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ADAPTED COMPLEX STRUCTURES AND GEOMETRIC QUANTIZATION

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Abstract. A compact Riemannian symmetric space admits a canonical complexification. This so called adapted complex manifold structure J_A is defined on the tangent bundle. For compact rank-one symmetric spaces another complex structure J_S is defined on the punctured tangent bundle. This latter is used to quantize the geodesic flow for such manifolds. We show that the limit of the push forward of J_A under an appropriate family of diffeomorphisms exists and agrees with J_S.

§0. Introduction

Different kind of geometric constructions naturally lead to complex structures defined on tangent bundles. The main purpose of this paper is to explore the relationship of two such constructions.

The first of these is the so called adapted complex structure and the second comes from geometric quantization.

Adapted complex structures (see Section 2 for the precise definition) were discovered by studying certain global solutions of the complex, homogeneous Monge-Ampère equation on Stein manifolds. One starts with a compact, real-analytic Riemannian manifold (M, g). The adapted complex structure lives on TM or perhaps only on a subdomain T^rM consisting of those vectors having length smaller than r for some fixed positive radius r.

These complex manifolds have many interesting properties. Among others the norm-square function is strictly plurisubharmonic and thus it defines a Kähler metric. The Kähler form of this metric is the symplectic form Ω of the tangent bundle (that comes from the canonical symplectic structure of the cotangent bundle using the metric to identify these two bundles).

The second kind of complex structures J_S live on the punctured tangent bundle of compact, rank-1 symmetric spaces. For the spheres it was Souriau

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who in [So] identified the regularized Kepler manifold with the cotangent bundle of the sphere minus its zero section ($\mathring{\mathrm{T}}^*S^n$). He also defined a complex manifold structure on this space identifying it with the singular affine hyperquadric.

Later it was observed by Rawnsley [Ra1], that the norm function is strictly plurisubharmonic with respect to the above complex structure and thus defines a Kähler metric on $\mathring{\mathrm{T}}S^n$ and the Kähler form is again Ω. He also observed the complex structure to be invariant w.r.t. the normalized geodesic flow that he used in [Ra2] to quantize the geodesic flow on the spheres.

Subsequently Furutani and Tanaka [F-T] defined a Kähler structure on the punctured cotangent bundle of complex and quaternionic projective spaces. Their Kähler structure is also invariant w.r.t. the normalized geodesic flow. Furutani and Yoshizawa [F-Y] used this Kähler structure to quantize the geodesic flow on complex and quaternionic projective spaces.

Furutani and Tanaka described their Kähler structures in terms of matrices in the spirit of Lie groups. A more geometric description of the complex (Kähler) structure on $\mathring{\mathrm{T}}M$ (where M now can be either a sphere, a complex or quaternionic projective space or their quotient w.r.t. a discrete group of isometries) was given by Kiyotaka Ii [Ii]. In this paper we shall use Ii's description of these structures (see Section 3 for more details).

The main result of this paper (Theorem 3.2) is to show that for compact, rank-1 symmetric spaces the family of complex structures obtained by pushing forward the adapted complex structure with respect to an appropriate family of diffeomorphisms has a limit and this limit complex structure coincides with J_S.

After this paper was finished, we received a preprint from R. Aguilar [Ag], discussing, among other things, similar limits. Instead of the punctured tangent bundle, he works on the unit sphere bundle.

§1. Geometry of the tangent bundle

Let M be a smooth n-dimensional manifold and TM its tangent bundle. π will denote the projection map $\pi : TM \to M$. We shall denote by $\mathring{\mathrm{T}}M$ the punctured tangent bundle i.e., $\mathring{\mathrm{T}}M = TM \setminus \{\text{zero section}\}$.

Suppose a Riemannian metric g is also given on M. The metric identifies the tangent and cotangent bundles and thus we can also talk about the

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canonical 1-form θ on TM, that is defined by

$$\theta(v) := g(z, \pi_* v), \quad v \in T_z(TM). \tag{1.1}$$

$\Omega := d\theta$ defines then the symplectic form of TM.

