediately. ■

From the proof of the above theorem we can see that there is a one-to-one correspondence between the equivalence classes of G-frames of $\mathbb{C}^N$ and the set

$$\{(a_1, \dots, a_k) : N = \sum_{j=1}^{k} a_j m_j, 0 \le a_j \le m_j\}.$$

We will use this fact in the next section.

Similarly we have the following classification theorem for frame representations:

Theorem 11 *Let G be a finite group and $N \le M = |G|$. Then the number of equivalence classes of all the N-dimensional frame representations of G is exactly the number of the integer vectors $(a_1, \dots, a_k)$ $(0 \le a_j \le m_j)$ such that $\sum_{j=0}^{k} a_j m_j = N$.*

Proof Let P and Q be two N-dimensional orthogonal projections in $\mathcal{A}_\rho$. We use the decomposition of $\mathcal{A}_\rho$ as in the proof of Theorem 10: $P = P_1 \oplus \cdots \oplus P_k$, $Q = Q_1 \oplus \cdots \oplus Q_k$, where $\{P_j\}$ and $\{Q_j\}$ are two families of orthogonal projections in $\mathcal{A}_{\rho_j}$. By re-arranging the order if necessary, we have that P and Q are equivalent in $\mathcal{A}_\rho$ if and only if P_j and Q_j are equivalent in $\mathcal{A}_{\rho_j}$ $(j = 1, \dots, k)$. Using Lemma 9 and the fact that $\mathcal{A}_\pi$ is the full-matrix algebra when π is irreducible, we have that P_j and Q_j are equivalent in $\mathcal{A}_{\rho_j}$ if and only if the have the same dimension.

Let a_j, b_j be nonnegative integers such that

$$\dim \mathrm{Range}(P_j) = a_j m_j \quad \text{and} \quad \dim \mathrm{Range}(Q_j) = b_j m_j.$$

Then $N = \sum_{j=1}^{k} a_j m_j = \sum_{j=1}^{k} b_j m_j$. From the above argument we have that P and Q are equivalent in $\mathcal{A}_\rho$ if and only if $(a_1, \dots, a_k) = (b_1, \dots, b_k)$. Hence the proof is complete. ∎

Corollary 12 *Let G be a finite group of order M.*

(i) *The cardinality of the set of all the equivalence classes of G-frames on $\mathbb{C}^N$ and the cardinality of the set of all the equivalence classes of the N-dimensional frame representations are the same only when we have finitely many equivalence classes of G-frames for $\mathbb{C}^N$.*

(ii) *If G is abelian, then there are $\binom{M}{N}$ number of equivalence classes of G-frames on $\mathbb{C}^N$ and the same number of N-dimensional frame representations. In particular, there are exactly $\binom{M}{N}$ classes of general harmonic frames of M-elements for the N-dimensional complex space $\mathbb{C}^N$.*

Proof Part (i) follows from Theorems 10, 11 and the fact that if $P = \sum_{j=0}^{k} \oplus P_j$ and $Q = \sum_{j=0}^{k} \oplus Q_j$ with P_j, Q_j being either 0 or I_j, then P and Q can never be equivalent unless they are the same projection. Part (ii) follows from (i) and the fact that $m_j = 1$ for all j when G is abelian. ∎

https://doi.org/10.4153/CMB-2007-008-9 Published online by Cambridge University Press

Example 1 Let $G = \mathcal{S}_3$ be the permutation group of three elements. Then the dimensions of the irreducible representations are $m_1 = m_2 = 1$ and $m_3 = 2$. Therefore the number of the equivalence classes of G-frames on $\mathbb{C}^N$ is 2 when $N = 1$ or 5, ∞ when $N = 2, 3$, or 4 and 1 when $N = 6$. The number of the equivalent frame representations of G is 2 when $N = 1, 2, 3, 4, 5$ and 1 when $N = 6$.

