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Nagoya Math. J.
Vol. 134 (1994), 65–73

COMMUTATIVE ALGEBRAS FOR ARRANGEMENTS

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1. Introduction

Let V be a vector space of dimension l over some field $\mathbf{K}$. A hyperplane H is a vector subspace of codimension one. An arrangement $\mathscr{A}$ is a finite collection of hyperplanes in V. We use [7] as a general reference. Let $M(\mathscr{A}) = V - \cup_{H\in\mathscr{A}} H$ be the complement of the hyperplanes. Let V^* be the dual space of V. Each hyperplane $H \in \mathscr{A}$ is the kernel of a linear form $\alpha_H \in V^*$, defined up to a constant. The product

$$Q(\mathscr{A}) = \prod_{H\in\mathscr{A}} \alpha_H$$

is called a *defining polynomial* of $\mathscr{A}$. Brieskorn [3] associated to $\mathscr{A}$ the finite dimensional skew-commutative algebra $R(\mathscr{A})$ generated by 1 and the differential forms $d\alpha_H/\alpha_H$ for $H \in \mathscr{A}$. When $\mathbf{K} = \mathbf{C}$, the algebra $R(\mathscr{A})$ is isomorphic to the cohomology algebra of the open manifold $M(\mathscr{A})$. The structure of $R(\mathscr{A})$ was determined in [6] as the quotient of an exterior algebra by an ideal. In particular this shows that $R(\mathscr{A})$ depends only on the intersection poset of $\mathscr{A}$, $L(\mathscr{A})$, and not on the individual linear forms α_H.

A subarrangement $\mathscr{B} \subseteq \mathscr{A}$ is called *independent* if $\cap_{H\in\mathscr{B}} H$ has codimension $|\mathscr{B}|$, the cardinality of $\mathscr{B}$. In a special lecture at the Japan Mathematical Society in 1992, Aomoto suggested the study of the graded $\mathbf{K}$-vector space

$$AO(\mathscr{A}) = \sum_{\mathscr{B}} \mathbf{K}Q(\mathscr{B})^{-1}, \quad \mathscr{B} \text{ independent.}$$

It appears as the top cohomology group of a certain 'twisted' de Rham chain complex [1]. When $\mathbf{K} = \mathbf{R}$, he conjectured that the dimension of $AO(\mathscr{A})$ is equal to the number of connected components (chambers) of $M(\mathscr{A})$, which he proved for generic arrangements. In this paper we prove this conjecture in general. We construct a

Received March 3, 1993.

[1] This work was supported in part by the National Science Foundation

https://doi.org/10.1017/S0027763000004852 Published online by Cambridge University Press

commutative algebra $W(\mathscr{A})$ which is isomorphic to $AO(\mathscr{A})$ as a graded vector space. Note that $AO(\mathscr{A})$ is not closed under multiplication because a product $Q(\mathscr{B})^{-1}Q(\mathscr{B}')^{-1}$ may contain the same linear form α_H more than once. In order to allow this, we need the following definition. A *multiarrangement* $\mathscr{E}$ is a finite set of hyperplanes where each hyperplane may occur more than once. The multiplicity of H in $\mathscr{E}$, $m(H, \mathscr{E})$ is the number of times H occurs in $\mathscr{E}$. The cardinality of $\mathscr{E}$, $|\mathscr{E}|$, is the total number of elements of $\mathscr{E}$, each hyperplane counted with its multiplicity. Let $\mathbf{E}_p(\mathscr{A})$ be the set of multisubarrangements $\mathscr{E}$ of $\mathscr{A}$ of cardinality p. Let $\mathbf{E}(\mathscr{A}) = \cup_{p \geq 0}\mathbf{E}_p(\mathscr{A})$. This union is disjoint. We write $\mathbf{E} = \mathbf{E}(\mathscr{A})$ when $\mathscr{A}$ is fixed. Let $\cap\mathscr{E} = \cap_{H\in\mathscr{E}}H$. We call $\mathscr{E} \in \mathbf{E}$ *independent* if codim $(\cup\mathscr{E}) = |\mathscr{E}|$, and *dependent* otherwise. Note that if $m(H, \mathscr{E}) > 1$ for some $H \in \mathscr{E}$, then $\mathscr{E}$ is dependent. Let $\mathbf{E}^i$ denote the set of independent multisubarrangements. This is a finite set. Let $\mathbf{E}^d$ denote the set of dependent multisubarrangements. This is an infinite set. There is a disjoint union $\mathbf{E} = \mathbf{E}^i \cup \mathbf{E}^d$. Let $S = S(V^*)$ be the symmetric algebra of V^*. Choose a basis $\{e_1, \ldots, e_l\}$ in V and let $\{x_1, \ldots, x_l\}$ be the dual basis in V^* so $x_i(e_j) = \delta_{i,j}$. We may identify $S(V^*)$ with the polynomial algebra $S = \mathbf{K}[x_1, \ldots, x_l]$. Let $Q(\mathscr{E}) = \Pi_{H\in\mathscr{E}}\,\alpha_H$ for $\mathscr{E} \in \mathbf{E}$. Note that α_H appears with multiplicity $m(H, \mathscr{E})$ in $Q(\mathscr{E})$. Let $S_{(0)}$ be the field of quotients of S, the field of rational functions on V.