When $M = \mathbb{R}$ we can canonically identify $T\mathbb{R}$ with the complex plane:

$$T_\sigma \mathbb{R} \ni \tau \frac{\partial}{\partial \sigma} \overset{\iota}{\longmapsto} \sigma + i\tau \in \mathbb{C}. \tag{1.2}$$

From now on we shall always tacitly assume this identification. Denote by $\mathcal{H}_+$ the upper half plane and by $T^+\mathbb{R}$ the corresponding part of $T\mathbb{R}$. For simplicity, we shall also assume that our manifold (M, g) is complete.

Let $\gamma : \mathbb{R} \to M$ be a geodesic and $\gamma_* : T\mathbb{R} \to TM$ the induced map. As γ runs through all geodesics, the images $\gamma_*(T^+\mathbb{R})$ define a foliation of $\mathring{\mathrm{T}}M$. This foliation is called the *Riemann foliation*. We shall denote the corresponding distribution on $\mathring{\mathrm{T}}M$ by $\mathcal{L}$.

Let $\rho : TM \to \mathbb{R}$ be the norm-square function. There is another natural distribution on $\mathring{\mathrm{T}}M$, denoted by $\mathcal{H}$, that at a point $z \in \mathring{\mathrm{T}}M$ is defined by

$$\mathcal{H}_z := \ker \theta_z \cap \ker d\rho. \tag{1.3}$$

$\mathcal{H}$ and $\mathcal{L}$ are complementary distributions, i.e., $T_z(\mathring{\mathrm{T}}M) = \mathcal{L}_z + \mathcal{H}_z$.

For a real number s define the map $N_s : TM \to TM$ by multiplying every vector in each fibre with s. For a non-zero s, N_s is a diffeomorphism.

The metric g determines the Levi-Cività connection on TM and thus a splitting of $T_z(TM)$ into horizontal and vertical subspaces. Let $v, z \in T_pM$. We shall denote the horizontal (resp. vertical) lift of the vector v to z by $\xi_z^v \in T_z(TM)$ (resp. $\eta_z^v \in T_z(TM)$).

For an $\lambda \neq 0$ we get

$$\begin{aligned} (N_\lambda)_* \xi_z^v &= \xi_{\lambda z}^v, \\ (N_\lambda)_* \eta_z^v &= \lambda \eta_{\lambda z}^v. \end{aligned} \tag{1.4}$$

Now let γ be a unit speed geodesic. Let $z = \tau_0 \dot{\gamma}(\sigma_0)$, $v = \dot{\gamma}(\sigma_0)$. Then we have

$$\begin{aligned} (\gamma_*)_* \left(\frac{\partial}{\partial \sigma}\Big|_{(\sigma_0, \tau_0)} \right) &= \xi_z^v, \\ (\gamma_*)_* \left(\frac{\partial}{\partial \tau}\Big|_{(\sigma_0, \tau_0)} \right) &= \eta_z^v. \end{aligned} \tag{1.5}$$

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§2. Adapted complex structures

For an $0 < r \leq \infty$ denote by $T^r M$ the tube of vectors having length smaller than r.

DEFINITION 2.1. Let (M, g) be a complete Riemannian manifold. A smooth complex structure on $T^r M$ is called *adapted* if for any geodesic $\gamma : \mathbb{R} \to M$, the map

$$\gamma_* : (\gamma_*)^{-1}(T^r M) \longrightarrow T^r M$$

is holomorphic, where $(\gamma_*)^{-1}(T^r M) \subset T\mathbb{R}$ and $T\mathbb{R}$ is endowed with the complex structure from (1.2).

The adapted complex structure (if it exists) is unique and when (M, g) is compact, real-analytic, we have existence for small enough r (see [G-S], [L-Sz], [Sz]).

