Canad. Math. Bull. Vol. **49** (2), 2006 pp. 270–280

A Characterization of Products of Projective Spaces

Gianluca Occhetta

Abstract. We give a characterization of products of projective spaces using unsplit covering families of rational curves.

1 Introduction

Since Mori's proof of the Hartshorne conjecture, families of rational curves have become a fundamental tool in the study of higher dimensional complex varieties, as is shown in Kollar's book [8], the basic reference for most of the techniques and results related to this subject.

Among these families a very special role is played by the so called unsplit families; roughly speaking these are families of rational curves whose degenerations don't split up into reducible cycles (for a precise definition see Section 2).

It was soon realized that the existence of these families could be related to bounds on the Picard number; for instance if on a variety X there exists an unsplit family of rational curves such that the curves in the family passing through a point cover the whole variety then $\rho_X = 1$ (see [2, Proof of Proposition 1.1]).

For Fano manifolds, which are covered by rational curves, Mukai [10] proposed the following conjecture:

$$\text{(M)} \qquad \rho_X(r_X - 1) \leq \dim X,$$

where r_X is the *index* of X, *i.e.,* the greatest integer m such that there exists $L \in \mathrm{Pic}(X)$ satisfying $-K_X = mL$. In [11] Wiśniewski introduced the related notion of *pseudoindex* i_X of a Fano manifold, as the minimum anticanonical degree of rational curves, and proved the following

Theorem Let X be a Fano manifold of index r_X and pseudoindex i_X.

(A) If $2i_X > \dim X + 2$ then $\rho_X = 1$.
(B) If $2r_X = \dim X + 2$ then $\rho_X = 1$ except if $X \simeq (\mathbb{P}^{r_X - 1})^2$.

Recently [4], a generalized version of conjecture (M) has been proposed in the following form:

$$\text{(GM)} \qquad \rho_X(i_X - 1) \leq \dim X,$$

Received by the editors July 8, 2004; revised September 25, 2004.
AMS subject classification: 14J40, 14J45.
Keywords: Rational curves, Fano varieties.

https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

with equality if and only if $X \simeq (\mathbb{P}^{i_X-1})^{\rho_X}$; in [4], conjecture (GM) is proved for $\dim X = 3, 4$, for some families of toric Fano manifolds and for homogeneous Fano manifolds.

Note that for $i_X = \dim X + 1$ this conjecture states that the only Fano manifold with pseudoindex $\dim X + 1$ is the projective space; this long-standing question recently has been answered [5, 7].

This paper proposes a characterization of products of projective spaces which arose from the study of the case $2i_X = \dim X + 2$, *i.e.,* our main result is the following:

Theorem 1.1 *A smooth complex projective variety X of dimension n is isomorphic to a product of projective spaces $\mathbb{P}^{n(1)} \times \cdots \times \mathbb{P}^{n(k)}$ if and only if there exist k unsplit covering families of rational curves $V^1, \dots, V^k$ of degrees $n(1)+1, \dots, n(k)+1$ with $\sum n(i) = n$ such that the numerical classes of $V^1, \dots, V^k$ are linearly independent in $N_1(X)$.*

In particular it follows from this theorem that for a Fano manifold with $2i_X = \dim X + 2$, we have $\rho_X = 1$ except if $X \simeq (\mathbb{P}^{i_X-1})^2$.

2 Families of Rational Curves

We recall some of our basic definitions; our notation is consistent with the one in [8] to which we refer the reader.

Let X be a projective variety and let $\mathrm{Hom}(\mathbb{P}^1, X)$ be the scheme parametrizing morphisms $f \colon \mathbb{P}^1 \to X$; let $\mathrm{Hom}_{\mathrm{bir}}(\mathbb{P}^1, X) \subset \mathrm{Hom}(\mathbb{P}^1, X)$ be the open subscheme corresponding to morphisms which are birational onto their image. The group $\mathrm{Aut}(\mathbb{P}^1)$ acts on the normalization $\mathrm{Hom}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ and the quotient exists.

Definition 2.1 The space $\mathrm{RatCurves}^n(X)$ is the quotient of $\mathrm{Hom}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ by the action of $\mathrm{Aut}(\mathbb{P}^1)$ and the space $\mathrm{Univ}(X)$ is the quotient of $\mathrm{Hom}^n_{\mathrm{bir}}(\mathbb{P}^1, X) \times \mathbb{P}^1$ by the product action of $\mathrm{Aut}(\mathbb{P}^1)$.