Definition 1.1. Let $\mathbf{K}[\alpha_{\mathscr{A}}^{-1}]$ be the $\mathbf{K}$-subalgebra of $S_{(0)}$ generated by

$$\{Q(\mathscr{E})^{-1} \mid \mathscr{E} \in \mathbf{E}\}.$$

Let $J(\mathscr{A})$ be the ideal of $\mathbf{K}[\alpha_{\mathscr{A}}^{-1}]$ generated by $\{Q(\mathscr{E})^{-1} \mid \mathscr{E} \in \mathbf{E}^d\}$.
Let $W(\mathscr{A}) = \mathbf{K}[\alpha_{\mathscr{A}}^{-1}]/J(\mathscr{A})$.

Consider the usual grading of $S_{(0)}$. Since $J(\mathscr{A})$ is a homogeneous ideal, $W(\mathscr{A})$ is a graded commutative algebra. There is a natural map of graded vector spaces $j: AO(\mathscr{A}) \to W(\mathscr{A})$ defined by $Q(\mathscr{B})^{-1} \mapsto [Q(\mathscr{B})^{-1}]$. It is clear that $AO(\mathscr{A})$ is finite dimensional because the set $\mathbf{E}^i$ is finite. Since the map j is surjective, the algebra $W(\mathscr{A})$ is also a finite dimensional $\mathbf{K}$-vector space. Its total dimension, Poincaré polynomial, or algebra structure are not obvious at this point. In the rest of this paper we determine these.

In Section 2 we define a polynomial algebra $\mathbf{K}[u_{\mathscr{A}}]$ based on $\mathscr{A}$ and a quotient algebra $U(\mathscr{A})$. We also study some properties of $U(\mathscr{A})$. Section 3 contains the proof that $U(\mathscr{A})$ and $W(\mathscr{A})$ are isomorphic graded algebras. In Section 4 we compute the Poincaré polynomial of $W(\mathscr{A})$ and prove that j is an isomorphism of vector spaces.

It is not clear from our results whether $W(\mathscr{A})$ depends only on $L(\mathscr{A})$ or not. Another interesting question is if $W(\mathscr{A})$ is the model for any topological invariant of $M(\mathscr{A})$.

2. The algebra $U(\mathscr{A})$

Let $\mathscr{A}$ be an arrangement. Let $L = L(\mathscr{A})$ be the set of all intersections of elements of $\mathscr{A}$. We agree that L includes V as the intersection of the empty collection of hyperplanes. We should remember that if $X \in L$, then $X \subseteq V$. Partially order L by reverse inclusion. Then L is a geometric lattice with rank function $r(X) = \mathrm{codim}(X)$ [7, Lemma 2.3].

Let $\mathbf{E}_X = \{\mathscr{E} \in \mathbf{E} \mid \cap \mathscr{E} = X\}$. Then we have the disjoint union

$$\mathbf{E} = \cup_{X \in L} \mathbf{E}_X.$$

We use notation such as $\mathbf{E}_{p,X} = \mathbf{E}_p \cap \mathbf{E}_X$, $\mathbf{E}^t_{p,X} = \mathbf{E}^t \cap \mathbf{E}_{p,X}$, etc.