3 Disjoint Group-Frames

Let G be a finite group of order M. In this section we discuss the existence of all the possible strongly disjoint L-tuple G-frames on $\mathbb{C}^N$. For this purpose, let $\pi_1, \dots, \pi_k$ be a complete set of irreducible representations of G with representation dimensions $m_1, \dots, m_k$. We have seen in the last section the importance of this set in classifying the G-frames for $\mathbb{C}^N$. We will see that this set also determines the largest number L such that there exists a strongly disjoint L-tuple of G-frames for $\mathbb{C}^N$. This number is called the G-frame *disjointness index* (depending on N)

For two integer vectors $\vec{a} = (a_1, \dots, a_k)$ and $\vec{b} = (b_1, \dots, b_k)$, we write $\vec{a} \le \vec{b}$ if $a_j \le b_j$ for $j = 1, \dots, k$. The *N-th level decompostion index* of $\vec{m} = (m_1, \dots, m_k)$ is the largest integer L such that there exist integer vectors $\vec{0} \le \vec{a}^{(n)} = \langle a_1^{(n)}, \dots, a_k^{(n)} \rangle$ $(n = 1, 2, \dots, L)$ such that $\sum_{j=1}^k a_j^{(n)} m_j = N$ for $1 \le n \le L$ and $\sum_{n=1}^L \vec{a}^{(n)} \le \vec{m}$.

Theorem 13 *The G-frame disjointness index is equal to the decomposition index of $\vec{m}$.*

Proof We adopt the notations of the decomposition of $\ell^2(G)$ in the proof of Theorem 10.

We first need to show that if $\bigl(\{\pi_1(g)\varphi_1\}, \dots, \{\pi_L(g)\varphi_L\}\bigr)$ is a list of strongly disjoint L-tuple G-frames on $\mathbb{C}^N$, then there exist integer vectors

$$\vec{0} \le \vec{a}^{(n)} = \langle a_1^{(n)}, \dots, a_k^{(n)} \rangle$$

$(n = 1, 2, \dots, L)$ such that $\sum_{j=1}^k a_j^{(n)} m_j = N$ for $1 \le n \le L$ and $\sum_{n=1}^L \vec{a}^{(n)} \le \vec{m}$.

In fact, let $P^{(n)}$ be the orthogonal projection from $\ell^2(G)$ onto the range space of $\{\pi_n(g)\varphi_n\}, (n = 1, \dots, L)$. Then $P^{(n)}$ is an N-dimensional projection in $\mathcal{A}_\rho$. Thus, by the proof of Theorem 10, there exist $0 \le a_j^{(n)} \le m_j$ such that $N = \sum_{j=1}^k a_j^{(n)} m_j$ and $P^{(n)} = \sum_{j=1}^k \oplus P_j^{(n)}$, where $P_j^{(n)}$ is an $a_j^{(n)} m_j$-dimensional projection on the subspace $\sum_{\ell=1}^{m_j} \oplus M_\ell^j$. The strong disjointness of the frames implies that $P_j^{(n)} \perp P_j^{(n')}$ if $n \ne n'$. Thus $\sum_{n=1}^L a_j^{(n)} m_j \le m_j^2$ which implies that $\sum_{n=1}^L \vec{a}^{(n)} \le \vec{m}$.

Secondly, we need to show that if there are integer vectors

$$\vec{0} \le \vec{a}^{(n)} = \langle a_1^{(n)}, \dots, a_k^{(n)} \rangle, \quad (n = 1, 2, \dots, L)$$

such that $\sum_{j=1}^k a_j^{(n)} = N$ for $1 \le n \le L$ and $\sum_{n=1}^L \vec{a}^{(n)} \le \vec{m}$, then we can find a strongly disjoint L-tuple of G-frames on $\mathbb{C}^N$. It suffices to show that there exist L mutually orthogonal N-dimensional projections in $\mathcal{A}_\rho$. For each fixed $1 \le j \le k$, using the fact $\sum_{n=1}^L a_j^{(n)} \le m_j$ and the structure of $\mathcal{A}_{\rho_j}$ (where ρ_j is as in the proof of

Theorem 10), we can construct projections $P_j^{(n)} \in \mathcal{A}_{\rho_j}$ such that $P_j^{(n)}$ has dimension $a_j^{(n)} m_j$ and $P_j^{(n)} \perp P_j^{(n')}$ when $n \neq n'$. Now let $P^{(n)} = \sum_{j=1}^{k} \oplus P_j^{(n)}$. Then $P^{(n)} \in \mathcal{A}_\rho$ is an N-dimensional projection and $\{P^{(n)} : n = 1, \dots, L\}$ are mutually orthogonal. ■

Remark The power of Theorem 10 and Theorem 13 is that without explicitly constructing any inequivalence classes of G-frames or any strongly disjoint G-frames, we can easily calculate the number of the equivalence classes and the disjoint G-frame index if we know $\vec{m}$. In some simple cases, it is not very hard to find the number of all the conjugacy classes of a group G and then use the property $m_j|M$ and $\sum_{j=1}^{k} m_j^2 = M$ to find $\vec{m} = \langle m_1, \dots, m_k \rangle$, where M is the order of G. In Example 1 we have $\vec{m} = \langle 1, 1, 2 \rangle$. So, from Theorem 13, we can easily find out that the G-frame disjointness index is 2 when $N = 1$; 3 when $N = 2$; 2 when $N = 3$; 1 when $N = 4, 5, 6$.