One particular example that shall concern us in Section 3 is the case of the round sphere. The standard realization of TS^n is

$$\{(e, X) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \mid \|e\| = 1, \langle e, X\rangle = 0\}. \tag{2.1}$$

Let $X_1, X_2 \in S^n$, $\langle X_1, X_2\rangle = 0$. Then $\gamma : \mathbb{R} \to S^n \subset \mathbb{R}^{n+1}$, $\gamma(\theta) = \cos\theta X_1 + \sin\theta X_2$ is a unit speed geodesic (i.e., a great circle) of S^n. Complexifying this map gives $\gamma_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}^{n+1}$,

$$\gamma_{\mathbb{C}}(\zeta) = \cos\zeta X_1 + \sin\zeta X_2.$$

The image $\gamma_{\mathbb{C}}(\mathbb{C})$ lies in the affine hyperquadric $Q_1 \subset \mathbb{C}^{n+1}$.

$$\begin{aligned} Q_1 &= \left\{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} z_j^2 = 1\right\} \\ &= \{X + iY \mid X, Y \in \mathbb{R}^{n+1}, \langle X, X\rangle - \langle Y, Y\rangle = 1, \langle X, Y\rangle = 0\}. \end{aligned} \tag{2.2}$$

If we can find a diffeomorphism $\delta : TS^n \to Q_1$ such that the following diagram commutes, the pullback complex structure on TS^n will be adapted (ι is from (1.2)).

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\gamma_{\mathbb{C}}} & Q \\ {\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\delta} \\ T\mathbb{R} & \xrightarrow{\gamma_*} & TS^n. \end{array}$$

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A little bit of calculation gives that the choice

$$\delta(e,X) := \cosh(\|X\|)e + i\frac{\sinh(\|X\|)}{\|X\|}X \tag{2.3}$$

will work.

It is always possible to express the almost complex tensor J_A of the adapted complex structure in terms of Jacobi fields and analytic continuation (see [L-Sz]). On a symmetric space Jacobi's equation can be explicitly solved. This suggests that for such spaces the formula for J_A should be very explicit as well. This is indeed the case.

Denote by $\mathcal{R}$ the curvature tensor of the metric and let $z \in T_pM$. The operator $R_z(\cdot) = \mathcal{R}(\cdot\,, z)z$ is the Jacobi operator associated to z. In fact a little bit of calculation together with [Sz, Theorem 2.5] yields:

PROPOSITION 2.2. *Let (M,g) be a compact symmetric space (or a locally symmetric quotient of such). Let $z \in \mathring{\mathrm{T}}M$, $v_1 = z/\|z\|$. Let $v_1, v_2, \ldots, v_n$ be an orthonormal basis of $T_{\pi z}M$ consisting of eigenvectors of the Jacobi operator R_{v_1} with eigenvalue Λ_j, $j = 1, \ldots, n$. Let $h(x) := x\coth x$. Let $\xi_z^{v_j}$ (resp. $\eta_z^{v_j}$) be the horizontal (resp. vertical) lift of v_j to the point z, $j = 1, \ldots, n$. Then*

$$J_A\xi_z^{v_j} = h\big(\sqrt{\Lambda_j}\|z\|\big)\eta_z^{v_j}. \tag{2.4}$$

i.e., with the positive, real-analytic function $t(x) := h(\sqrt{x})$, the matrix of J_A in the basis $\xi_z^{v_j}, \eta_z^{v_j}$, $j = 1, \ldots, n$ is:

$$J_A|_z = \begin{bmatrix} 0 & -(t(R_z))^{-1} \\ t(R_z) & 0 \end{bmatrix}. \tag{2.5}$$

§3. Complex structures on $\mathring{\mathrm{T}}M$

In this section we discuss the complex structures on the punctured tangent bundles of compact, rank-1 symmetric spaces that arose in geometric quantization, using Ii's description of these structures. We start first with the spheres. We identify (using the metric) the tangent and cotangent bundles and take the standard representation (2.1) of the punctured tangent bundle of S^n. The singular affine hyperquadric Q_0 is defined as

$$Q_0 = \left\{ z = (z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} z_j^2 = 0 \right\}.$$

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The complex structure J_S on $\mathring{\mathrm{T}}S^n$ is obtained by identifying it with Q_0. More precisely, the map $\mathring{\mathrm{T}}S^n \to Q_0 \setminus \{0\}$ defined by

$$(e, X) \longmapsto z = \|X\|e + iX \tag{3.1}$$

is a diffeomorphism. The pull back gives us the desired complex structure on $\mathring{\mathrm{T}}S^n$.