We have the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Hom}^n_{\mathrm{bir}}(\mathbb{P}^1, X) \times \mathbb{P}^1 & \xrightarrow{\ U_X\ } & \mathrm{Univ}(X) \\
\downarrow & & \downarrow \pi \\
\mathrm{Hom}^n_{\mathrm{bir}}(\mathbb{P}^1, X) & \xrightarrow{\ u_X\ } & \mathrm{RatCurves}^n(X)
\end{array}$$

where u_X and U_X are principal $\mathrm{Aut}(\mathbb{P}^1)$-bundles and π is a $\mathbb{P}^1$-bundle.

Definition 2.2 A *family of rational curves* is a closed irreducible subvariety $V \subset \mathrm{RatCurves}^n(X)$. The family V is called an *unsplit family* of rational curves if V is a proper subvariety.

https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

Given a family of rational curves, we have the following basic diagram

$$\begin{array}{ccc} U & \xrightarrow{\ i\ } & X \\ {\scriptstyle \pi}\big\downarrow & & \\ V & & \end{array}$$

where i is the map induced by the evaluation ev: $\mathrm{Hom}^n_{\mathrm{bir}}(\mathbb{P}^1, X) \times \mathbb{P}^1 \to X$ and π is a $\mathbb{P}^1$-bundle; we say that V is a *covering family* if i is dominant, otherwise we denote the closure of $i(U)$ by $\mathrm{Locus}(V)$. If V is proper, *i.e.*, if the family is unsplit, then i is a proper morphism [8, II.2.3]. Finally we denote by V_x the subfamily parametrizing rational curves of the family V passing through x.

3 Chains of Rational Curves

In this section we give slight modifications of some results in [4] that we need for the proof of Theorem 1.1.

Let X be a projective variety, $V^1, \dots, V^k$ unsplit families of rational curves on X, and Y a subset of X.

Definition 3.1 We denote by $\mathrm{Locus}(V^1, \dots, V^k)_Y$ the set of points $x \in X$ such that there exist curves $C_1, \dots, C_k$ with the following properties:

- C_i belongs to the family V^i,
- $C_i \cap C_{i+1} \neq \varnothing$,
- $C_1 \cap Y \neq \varnothing$ and $x \in C_k$.

i.e., $\mathrm{Locus}(V^1, \dots, V^k)_Y$ is the set of points that can be joined to Y by a connected chain of k curves belonging, respectively, to the families $V^1, \dots, V^k$.

The following lemma is well known (see for istance [8, IV.3.13.3]), but we give a sketch of the proof since we will need it for the crucial Remark 3.3.

Lemma 3.2 *Let $Y \subset X$ be a closed subset and let V be an unsplit family of rational curves. Then* $\mathrm{Locus}(V)_Y$ *is closed and every curve contained in* $\mathrm{Locus}(V)_Y$ *is numerically equivalent to a linear combination with rational coefficients of curves in Y and curves parametrized by V.*

Proof Let U, V, π and i be as in Definition 2.2 and let C be a curve contained in $\mathrm{Locus}(V)_Y$. If $C \subset Y$ or C is a curve parametrized by V we have nothing to prove, so we can suppose that this is not the case.

In particular we have that $i^{-1}(C)$ contains an irreducible curve C' which is not contained in a fiber of π and dominates C via i; let B' be the curve $\pi(C') \subset V$, let $\nu\colon B \to B'$ be the normalization of B', and let S be the normalization of $B \times_V U$.

https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

By standard arguments it can be shown that S is a ruled surface over the curve B; we thus have a diagram:

$$\begin{array}{ccc} S & \xrightarrow{\ j\ } & X \\ \big\downarrow{\scriptstyle p} & & \\ B & & \end{array}$$

Let f be a fiber of p and let C_Y be a curve in S which dominates B and whose image via j is contained in Y; such a curve exists since the image via j of every fiber of p meets Y.

Since S is a ruled surface, every curve in S is algebraically equivalent to a linear combination with rational coefficients of C_Y and f.

Therefore every curve in $j(S)$ is algebraically equivalent in X to a linear combination with rational coefficients of $j_*(C_Y)$ and $j_*(f)$:

$$C \equiv \lambda j_*(C_Y) + \mu j_*(f),$$

where $j_*(C_Y)$ is a curve in Y or is the zero cycle, and $j_*(f)$ is a curve of the family V. Since algebraic equivalence implies numerical equivalence we conclude the proof. ■

Remark 3.3 Note that the proof of the above lemma actually yields that a curve C in $\text{Locus}(V)_Y$ is algebraically equivalent to a linear combination with rational coefficients

$$\lambda j_*(C_Y) + \mu j_*(f)$$

such that $\lambda \geq 0$; in fact, let C_S be an irreducible curve in S which dominates C via j as in the proof of the lemma. In S we write $C_S \equiv \lambda C_Y + \mu f$ and, intersecting with f we have $\lambda \geq 0$. To the author's knowledge this was first noted by Wiśniewski in [3, Proof of Lemma 1.4.5].