From the proof of Theorem 13, we also have

Corollary 14 *Let G be a finite group with $|G| = M$ and L be its G-frame disjointness index. If $\{\pi_1(g)\varphi_1 : G \in G\}$ is a normalized tight frame for $\mathbb{C}^N$, then there exist $L - 1$ G-frames $\{\pi_j(g)\varphi_j : g \in G\}$ $(j = 2, \dots, L)$ such that $\{\sum_{j=1}^{L} \oplus \pi_j(g)\varphi_j : g \in G\}$ is a normalized tight frame for direct space $\sum_{j=1}^{L} \oplus \mathbb{C}^N$.*

In general, those representations π_j in Corollary 14 are different. This can be seen clearly from the following result, which was proved in [16].

Proposition 15 *Let $\{\pi(g)\varphi : g \in G\}$ and $\{\pi(g)\psi : g \in G\}$ be two frames for $\mathbb{C}^N$, and let P and Q be the orthogonal projections onto the range spaces of their analysis operators, respectively. Then $\{\pi(g)\varphi : g \in G\}$ and $\{\pi(g)\psi : g \in G\}$ are strongly disjoint if and only if $P \perp Q$ and P and Q are equivalent in $\mathcal{A}'_\rho$. In particular, if G is an abelian group, then $\{\pi(g)\varphi : g \in G\}$ and $\{\pi(g)\psi : g \in G\}$ can never be strongly disjoint (in fact they are always equivalent).*

Note that group-frames are special uniform frames. Therefore for each $M \geq N$, we can construct $[M/N]$ strongly disjoint uniform normalized tight frames for $\mathbb{C}^N$ (by using Corollary 14 in the case that G is a cyclic group of order M). However, in general we still do not know the answer to the following question:

> *Given a uniform normalized tight frame $\{\varphi_1^{(1)}, \dots, \varphi_M^{(1)}\}$ for $\mathbb{C}^N$, what is the largest number L such that there exist $L - 1$ uniform normalized tight frames $\{\varphi_1^{(j)}, \dots, \varphi_M^{(j)}\}$ for $\mathbb{C}^N$ such that $\{\sum_{j=1}^{L} \oplus \varphi_1^{(j)}, \dots, \sum_{j=1}^{L} \oplus \varphi_M^{(j)}\}$ is a normalized tight frame for $\sum_{j=1}^{L} \oplus \mathbb{C}^N$?*

We expect, as in the normalized tight frame case, that $L = [M/N]$. If we rephrase the above question in terms of projections, then the problem becomes: given an N-dimensional projection P_1 in C^M such that $P_1 e_j$ $(j = 1, \dots, M)$ have the same norm, is it always possible that we can find $[M/N] - 1$ N-dimensional projections in

$\mathbb{C}^M$ such that $\{P_j : j = 1, \dots, [M/N]\}$ are mutually orthogonal, and $P_j e_k$ have the same norm for all j, k? A naive idea is to prove this by using the step-down method: for any projection P such that $\|Pe_1\| = \cdots = \|Pe_M\|$, there exists a non-trivial sub-projection Q of P satisfying $\|Qe_1\| = \cdots = \|Qe_M\|$. However, this statement is false in general. Peter Casazza and J. Kovačević constructed a counterexample [8] for a different purpose in the real case, but their example does not serve as a counterexample for the complex case. Here is an example for the complex case:

Example 2 Let P be the orthogonal projection from $\mathbb{C}^6$ onto the 2-dimensional space spanned by

$$\xi = (1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{6}, 1/\sqrt{3}, 0),$$
$$\eta = (1/\sqrt{6}, -1/\sqrt{6}, i/\sqrt{6}, -i/\sqrt{6}, 0, 1/\sqrt{3}).$$

Then $\|Pe_1\| = \cdots = \|Pe_6\| = 1/\sqrt{3}$. If Q is a one-dimensional subprojection of P such that $\|Qe_1\| = \cdots = \|Qe_6\|$, then there is a vector $a\xi + b\eta$ such that all of its components have equal modules. In particular, we have $|a+b| = |a-b| = |a+ib| = |a-ib|$, which implies $a = 0$ or $b = 0$. Thus $a = b = 0$, which is a contradiction.