Ii's description of J_S on $\mathring{\mathrm{T}}S^n$ is as follows. For $0 \neq z \in T_pS^n, v \in T_pM$, denote by ξ_z^v (resp. η_z^v) the horizontal (resp. vertical) lift of the vector v to the point z. Then

$$J_S\xi_z^v = \|z\|\eta_z^v. \tag{3.2}$$

Our first observation about J_A (the adapted complex tensor) and J_S is that they are genuinely different complex structures on $\mathring{\mathrm{T}}S^n$.

PROPOSITION 3.1. *The complex manifolds* $(\mathring{\mathrm{T}}S^n, J_A)$ *and* $(\mathring{\mathrm{T}}S^n, J_S)$ *are not biholomorphic.*

Proof. The statement is equivalent to showing that $X = Q_0 \setminus \{0\}$ and $Y = Q_1 \setminus S^n$ (where Q_1 is the affine hyperquadric (see (2.2)) are not biholomorphic. Suppose on the contrary that $\psi : Y \to X$ is a biholomorphism. We can apply a theorem of Hartogs (see [Sh]), to conclude that ψ has a holomorphic extension $\tilde{\psi} : Q_1 \to \mathbb{C}^{n+1}$ (the theorem we use here says that if N is a complex manifold, $L \subset N$ a real submanifold of real codimension at least 2 and $\psi : N \setminus L \to \mathbb{C}$ is holomorphic which does not extend holomorphically to a point $q \in L$, then in a neighborhood of q, L is a complex submanifold).

Since Q is defined as the zero set of a holomorphic function, therefore $\tilde{\psi}$ also maps into Q_0. Since ψ is biholomorphic and $\tilde{\psi}$ is holomorphic, $\tilde{\psi}(S^n)$ must be the point 0. But S^n is a maximal dimensional totally real submanifold in Q, hence $\tilde{\psi}$ must be constant, a contradiction. ∎

Despite of the negative result of Proposition 3.1, J_A and J_S do have something to do with each other. To discover their relation, consider now for any $t \geq 0$ the quadric

$$Q_t = \left\{ z \in \mathbb{C}^{n+1} \;\middle|\; \sum_{j=1}^{n+1} z_j^2 = t \right\}.$$

For $t, s > 0$, Q_t and Q_s are biholomorphic complex submanifolds and $Q_t \setminus \mathbb{R}^n$ and $Q_s \setminus \mathbb{R}^n$ are both biholomorphic to $(\mathring{\mathrm{T}}S^n, J_A)$. The complex manifold

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$Q_0 \setminus \mathbb{R}^n = Q_0 \setminus \{0\}$ is in some sense the limit of the complex manifolds $Q_t \setminus \mathbb{R}^n$ when t goes to zero. This gives the idea to try to push forward J_A by a family of diffeomorphisms in such a way that the limit of these push forwards is J_S. This indeed can be done as the following theorem shows.

THEOREM 3.2. *Let (M, g) be a compact, rank-1 symmetric space. Denote by Φ_ε the diffeomorphism* : $\mathring{\mathrm{T}}M \to \mathring{\mathrm{T}}M$

$$\Phi_\varepsilon(v) = \varepsilon \exp(\|v\|)\frac{v}{\|v\|}. \tag{3.3}$$

Then the following limit complex structure J_0 exits

$$\lim_{\varepsilon \to 0} (\Phi_\varepsilon)_* J_A = J_0.$$

The maps $N_s, s \neq 0$ and the normalized geodesic flow are J_0 holomorphic. When M is the sphere, J_0 agrees with J_S and for the complex and quaternionic projective spaces it coincides with the structure studied by Furutani-Tanaka-Ii ([F-T], [Ii]).

Remark. Since the entire construction and the proof is compatible with taking quotients w.r.t. a discrete group of the isometry group, the result of Theorem 3.2 is valid for such quotients as well.

When (M, g) is a round sphere, it was already noticed by Su-Jen Kan in [Ka], that the CR-manifolds $S_r = \{\|v\| = r\}$ posses a limit CR structure as $r \to \infty$.