Definition 2.1. Let $\mathbf{K}[u_{\mathscr{A}}]$ be the polynomial ring in the indeterminates u_H, $H \in \mathscr{A}$. Write $u_{\mathscr{E}} = \Pi_{H \in \mathscr{E}} u_H$. Define

$$\mathbf{K}[u_{\mathscr{A}}]_p = \sum_{\mathscr{E} \in \mathbf{E}_p} \mathbf{K} u_{\mathscr{E}}, \quad \mathbf{K}[u_{\mathscr{A}}]_X = \sum_{\mathscr{E} \in \mathbf{E}_X} \mathbf{K} u_{\mathscr{E}},$$
$$\mathbf{K}[u_{\mathscr{A}}]^i = \sum_{\mathscr{E} \in \mathbf{E}^i} \mathbf{K} u_{\mathscr{E}}, \quad \mathbf{K}[u_{\mathscr{A}}]^d = \sum_{\mathscr{E} \in \mathbf{E}^d} \mathbf{K} u_{\mathscr{E}}.$$

We have the following direct sum decompositions:

$$\begin{aligned}
\mathbf{K}[u_{\mathscr{A}}] &= \oplus_{p \geq 0} \mathbf{K}[u_{\mathscr{A}}]_p, \\
\mathbf{K}[u_{\mathscr{A}}] &= \oplus_{X \in L} \mathbf{K}[u_{\mathscr{A}}]_X, \\
\mathbf{K}[u_{\mathscr{A}}] &= \mathbf{K}[u_{\mathscr{A}}]^i \oplus \mathbf{K}[u_{\mathscr{A}}]^d.
\end{aligned}$$

Let π_p, π_X, π^i, and π^d be the respective projections. These maps commute pairwise. We use notation such as $\mathbf{K}[u_{\mathscr{A}}]_{p,X} = \mathbf{K}[u_{\mathscr{A}}]_p \cap \mathbf{K}[u_{\mathscr{A}}]_X$, $\mathbf{K}[u_{\mathscr{A}}]^t_{p,X} = \mathbf{K}[u_{\mathscr{A}}]^t \cap \mathbf{K}[u_{\mathscr{A}}]_{p,X}$, etc.

Definition 2.2. Let $I(\mathscr{A})$ be the ideal of $\mathbf{K}[u_{\mathscr{A}}]$ generated by

(i) the elements of $\mathbf{K}[u_{\mathscr{A}}]^d$,

(ii) when $\sum_{H \in \mathscr{E}} c_H \alpha_H = 0$ with $c_H \in \mathbf{K}$, the element $\sum_{H \in \mathscr{E}} c_H u_{\mathscr{E} - \{H\}}$.

Let $U(\mathscr{A}) = \mathbf{K}[u_{\mathscr{A}}] / I(\mathscr{A})$.

https://doi.org/10.1017/S0027763000004852 Published online by Cambridge University Press

Grade $\mathbf{K}[u_{\mathscr{A}}]$ by $\deg u_H = -1$. Since $I(\mathscr{A})$ is a homogeneous ideal, $U(\mathscr{A})$ is a graded commutative $\mathbf{K}$-algebra. The isomorphism $U(\mathscr{A}) \simeq \mathbf{K}[u_{\mathscr{A}}]^i / I(\mathscr{A}) \cap \mathbf{K}[u_{\mathscr{A}}]^i$ shows that $U(\mathscr{A})$ is a finite dimensional graded commutative $\mathbf{K}$-algebra.

A *circuit* $C = \{H_1, \ldots, H_k\}$ is a minimally dependent subset of hyperplanes: C is dependent, but $C - \{H_i\}$ is independent for all i.

PROPOSITION 2.3. *The ideal $I(\mathscr{A})$ is generated by the following finite set*:
(a) u_H^2 *for* $H \in \mathscr{A}$,
(b) *for each circuit* $C = \{H_1, \ldots, H_k\}$ *with* $\sum_{i=1}^k c_i \alpha_{H_i} = 0$, *the element* $\sum_{i=1}^k c_i u_{C-H_i}$.

Proof. Let I' denote the ideal generated by elements of type (a) and (b) of the proposition. It is clear that $I' \subseteq I$. To prove the converse, we argue separately for elements of type (i) and (ii) of the definition.