References

[1] A. Aldroubi, D. Larson, W.-S. Tang and E. Weber, *Geometric aspects of frame representations of abelian groups* Trans. Amer. Math. Soc. **356**(2004), no. 12, 4767–4786.

[2] R. Balan, A study of Weyl-Heisenberg and wavelet frames, Ph.D. Thesis, Princeton University, 1998

[3] R. Balan, *Density and redundancy of the noncoherent Weyl-Heisenberg superframes.* In: The Functional and Harmonic Analysis of Wavelets and Frames, Contemp. Math. 247, American Mathematical Society, Providence, RI, 1999, pp. 29–41.

[4] J. Benedetto and M. Fickkus, *Finite tight frames.* Adv. Comput. Math. **18**(2003), 357–385.

[5] P. Casazza, *Custom building finite frames.* In: Wavelets, Frames and Operator Theory, Contemp. Math. 345, American Mathematical Society, Providence, RI, 2004, pp. 61–86.

[6] P. Casazza, M. Fickus, J. C. Tremain, and E. Weber, *The Kadison-Singer problem in mathematics and engineering. A detailed account.* In: Operator Theory, Operator Algebras and Applications. Contemporary Math. American mathematical Society, Providence, RI, 2006, to appear.

[7] P. Casazza and J. Kovačević, *Uniform tight Frames for signal processing and communications.* In: Proc. SPIE Conference on Wavelet Applications in Signal and Image Processing, San Diego, CA, 2001, pp. 512–521.

[8] P. Casazza and J. Kovačević, *Uniform tight frames with erasures.* Adv. Comput. Math. **18**(2003), 387–430.

[9] Y. Eldar and H. Bölcskei, *Geometrically uniform frames.* IEEE Trans. Inform. Theory **49**(2003), no. 4, 993–1006.

[10] J.-P. Gabardo and D. Han, *Frame representations for group-like unitary operator systems.* J. Operator Theory, **49**(2003), no. 2, 223–244.

[11] ———, *Subspace Weyl-Heisenberg frames.* J. Fourier Anal. Appl. **7**(2001), no. 4, 419–433.

[12] ———, *The uniqueness of the dual of Weyl-Heisenberg subspace frames.* Appl. Comput. Harmon. Anal. **17**(2004), no. 2, 226–240.

[13] J.-P. Gabardo, D. Han and D. Larson, *Gabor frames and operator algebras.* In: Proc. SPIE Conference on Wavelet Applications in Signal and Image Processing, San Diego, CA, 2000, pp. 337–345.

[14] V. K. Goyal, J. Kovačević and J. A. Kelner, *Quantized frame expansions with erasures* Appl. Comput. Harmon. Anal. **10**(2001), no. 3, 203–233.

[15] D. Han, *Approximations for Gabor and wavelet frames.* Trans. Amer. Math. Soc. **355**(2003), no. 8, 3329–3342.

[16] D. Han and D. Larson, *Frames, Bases and Group Parametrizations.* Memoirs Amer. Math. Soc. 697, 2000.

[17] A. Shokrollabhi, B. Hassibi, B. Hochwald, and W. Sweldens, *Representation theory for high-rate multiple-antenna code design.* IEEE Trans. Inform. Theory **47**(2001), no. 6, 2355–2367.
[18] M. A. Naimark and A. I. Stern, *Theory of Group Representations.* Grundlehren der Mathematischen Wissenschaften 246, Springer-Verlag, New York, 1982.
[19] T. Strohmer, *Approximation of dual Gabor frames, window decay, and wireless communications.* Appl. Comput. Harmon. Anal. **11**(2001), no. 2, 243–262.

Department of Mathematics
University of Central Florida
Orlando, FL 32163
U.S.A.
e-mail: dhan@pegasus.cc.ucf.edu

https://doi.org/10.4153/CMB-2007-008-9 Published online by Cambridge University Press