Corollary 3.4 *Let $V^1, \dots, V^k$ be unsplit families of rational curves and let x be a point in X such that $\text{Locus}(V^1, \dots, V^k)_x$ is not empty. Then every curve contained in $\text{Locus}(V^1, \dots, V^k)_x$ is numerically equivalent to a linear combination with rational coefficients of curves in $V^1, \dots, V^k$.*

Proof To prove the corollary we apply Lemma 3.2 k times, taking $Y_1 = x$ and $Y_i = \text{Locus}(V^1, \dots, V^{i-1})_x$. ■

If Y is a point and X is smooth, we have the following dimension bound which is a generalization of [8, Proposition IV.2.6]

Theorem 3.5 ([4, Theorem 5.2]) *Let $V^1, \dots, V^k$ be linearly independent unsplit families of rational curves on a smooth variety X and let x be a point in X such that $\text{Locus}(V^1, \dots, V^k)_x$ is not empty. Then*

$$\dim \text{Locus}(V^1, \dots, V^k)_x \geq -\sum K_X \cdot V^i - k.$$

https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

4 Products of Projective Spaces

In the proof of Theorem 1.1 we will use the following well-known lemmata:

Lemma 4.1 *Let $p\colon Y \to B$ be a morphism from a smooth variety to a smooth curve, such that $\rho(Y/B) = 1$ and the general fiber of p is a projective space. Then there exists a vector bundle $\mathcal{F}$ of rank $= \dim Y$ on B such that $Y = \mathbb{P}_B(\mathcal{F})$ and p is the natural projection.*

Lemma 4.2 *Let $Y = \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ be the projectivization of a vector bundle of rank $r+1$ over $\mathbb{P}^1$. Then the Mori cone $NE(Y)$ is generated by the class of a line l in a fiber of the natural projection $p\colon Y \to \mathbb{P}^1$ and the class of a section whose intersection with the tautological line bundle $\xi_{\mathcal{E}}$ is minimal (a minimal section). Moreover, denoting by C_0 a minimal section, any curve whose numerical class is a multiple of $[C_0]$ is (set-theoretically) the union of disjoint minimal sections.*

Proof of Theorem 1.1 The "only if" part of the theorem is clear, taking as V^i the family of lines in $\mathbb{P}^{n(i)}$; to prove the "if" part first of all we observe that, in the assumptions of the theorem, we have $\rho_X = k$.

In fact, since on X there exists k numerically independent curves we have $\rho_X \geq k$. On the other hand $\operatorname{Locus}(V^1, \dots, V^k)_x$ is not empty for every $x \in X$ since the families V^j are covering families, and by Theorem 3.5 we have

$$\dim \operatorname{Locus}(V^1, \dots, V^k)_x \geq -\sum K_X \cdot V^i - k = n,$$

so $X = \operatorname{Locus}(V^1, \dots, V^k)_x$ and $\rho_X \leq k$ by Corollary 3.4.

We will now proceed by induction on the Picard number ρ_X. If $\rho_X = 1$ then $X \simeq \mathbb{P}^n$ by [7, Theorem 1.1] or [5, Corollary 0.4]; so let us suppose that $\rho_X = k$. We denote by $\operatorname{NE}(X)$ the Mori cone of X, *i.e.,* the cone in $N_1(X)$ generated by the classes of effective curves on X.

Step 1 $\operatorname{NE}(X) = \mathbb{R}_+[V^1] + \dots + \mathbb{R}_+[V^k]$.