(i) Suppose $f \in \mathbf{K}[u_{\mathscr{A}}]^d$. Since $\mathbf{K}[u_{\mathscr{A}}]^d$ is an ideal, it suffices to assume that $f = u_C$, where $C = \{H_1, \ldots, H_k\}$ is a circuit. Suppose $\sum_{i=1}^k c_i \alpha_{H_i} = 0$. It follows from (b) that $\sum_{i=1}^k c_i u_{C-H_i} \in I'$ and $u_{H_1} \sum_{i=1}^k c_i u_{C-H_i} \in I'$. We use the distributive law and (a) to conclude that $c_1 u_C \in I'$. Since C is a circuit, $c_1 \neq 0$. Thus $f \in I'$.

(ii) We show that for each relation $\sum_{H \in \mathscr{E}} c_H \alpha_H = 0$ with $c_H \in \mathbf{K}$, the corresponding element $\sum_{H \in \mathscr{E}} c_H u_{\mathscr{E} - \{H\}} \in I'$. Suppose not. Choose a counterexample with minimal $|\mathscr{E}|$. Note that minimality implies that for every $H \in \mathscr{E}$, $m(H, \mathscr{E}) = 1$. Let $\mathscr{E} = \{H_1, \ldots, H_m\}$ with distinct H_j. Let $\sum_{i=1}^m c_i \alpha_{H_i} = 0$ be the corresponding relation. Since $\mathscr{E}$ is dependent, it contains a circuit. We may assume that $C = \{H_1, \ldots, H_k\}$, $k \leq m$, is a circuit. Thus $\sum_{i=1}^k a_i \alpha_{H_i} = 0$ and $a_i \neq 0$ for $1 \leq i \leq k$. Define $a_i = 0$ for $k+1 \leq i \leq m$. Then we have

$$\sum_{i=1}^m c_i \alpha_{H_i} - \frac{c_1}{a_1} \sum_{i=1}^k a_i \alpha_{H_i} = \sum_{i=2}^m \left(c_i - \frac{c_1}{a_1} a_i\right) \alpha_{H_i} = 0.$$

The index set of the last relation is $\mathscr{E} - \{H_1\}$. It follows from the minimality assumption, that the corresponding element

$$\sum_{i=2}^m \left(c_i - \frac{c_1}{a_1} a_i\right) u_{\mathscr{E} - \{H_1, H_i\}} \in I'.$$

Multiply by u_{H_1} and rewrite to get

$$\sum_{i=1}^m c_i u_{\mathscr{E} - \{H_i\}} - \frac{c_1}{a_1} u_{\mathscr{E} - C} \sum_{i=1}^k a_i u_{C - \{H_i\}} \in I'.$$

Since the second sum is in I' by (b), so is the first sum. This contradiction completes the argument. $\square$

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3. The isomorphism

DEFINITION 3.1. Let $\Phi_0 : \mathbf{K}[u_{\mathscr{A}}] \to S_{(0)}$ be the $\mathbf{K}$-algebra homomorphism induced by $u_H \mapsto \alpha_H^{-1}$. Since $\mathrm{im}(\Phi_0) = \mathbf{K}[\alpha_{\mathscr{A}}^{-1}]$, we have a surjective graded algebra homomorphism

$$\Phi : \mathbf{K}[u_{\mathscr{A}}] \to \mathbf{K}[\alpha_{\mathscr{A}}^{-1}].$$

Let $K = \ker(\Phi)$.

LEMMA 3.2. (1) $\pi_p(K) \subseteq K$,
(2) $\pi_X(K) \subseteq K$,
(3) $\pi^i(K) \subseteq K$,
(4) $\pi^d(K) \subseteq K$.

Proof. (1) If a $\mathbf{K}$-linear combination of $Q(\mathscr{E})^{-1}$ is zero, then each homogeneous component of it is zero.