Before we go on with the proof, we give another description of the complex structures studied in [F-T], [Ii], [Ra1], [So]. Recall the notion of Jacobi operator R_z from Section 2. R_z is self-adjoint and z always belongs to its kernel. We shall need a modified operator $\widetilde{R}_z : T_pM \to T_pM$, defined as follows

$$\widetilde{R}_z(X) = \langle X, z\rangle z + R_z(X) = \langle X, z\rangle z + \mathcal{R}(X, z)z. \tag{3.4}$$

Define the almost complex tensor $J_0 : T_z(TM) \to T_z(TM)$ in the horizontal and verical decomposition $T_z(TS^n) = H_z + V_z$, by the formula

$$J_0 = \begin{bmatrix} 0 & -\left(\sqrt{\widetilde{R}_z}\right)^{-1} \\ \sqrt{\widetilde{R}_z} & 0 \end{bmatrix}. \tag{3.5}$$

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EXAMPLE.

1) Let (M, g) have constant curvature 1. Then the curvature tensor $\mathcal{R}$ has a particularly simple form $\mathcal{R}(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y$.

Therefore in this case $\widetilde{R}_z(X) = \langle z,z\rangle X$. This implies that for $z \neq 0$ the complex structure J_S of (3.2), agrees with J_0.

2) Let now (M, g, J) be a Kähler manifold of constant holomorphic sectional curvature 4. Then the curvature tensor has the following form

$$\begin{aligned}\mathcal{R}(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y - \langle Y,JZ\rangle JX \\ +\langle X,JZ\rangle JY + 2\langle X,JY\rangle JZ.\end{aligned}$$

This yields

$$\widetilde{\mathcal{R}}_z v = \langle z,z\rangle v + 3\langle v,Jz\rangle Jz.$$

From this it follows:

$$\sqrt{\widetilde{\mathcal{R}}_z} v = \frac{\langle v,Jz\rangle}{\|z\|} Jz + \|z\| v, \quad z \neq 0. \tag{3.6}$$

The complex structure on $\mathring{\mathrm{T}}M$ studied by Furutani and Tanaka in [F-T] is defined by (see [Ii])

$$J_1(\xi_z^v) = \begin{cases} 2\|z\|\eta_z^v, & \text{if } v = \alpha Jz, \text{ for some } \alpha \in \mathbb{R}, \\ \|z\|\eta_z^v, & \text{if } v \perp Jz. \end{cases}$$

It follows from formula (3.6) that J_1 in fact has the form of (3.5).

3) Let (M, g, I_1, I_2, I_3) be a quaternion Kähler manifold of constant Q-sectional curvature 4. Then

$$\begin{aligned}\mathcal{R}(X,Y)Z = \langle Y,Z\rangle X - \langle X,Z\rangle Y \\ -\sum_{i=1}^{3} \langle Y,I_iZ\rangle I_iX + \langle X,I_iZ\rangle I_iY + 2\langle X,I_iY\rangle I_iZ.\end{aligned}$$

This yields

$$\widetilde{\mathcal{R}}_z v = \langle z,z\rangle v + \sum_{i=1}^{3} \langle v,I_iz\rangle I_iz.$$

Thus

$$\sqrt{\widetilde{\mathcal{R}}_z} v = \|z\| v + \sum_{i=1}^{3} \frac{\langle v,I_iz\rangle}{\|z\|} I_iz. \tag{3.7}$$

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The complex structure of Furutani-Tanaka is defined on $\mathring{\mathrm{T}}M$ by (see [Ii])

$$J_2\xi_z^v = \begin{cases} 2\|z\|\eta_z^v, & \text{if } v = \sum_{i=1}^3 \alpha_i I_i z \text{ for some } \alpha_i \in \mathbb{R}, \\ \|z\|\eta_z^v, & \text{if } v \perp I_i z,\ i = 1,2,3. \end{cases}$$

Comparing this to formula (3.7) yields that J_2 again has the form (3.5).

Remark. The formula (3.5) for J_0 is so simple that it cries for a generalization for higher rank symmetric spaces. Of course when the rank is larger than 1, the operators $\sqrt{\widetilde{R}_z}$ are no longer invertible, so formula (3.5) makes no sense literaly. But an appropriate formulation of Theorem 3.2 is still valid. We shall discuss these circle of questions in another publication.

Proof of Theorem 3.2. We shall prove that $\lim_{\varepsilon\to 0}(\Phi_\varepsilon)_* J_A$ exists and has the form of (3.5).

STEP 1.