For every point $x \in X$ and for every permutation $i(1), \dots, i(k)$ of the integers $1, \dots, k$ we have $X = \operatorname{Locus}(V^{i(1)}, \dots, V^{i(k)})_x$. In fact, $\operatorname{Locus}(V^{i(1)}, \dots, V^{i(k)})_x$ is not empty since the families V^j are covering families, and by Theorem 3.5 we have

$$\dim \operatorname{Locus}(V^{i(1)}, \dots, V^{i(k)})_x \geq -\sum K_X \cdot V^i - k = n.$$

Now let C be an effective curve in X; by Corollary 3.4 and Remark 3.3, for every permutation $i(1), \dots, i(k)$ we find coefficients $\lambda_{i(1)}, \dots, \lambda_{i(k)}$, with $\lambda_{i(1)} \geq 0$ such that

$$[C] = \lambda_{i(1)}[C_{i(1)}] + \dots + \lambda_{i(k)}[C_{i(k)}].$$

Since $V^1, \dots, V^k$ are independent in $N_1(X)$, the decomposition of $[C]$ is unique and we get that $\lambda_{i(j)} \geq 0$ for all $j \in 1, \dots, k$. It follows that $\operatorname{NE}(X) = \mathbb{R}_+[V^1] + \dots + \mathbb{R}_+[V^k]$; moreover by the Kleiman criterion $-K_X$ is ample, so X is a Fano manifold.

https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

Notation We will usually denote by $\varphi_\sigma \colon X \to Y_\sigma$ the contraction corresponding to the extremal face σ. If $\tau \subsetneq \sigma$ is a (possibly empty) subface, then $\varphi_\sigma \colon X \to Y_\sigma$ factors through $\varphi_\tau \colon X \to Y_\tau$ and a morphism $Y_\tau \to Y_\sigma$ which we will call $\psi_{\tau\sigma}$.

Note that, if $\tau = \varnothing$ then $Y_\tau = X$ and $\psi_{\tau\sigma} = \varphi_\sigma$.

Step 2 Every extremal contraction of X is equidimensional and its general fiber is a product of projective spaces.

By Step 1 the cone $\mathrm{NE}(X)$ is k-dimensional and it is spanned by k extremal rays, so to every subset $I \subset \{1, \dots, k\}$ corresponds an extremal face, spanned by the rays indexed by I and an extremal contraction.

Let $\sigma = \langle R_{i_1} \cdots R_{i_l} \rangle$ be an extremal face of $\mathrm{NE}(X)$ and let $\sigma^\perp$ be the face spanned by the extremal rays of $\mathrm{NE}(X)$ which are not in σ; these two faces are clearly disjoint.

We claim that for every fiber G_σ of φ_σ we have

$$\dim G_\sigma = \sum_{j=1,\dots,l} n(i_j); \tag{1}$$

since $G_\sigma \supseteq \mathrm{Locus}(V^{i_1}, \dots, V^{i_l})_x$, by Theorem 3.5 we know that

$$\dim G_\sigma \geq - \sum_{j=1,\dots,l} K_X \cdot V^{i_j} - l = \sum_{j=1,\dots,l} n(i_j).$$

Let x be a point of G_σ and let $G_{\sigma^\perp}$ be the fiber of $\varphi_{\sigma^\perp}$ passing through x; we have

$$\dim G_{\sigma^\perp} \geq - \sum_{j=l+1,\dots,k} K_X \cdot V^{i_j} - (k-l) = \sum_{j=l+1,\dots,k} n(i_j).$$

Moreover, since the numerical class of every curve in G_σ lies in σ and the numerical class of every curve in $G_{\sigma^\perp}$ lies in $\sigma^\perp$ we have $\dim(G_\sigma \cap G_{\sigma^\perp}) = 0$, so that, by Serre's inequality, $\dim G_\sigma + \dim G_{\sigma^\perp} \leq n$. Hence

$$n = \sum_{j=1,\dots,k} n(i_j) \leq \dim G_\sigma + \dim G_{\sigma^\perp} \leq n,$$

forcing

$$\dim G_\sigma = \sum_{j=1,\dots,l} n(i_j).$$

In particular, if we take G_σ to be a general fiber of φ_σ, then G_σ is smooth and satisfies the assumptions of Theorem 1.1, and so we have $G_\sigma \simeq \mathbb{P}^{n(i_1)} \times \cdots \times \mathbb{P}^{n(i_l)}$ by the induction assumption.

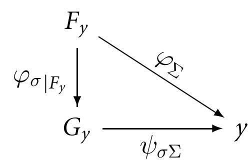
We note here, for later use, that from the proof of Step 2 it follows that

$$\dim Y_\sigma = \dim X - \dim G_\sigma = \dim G_{\sigma^\perp}. \tag{2}$$

Step 3 Let Σ be an extremal face of $\mathrm{NE}(X)$ of dimension $s < k$ and let $\sigma \subset \Sigma$ be a subface of codimension one; then $\psi_{\sigma\Sigma} \colon Y_\sigma \to Y_\Sigma$ is a projective bundle outside a set of codimension two in Y_Σ.