(2) Fix $f \in K$. By (1), we may assume that $f \in \mathbf{K}[u_{\mathscr{A}}]_p$. Write

(i) $$f = \sum_{\mathscr{E} \in \mathbf{E}_p} c_{\mathscr{E}} u_{\mathscr{E}}.$$

Let $f_Z = \pi_Z(f)$ for $Z \in L$. If $f_Z \in K$ for all Z, we are done. Suppose there exist some $Z \in L$ with $f_Z \notin K$. Among these Z we choose one with minimal rank and call it X. Thus we may assume $f_Y \in K$ for all Y with $r(Y) < r(X)$. We may write $\Phi(f) = 0$ as

(ii) $$\sum_{\substack{\mathscr{E} \in \mathbf{E}_p \\ \cap \mathscr{E} = X}} c_{\mathscr{E}} Q(\mathscr{E})^{-1} = - \sum_{\substack{Y \neq X \\ f_Y \notin K}} \sum_{\substack{\mathscr{E} \in \mathbf{E}_p \\ \cap \mathscr{E} = Y}} c_{\mathscr{E}} Q(\mathscr{E})^{-1}.$$

Multiply both sides of (ii) by $Q(\mathscr{A})^p$. All the resulting terms are in S. We count zeros on both sides separately. We may choose coordinates so that $X = \{x_1 = \cdots = x_r = 0\}$. Let M be the ideal of S generated by $\{x_1, \ldots, x_r\}$.

Let $c_{\mathscr{E}} Q(\mathscr{E})^{-1}$ be a term from the right side of (ii). Let $Y = \cap \mathscr{E} \neq X$. Note that $Y \not\leq X$ because if $Y < X$, then $r(Y) < r(X)$ so $f_Y \in K$ by the minimality assumption. Thus $\mathscr{E} \cap (\mathscr{A} - \mathscr{A}_X) \neq \emptyset$. It follows that $Q(\mathscr{A})^p Q(\mathscr{E})^{-1} \in M^{p|\mathscr{A}_X|-p+1}$. Now consider the left side of (ii). Since $Q(\mathscr{A}) / Q(\mathscr{A}_X) \notin M$ and $M^{p|\mathscr{A}_X|-p+1}$ is an M-primary ideal, we have

$$Q(\mathscr{A}_X)^p \sum_{\substack{\mathscr{E} \in \mathbf{E}_p \\ \cap \mathscr{E} = X}} c_{\mathscr{E}} Q(\mathscr{E})^{-1} \in M^{p|\mathscr{A}_X|-p+1}.$$

The degree of this nonzero polynomial is equal to $p\,|\,\mathscr{A}_X\,|-p$. This contradiction completes the argument.

(3) Suppose $f\in K$. It follows from (1) and (2) that we may assume $f\in \mathbf{K}[u_{\mathscr{A}}]_{p,X}$. If $p>r(X)$, then $f\in \mathbf{K}[u_{\mathscr{A}}]^d$. Thus $\pi^i f=0\in K$. If $p=r(X)$, then $f\in \mathbf{K}[u_{\mathscr{A}}]^i$. Thus $\pi^i f=f\in K$.

(4) follows from (3) because $\pi^i+\pi^d=1$ □.

THEOREM 3.3. *The map* $\Phi:\mathbf{K}[u_{\mathscr{A}}]\to\mathbf{K}[\alpha_{\mathscr{A}}^{-1}]$ *induces an isomorphism of graded algebras* $\phi: U(\mathscr{A})\to W(\mathscr{A})$.

Proof. Since $\ker(\phi)=K+\mathbf{K}[u_{\mathscr{A}}]^d$, it is enough to show that $I=K+\mathbf{K}[u_{\mathscr{A}}]^d$. It is clear that $I\subseteq K+\mathbf{K}[u_{\mathscr{A}}]^d$ because the generators of I of the second kind belong to K. Since $\mathbf{K}[u_{\mathscr{A}}]^d\subseteq I$, it suffices to show for the converse that if $f\in K\cap\mathbf{K}[u_{\mathscr{A}}]^i$, then $f\in I$. It follows from Lemma 3.2 (1) and (2) that we may assume $f\in K\cap\mathbf{K}[u_{\mathscr{A}}]^i_{p,X}$. We argue by induction on p. If $p=0$, then $f=0\in I$. If $p>0$, then $\mathscr{A}_X\neq\emptyset$. Let $H_0\in\mathscr{A}_X$. Write $f=\sum_{\mathscr{E}\in\mathbf{E}^i_{p,X}}c_{\mathscr{E}}u_{\mathscr{E}}$. If $\mathscr{E}\in\mathbf{E}^i_{p,X}$, then $\{\mathscr{E},H_0\}$ is a dependent set. Thus we have