The diffeomorphisms Φ_ε map any leaf of the Riemann foliation onto itself. First we shall prove the existence of the limit complex structure along a leaf.

Let $\gamma : \mathbb{R} \to M$ be a unit speed geodesic. Denote by $\mathcal{H}_+$ the upper half plane. Then $\gamma_* : \mathcal{H}_+ \to TM$, $(\sigma + i\tau) \mapsto \tau\dot{\gamma}(\sigma)$ parametrizes a leaf. Define $\psi_\varepsilon : \mathcal{H}_+ \to \mathcal{H}_+$ by $\psi_\varepsilon(\sigma + i\tau) = \sigma + i\varepsilon \exp\tau$. Then ψ_ε is a diffeomorphism of $\mathcal{H}_+$ and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_+ & \xrightarrow{\gamma_*} & (TM, J_A) \\ {\scriptstyle\psi_\varepsilon}\downarrow & & \downarrow{\scriptstyle\Phi_\varepsilon} \\ \mathcal{H}_+ & \xrightarrow{\gamma_*} & (TM, J_A). \end{array}$$

Denote by J_+ the complex tensor on $\mathcal{H}_+$. Since γ_* is holomorphic, we get

$$(\Phi_\varepsilon)_* J_A = (\Phi_\varepsilon)_*(\gamma_*)_* J_+ = (\gamma_*)_*(\psi_\varepsilon)_* J_+. \tag{3.8}$$

Now let $\sigma_1 + i\tau_1 \in \mathcal{H}_+$. It follows from the definition of ψ_ε, that

$$(\psi_\epsilon)_* \left(\partial\sigma|_{\sigma_1+i\tau_1}\right) = \partial\sigma|_{\sigma_1+i\varepsilon\exp\tau_1}$$

and

$$(\psi_\epsilon)_* \left(\partial\tau|_{\sigma_1+i\tau_1}\right) = \varepsilon e^{\tau_1}\partial\tau|_{\sigma_1+i\varepsilon\exp\tau_1}.$$

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This implies that the push forward complex structure at a fixed point $\sigma_0+i\tau_0$ can be computed as follows

$$
\begin{aligned}
(3.9)\qquad & [(\psi_\varepsilon)_* J_+]\left(\partial\sigma|_{(\sigma_0+i\tau_0)}\right) \\
&= (\psi_\varepsilon)_*\left(J_+|_{(\sigma_0+i\log(\tau_0/\varepsilon))}\left(\partial\sigma|_{(\sigma_0+i\log(\tau_0/\varepsilon))}\right)\right) \\
&= (\psi_\varepsilon)_*\left(\partial\tau|_{(\sigma_0+i\log(\tau_0/\varepsilon)}\right) \\
&= \varepsilon\exp(\log(\tau_0/\varepsilon))\,\partial\tau|_{(\sigma_0+i\tau_0)} \\
&= \tau_0\,\partial\tau|_{(\sigma_0+i\tau_0)}
\end{aligned}
$$

Let $Y=\dot\gamma(\sigma_0)$ and $X=\tau_0 Y$. Recall from (1.5) that

$$(\gamma_*)_*\big(\partial\sigma|_{(\sigma_0+i\tau_0)}\big)=\xi^Y_X,\quad (\gamma_*)_*\big(\partial\tau|_{(\sigma_0+i\tau_0)}\big)=\eta^Y_X.$$

Furthermore

$$\widetilde{R}_X(Y)=\langle Y,X\rangle X,$$

hence

$$\sqrt{\widetilde{R}_X}Y=\frac{\langle Y,X\rangle}{\|X\|}X=X.$$

So we get

$$(3.10)\qquad J_0\left((\gamma_*)_*\big(\partial\sigma|_{(\sigma_0+\tau_0)}\big)\right)=J_0\xi^Y_X=\eta^X_X=(\gamma_*)_*\big(\tau_0\partial\tau|_{(\sigma_0+i\tau_0)}\big).$$

(3.8), (3.10), (3.10) together imply that the image $\gamma_*(\mathcal{H}_+)$ is a complex submanifold w.r.t. both structures, J_0 and $(\Phi_\varepsilon)_*J_A$ and in fact these two complex structures coincide.

Step 2.