Recall that $\varphi_\Sigma = \psi_{\sigma\Sigma} \circ \varphi_\sigma$ and that both φ_Σ and φ_σ are equidimensional and with connected fibers, so also $\psi_{\sigma\Sigma}$ is equidimensional and has connected fibers. Since Y_σ is a normal variety, also the general fiber of $\psi_{\sigma\Sigma}$ is normal; we claim that it is a projective space.

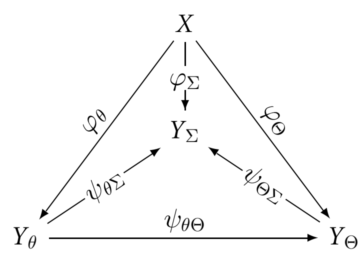
To prove the claim let y be a general point of Y_Σ, let G_y be the fiber of $\psi_{\sigma\Sigma}$ over y, let F_y be the fiber of φ_Σ over y, and consider the following diagram:



By Step 2 we know that F_y is a product of s projective spaces; the morphism $\varphi_{\sigma|F_y}$ is a proper morphism with connected fibers onto a normal variety, so it is a contraction of F_y. But on a product of projective spaces the only contractions are projections onto some factors. Since the general fiber of φ_σ is a product of $s-1$ projective spaces, the claim follows.

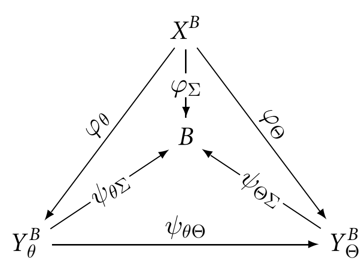
To prove that $\psi_{\sigma\Sigma}$ is a projective bundle outside a set of codimension two we have to prove that for a general curve $B \subset Y_\Sigma$ the variety $Y_\sigma^B := \psi_{\sigma\Sigma}^{-1}(B)$ is a projective bundle over B. To show this last statement, by Lemma 4.1, it is enough to prove that Y_σ^B is smooth.

Let θ and Θ be two subfaces of $\mathrm{NE}(X)$ such that $\theta \subset \Theta \subset \Sigma$, and consider the following commutative diagram:



Take $\dim Y_\Sigma - 1$ general very ample divisors on Y_Σ; by Bertini's theorem their intersection is a smooth curve B and also $X^B = \varphi_\Sigma^{-1}(B)$ is smooth.

We can consider the restriction to X^B of the previous diagram, where for the restricted morphisms we keep the same notation we used for the original ones and where $Y_\theta^B := \psi_{\theta\Sigma}^{-1}(B)$ and $Y_\Theta^B := \psi_{\Theta\Sigma}^{-1}(B)$.



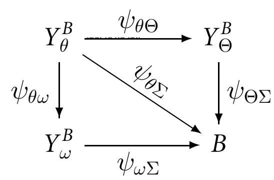
https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

We will show that, for every extremal face $\Theta \subsetneq \Sigma$ in $\mathrm{NE}(X)$, associated extremal contraction $\varphi_\Theta \colon X \to Y_\Theta$ and restricted morphism $\varphi_\Theta \colon X^B \to Y^B_\Theta$ the target variety Y^B_Θ is smooth. In particular it will follow that Y^B_σ is smooth.

We proceed by induction on the dimension of Θ; if $\dim \Theta = 0$ then φ_Θ does not contract anything and X^B is smooth.

Now let $\theta \subsetneq \Sigma$ be an extremal face of $\mathrm{NE}(X)$ of dimension $t-1 < s-1$, let $R \subset \Sigma$ be an extremal ray independent from θ, let Θ be the face spanned by θ and R and finally let ω the face spanned by all the rays in Σ but R.

We have a commutative diagram



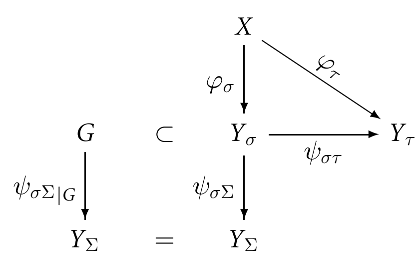
The general fiber of $\psi_{\omega\Sigma} \colon Y^B_\omega \to B$ is a projective space, so over an open Zariski subset U of B the morphism $\psi_{\omega\Sigma}$ is a projective bundle. We take H to be the closure of a hyperplane section of $\psi_{\omega\Sigma}$ defined over the open set U and $\mathcal{H} = \psi^{-1}_{\theta\omega}(H)$. Since Y^B_θ is smooth by induction, $\mathcal{H}$ is a Cartier divisor, which restricts to $\mathcal{O}(1)$ on the general fiber of $\psi_{\theta\Theta}$, so $\psi_{\theta\Theta}$ is a projective bundle globally by [6, Lemma 2.12] and so Y^B_Θ is smooth.

Step 4 Let Σ be a $(k-1)$-dimensional face of $\mathrm{NE}(X)$ obtained by removing a ray R_{i_1} and let $\varphi_\Sigma \colon X \to Y_\Sigma$ be the associated contraction. Then $Y_\Sigma \simeq \mathbb{P}^{\dim Y_\Sigma}$.

Let R_{i_2} be a ray in Σ and σ the (possibly empty) subface of Σ obtained removing the ray R_{i_2}; finally let $\tau =< \sigma, R_{i_1} >$.

The contraction φ_τ factors through φ_σ and a morphism $\psi_{\sigma\tau} \colon Y_\sigma \to Y_\tau$, the contraction φ_Σ factors through φ_σ and a morphism $\psi_{\sigma\Sigma} \colon Y_\sigma \to Y_\Sigma$ and both the morphisms $\psi_{\sigma\tau}$ and $\psi_{\sigma\Sigma}$ are, by Step three, projective bundles outside a set of codimension two.

The situation is illustrated by the commutative diagram



where G is a general fiber of $\psi_{\sigma\tau}$ and so a projective space.

Note that, by equations (1) and (2) we have $\dim Y_\sigma = n(i_1) + n(i_2)$, $\dim Y_\Sigma = n(i_1)$ and $\dim Y_\tau = n(i_2)$; it follows that $\dim G = \dim Y_\Sigma$, so that $\psi_{\sigma\Sigma}$ restricted to G is dominating.

First of all we will prove that $\psi_{\sigma\Sigma}$ restricted to G is not ramified outside a subset of codimension two.

https://doi.org/10.4153/CMB-2006-028-3 Published online by Cambridge University Press

Let l be a general line in G such that its image $C := \psi_{\sigma\Sigma}(l) \subset Y_\Sigma$ is not contained in the branch locus of $\psi_{\sigma\Sigma}$ and such that, over C, the morphism $\psi_{\sigma\Sigma}$ is a projective bundle. Let $\nu\colon \mathbb{P}^1 \to C \subset Y_\Sigma$ be the normalization of C and let Y_C be the fiber product $Y_C = \mathbb{P}^1 \times_C Y_\sigma$:

$$\begin{array}{ccc} Y_C & \xrightarrow{\bar{\nu}} & Y_\sigma \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \psi_{\sigma\Sigma}} \\ \mathbb{P}^1 & \xrightarrow[\nu]{} & Y_\Sigma \end{array}$$

The variety Y_C is a projective bundle over $\mathbb{P}^1$, so by Lemma 4.2 its cone of curves $\mathrm{NE}(Y_C)$ is generated by the class of a line in a fiber of p and by the class of a minimal section C_0.

The cone of curves $\mathrm{NE}(Y_\sigma)$ is generated by the class of a line in a fiber of $\psi_{\sigma\Sigma}$ and by the class of a line in a fiber of $\psi_{\sigma\tau}$, *i.e.*, the class of l.

The morphism $\bar{\nu}$ induces a map of spaces of cycles $N_1(Y_C) \to N_1(Y_\sigma)$ which allows us to identify $\mathrm{NE}(Y_C)$ with a subcone of $\mathrm{NE}(Y_\sigma)$. Since $\bar{\nu}(Y_C)$ contains lines in the fibers of $\psi_{\sigma\Sigma}$ and contains l, a line in G, we have an identification $\mathrm{NE}(Y_C) \simeq \mathrm{NE}(Y_\sigma)$.

In particular $G \cap \bar{\nu}(Y_C)$, which is a curve whose numerical class in Y_σ is a multiple of $[l]$, is the image of a curve Γ whose numerical class in Y_C is a multiple of $[C_0]$.

By Lemma 4.2 the curve Γ is the union of disjoint minimal sections, so $G \cap \bar{\nu}(Y_C)$ consists of the images via $\bar{\nu}$ of disjoint minimal sections. These images are disjoint curves since $\bar{\nu}$ is one to one on the fibers of p, so every point in C has the same number of preimages via $\psi_{\sigma\Sigma|G}$.