To complete the proof of the theorem we need to show the existence of the limit $(\Phi)_*J_A$, when we restrict ourselves to the distribution $\mathcal{H}$, the complementary distribution to $\mathcal{L}$ (see (1.3). Let $\varepsilon>0$ and $p\in\mathring{\mathrm{T}}M$ be fixed, $\|p\|=c$, define $v_1=p/c$, $r=\log(c/\varepsilon)$, $q=(r/c)p$, $\lambda=(\varepsilon e^r)/r$. Then Φ_ε maps the hypersurface $S_r=\{u=r\}$ onto $S_c=\{u=c\}$.

As before denote by N_λ the diffeomorphism of TM that is multiplication by λ in the fibers. It is clear that

$$(3.11)\qquad \Phi_\varepsilon|_{S_r}=N_\lambda|_{S_r}.$$

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Choose eigenvectors $v_2, \dots, v_n$ of R_{v_1} such that together with v_1 they form an orthonormal basis for $T_{\pi(p)}M$. Recall from (2.4)

$$J_A \xi_q^{v_j} = h\big(\sqrt{\Lambda_j} r\big) \eta_q^{v_j}, \tag{3.12}$$

where $h(x) = x \coth x$. From (1.4) we know

$$(N_\lambda)_* \xi_q^{v_j} = \xi_p^{v_j}, \quad (N_\lambda)_* \eta_q^{v_j} = (c/r) \eta_p^{v_j}. \tag{3.13}$$

From (3.11), (3.12) and (3.13) we get

$$\begin{aligned} (\Phi_\varepsilon) * J_A \xi_p^{v_j} &= (\Phi_\varepsilon)_* \left(J_A ((\Phi_\varepsilon)_*^{-1} \xi_p^{v_j}) \right) \\ &= (\Phi_\varepsilon)_* (J_A \xi_q^{v_j}) \\ &= (\Phi_\varepsilon)_* \left(h\big(\sqrt{\Lambda_j} r\big) \eta_q^{v_j} \right) \\ &= h\big(\sqrt{\Lambda_j} r\big)(c/r) \eta_p^{v_j}. \end{aligned} \tag{3.14}$$

Now $r \to \infty$ as $\varepsilon \to 0$. Hence $h\big(\sqrt{\Lambda_j} r\big)(c/r) = \sqrt{\Lambda_j} \coth\big(\sqrt{\Lambda_j} r\big) c \to \sqrt{\Lambda_j} c$. This together with (3.14) implies

$$\lim_{\varepsilon \to 0} (\Phi_\varepsilon)_* J_A \xi_p^{v_j} = \sqrt{\Lambda_j} c \eta_p^{v_j}. \tag{3.15}$$

On the other hand

$$\begin{aligned} \widetilde{R}_p v_j &= \langle v_j, p \rangle p + R_p(v_j) \\ &= \langle v_j, c v_1 \rangle c v_1 + c^2 R_{v_1} v_j \\ &= c^2 \Lambda_j v_j. \end{aligned}$$

This implies

$$J_0 \xi_p^{v_j} = c \sqrt{\Lambda_j} \eta_p^{v_j}. \tag{3.16}$$

Finally (3.15) and (3.16) together imply that indeed $\lim_{\varepsilon \to 0} (\Phi_\varepsilon)_* J_A$ exists and equals to J_0. The examples after (3.5) show that J_0 in fact coincides with the complex structure studied by Furutani-Tanaka-Ii. The normalized geodesic flow was shown to be J_0-holomorphic in [Ii].

Let now $0 \neq s, 0 \neq p \in T_m M$ and $v \in T_m M$ be arbitrary. Define w by $w = \sqrt{\widetilde{R}_{sp}} v$. From the definition of $\widetilde{R}_{sp}$ we get that

$$w = s \sqrt{\widetilde{R}_p} v.$$

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This together with the action (1.4) of $(N_s)_*$ and the definition (3.5) of J_0 yields:

$$J_0|_{sp}\,(N_s)_*\xi_p^v = J_0|_{sp}\,\xi_{sp}^v = \xi_{sp}^w = (N_s)_* J_p \xi_p^v,$$

i.e., N_s is indeed J_0-holomorphic. □

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