Now, recalling that C was not contained in the branch locus of $\psi_{\sigma\Sigma|G}$, we can conclude that $\psi_{\sigma\Sigma|G}$ is not ramified outside a subset of codimension two, for otherwise, the ramification divisor would be effective, hence ample on G and so it would meet l, a contradicition. We now prove that Y_Σ is a quotient of a projective space; if we remove the ramification and branch sets of $\psi_{\sigma\Sigma|G}$, the finite map

$$\psi_{\sigma\Sigma|G\setminus \mathrm{Ram}}\colon G\setminus \mathrm{Ram} \to Y_\Sigma \setminus \mathrm{Br}$$

is a topological covering.

Since the covering space, $\mathbb{P}^{\dim G}\setminus \{\text{set of codimension two}\}$, is simply connected, this is just the universal cover of $Y_\Sigma\setminus\mathrm{Br}$. The deck transformation group of this covering defines birational maps of $\mathbb{P}^{\dim G}$, which are isomorphisms outside a set of codimension two and therefore the strict transform of divisors is defined and it is the identity on the Picard group; it follows that the deck transformations are linear transformations. In particular $\psi_{\sigma\Sigma|G}\colon \mathbb{P}^{\dim G} \to Y_\Sigma$ is just a quotient by a finite subgroup of $PGL(\dim G)$.

Finally, we can conclude that Y_Σ is smooth by [1, Proposition 1.3], which asserts that if the target of an extremal equidimensional contraction has quotient singularities, then it is smooth. We can thus apply a result of Lazarsfeld [9, Theorem 4.1] which shows that the only smooth variety dominated by a projective space is the projective space itself, and conclude that $Y_\Sigma \simeq \mathbb{P}^{\dim Y_\Sigma}$.

Final step $X = \mathbb{P}^{n(1)} \times \cdots \times \mathbb{P}^{n(k)}$. We finish the proof using the notation of Step 4 and denoting by R the ray R_{i_1}. By the smoothness of Y_Σ and the purity of the branch locus we have that φ_Σ restricted to G is one to one; this implies that the line bundle $H = \varphi_\Sigma^* \mathcal{O}_{\mathbb{P}}(1)$ restricts to $\mathcal{O}_{\mathbb{P}}(1)$ on the fibers of φ_R, and so φ_R is a projective bundle by [6, Lemma 2.12].

In particular we get that Y_R is smooth, and so, by the induction hypothesis, it is a product of projective spaces.

We can write $X = \mathbb{P}_{Y_R}(\mathcal{E}_R)$ where $\mathcal{E}_R$ is vector bundle on Y_R of rank $n(i_1) + 1$.

We claim that the restriction of $\mathcal{E}_R$ to every line in Y_R is trivial; in fact, let l be a line in Y_R and let $\mathcal{E}_l$ be the restriction of $\mathcal{E}_R$ to l; $\mathbb{P}(\mathcal{E}_l)$ has a morphism onto $Y_\Sigma \simeq \mathbb{P}^{\dim Y_\Sigma}$; since $n(i_1) = \dim Y_\Sigma < \dim \mathbb{P}(\mathcal{E}_l) = n(i_1) + 1$ this is possible only if $\mathbb{P}(\mathcal{E}_l) \simeq \mathbb{P}^1 \times \mathbb{P}^{\dim Y_\Sigma}$ (otherwise $\mathbb{P}(\mathcal{E}_l)$ has morphisms either on $\mathbb{P}^1$ or onto a variety of dimension equal to the dimension of $\mathbb{P}(\mathcal{E}_l)$).

Now we proceed by induction on the rank of $\mathcal{E}_R$ to prove that $\mathcal{E}_R$ is a trivial bundle, following [2, Proof of Proposition 1.2].

Let G_Σ be a general fiber of φ_Σ and consider the pullback $\widetilde{\varphi_R}\colon \mathbb{P}(\varphi_\Sigma^* \mathcal{E}_R) \to G_\Sigma$, as in the following diagram:

$$\begin{array}{ccc}
\mathbb{P}(\varphi_R^* \mathcal{E}_R) & \xrightarrow{\widetilde{\varphi_{R|G_\Sigma}}} & X = \mathbb{P}(\mathcal{E}_R) \\
\Big\downarrow \widetilde{\varphi_R} & & \Big\downarrow \varphi_R \\
G_\Sigma & \xrightarrow[\varphi_{R|G_\Sigma}]{} & Y_R
\end{array}$$

By the universal property of the fiber product, the $\mathbb{P}$-bundle $\widetilde{\varphi_R}$ admits a section $s\colon G_\Sigma \to \mathbb{P}(\varphi_R^* \mathcal{E}_i)$ such that $\widetilde{\varphi_{R|G_\Sigma}} \circ s$ is the embedding of G_Σ into X, and so there exists a sequence of vector bundles over G_Σ

$$0 \longrightarrow \mathcal{E}'_R \longrightarrow \varphi_\Sigma^* \mathcal{E}_R \longrightarrow \mathcal{O}_{G_\Sigma} \longrightarrow 0$$

with $\mathcal{E}'_R$ trivial on every line in G_Σ. By induction we get that $\mathcal{E}'_R$ is trivial, and, since $H^1(G_\Sigma, \mathcal{O}_{G_\Sigma}) = 0$, the above sequence splits; we conclude the proof using [2, Lemma 1.2.2]. ■

Corollary 4.3 *Let X be a Fano manifold of pseudoindex i_X. If $2i_X \geq \dim X + 2$ then $\rho_X = 1$ except if $X \simeq (\mathbb{P}^{i_X - 1})^2$.*

Proof Since X is a Fano manifold, through every point of X there exists a rational curve of anticanonical degree $\leq \dim X + 1$, hence there exists an irreducible component $V^1 \subset \mathrm{RatCurves}^n_d(X)$ of anticanonical degree $d \leq \dim X + 1$ which is a covering family; note that, by our assumptions on the pseudoindex, the family V^1 is an unsplit family.

Let R be an extremal ray of X which does not contract curves of V^1; and let V^2 be the family of rational curves corresponding to a minimal degree curve whose numerical class is in R; if such a ray does not exist, then $\rho_X = 1$ and we are done.

By Theorem 3.5 we have

$$\dim X \geq \dim \mathrm{Locus}(V^1, V^2)_x \geq -\sum K_X.V^i - 2 \geq 2i_X - 2 \geq \dim X.$$

We get that $-K_X.V^1 = -K_X.V^2 = i_X - 1$, $\mathrm{Locus}(V^1, V^2)_x = X$ and that V^2 is a covering family, so we can apply Theorem 1.1. ■

Acknowledgements I wish to thank Jarosław Wiśniewski for some helpful suggestions.

References

[1] M. Andreatta, and J. A. Wiśniewski, *A view on contractions of higher dimensional varieties.* In: Algebraic Geometry, Proc. Sympos. Pure Math. 62, American Mathematical Society, Providence, RI, 1997, pp. 153–183.

[2] ———, *On manifolds whose tangent bundle contains an ample subbundle. Invent. Math.* **146**(2001), no. 1, 209–217.

[3] M. C. Beltrametti, A. J. Sommese, and J. A. Wiśniewski, *Results on varieties with many lines and their applications to adjunction theory.* In: Complex Algebraic Varieties 1507, Lecture Notes in Math. 1507, Springer, Berlin, 1992, pp. 16–38.

[4] L. Bonavero, C. Casagrande, O. Debarre, and S. Druel, *Sur une conjecture de Mukai.* Comment. Math. Helv. **78**(2003), no. 3, 601–626.

[5] K. Cho, Y. Miyaoka, and N. Shepherd-Barron, *Characterizations of projective space and applications to complex symplectic manifolds.* In: Higher Dimensional Birational Geometry, Adv. Stud. Pure Math. 35, Math. Soc. Japan, Tokyo, 2002, 1–88.

[6] T. Fujita, *On polarized manifolds whose adjoint bundles are not semipositive.* In: Algebraic Geometry, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam, 1987, pp. 167–178.

[7] S. Kebekus, *Characterizing the projective space after Cho, Miyaoka and Sheperd-Barron.* In: Complex Geometry, Springer, Berlin, 2002, pp. 147–155.

[8] J. Kollár, *Rational Curves on Algebraic Varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete 32, Springer-Verlag, Berlin, 1996.

[9] R. Lazarsfeld, *Some applications of the theory of positive vector bundles.* In: Complete Intersections, Lecture Notes in Math. 1092, Springer, Berlin, 1984, pp. 29–61.

[10] S. Mukai, *Open problems.* In: Birational Geometry of Algebraic Varieties, Cambridge Tracts in Mathematics 134, Cambridge University Press, Cambridge, 1988, pp. 67–60.

[11] J. A. Wiśniewski, *On a conjecture of Mukai.* Manuscripta Math. **68**(1990), no. 2, 135–141.

Dipartimento di Matematica
Università degli Studi di Trento
Via Sommarive, 14
I-38050 Povo (Trento)
Italy
e-mail: occhetta@science.unitn.it