$$c_0\alpha_{H_0}+\sum_{H\in\mathscr{E}}c_H\alpha_H=0.$$

Since $\mathscr{E}$ is independent, $c_0\neq 0$ and we may assume that $c_0=1$. By definition we have

$$u_{\mathscr{E}}+u_{H_0}\sum_{H\in\mathscr{E}}c_Hu_{\mathscr{E}-\{H\}}\in K.$$

It follows that $f-u_{H_0}g\in K$ for $g\in\mathbf{K}[u_{\mathscr{A}}]^i_{p-1}$, so $u_{H_0}g\in K$. Since $K=\ker(\Phi)$ is a prime ideal, $f\in K$ and $u_{H_0}\notin K$, we conclude that $g\in K$. Let $g_Y=\pi_Y(g)$ and write $g=\sum g_Y$ where $g_Y\in\mathbf{K}[u_{\mathscr{A}}]^i_{p-1,Y}$. It follows from Lemma 3.2 (2) that $g_Y\in K$ for all Y. By the induction hypothesis, $g_Y\in I$. Thus $g\in I$ and $f\in I$. □

4. Structure theorems

Let $U(\mathscr{A})_X=\mathbf{K}[u_{\mathscr{A}}]_X/\mathbf{K}[u_{\mathscr{A}}]_X\cap I$ and $W(\mathscr{A})_X=\mathbf{K}[\alpha_{\mathscr{A}}^{-1}]_X/\mathbf{K}[\alpha_{\mathscr{A}}^{-1}]_X\cap J$.

THEOREM 4.1. *The algebra* $U(\mathscr{A})$ *is the direct sum* $U(\mathscr{A})=\bigoplus_{X\in L}U(\mathscr{A})_X$.

Proof. It follows from Theorem 3.3 and Lemma 3.2 that $\pi_X(I)=\pi_X(K+\mathbf{K}[u_{\mathscr{A}}]^d)\subseteq K+\mathbf{K}[u_{\mathscr{A}}]^d=I$. It implies that $\pi_X(y)\in I\cap\mathbf{K}[u_{\mathscr{A}}]_X$ for any $y\in I$.

https://doi.org/10.1017/S0027763000004852 Published online by Cambridge University Press

Thus we have $I = \oplus_{X \in L}(I \cap \mathbf{K}[u_{\mathscr{A}}]_X)$. The result follows. $\square$

THEOREM 4.2. *The map $j : AO(\mathscr{A}) \to W(\mathscr{A})$ is an isomorphism of graded vector spaces.*

Proof. Observe that j is a graded, $\mathbf{K}$-linear, surjective map. It remains to show that j is injective. Note that $AO(\mathscr{A}) = \Phi(\mathbf{K}[u_{\mathscr{A}}]^{\iota})$. Thus we have a commuting diagram.

$$\begin{array}{ccccc} AO(\mathscr{A}) & = & \Phi(\mathbf{K}[u_{\mathscr{A}}]^{\iota}) & \simeq & \mathbf{K}[u_{\mathscr{A}}]^{\iota}/K \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota} \\ j\downarrow & & & & \downarrow\tau \\ W(\mathscr{A}) & \simeq & U(\mathscr{A}) & = & \mathbf{K}[u_{\mathscr{A}}]/I \end{array}$$

where τ is induced by the diagram. We show that τ is injective. It suffices to show that $I \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota} = K \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota}$. Theorem 3.3 implies that $I \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota} \supseteq K \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota}$. For the converse, write an element of $I = K + \mathbf{K}[u_{\mathscr{A}}]^{d}$ as $a + b$, where $a \in K$ and $b \in \mathbf{K}[u_{\mathscr{A}}]^{d}$. Then an arbitrary element of $I \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota}$ is $\pi^{\iota}(a+b) = \pi^{\iota}(a) \in K \cap \mathbf{K}[u_{\mathscr{A}}]^{\iota}$ by Lemma 3.2 (2). $\square$

If $M = \oplus_{p \geq 0} M_p$ is a finite dimensional graded vector space, we let $\mathrm{Poin}(M, t) = \sum_{p \geq 0}(\dim M_p)t^p$ be its Poincaré polynomial. Recall [7, 2.42] the (one variable) Möbius function $\mu : L(\mathscr{A}) \to \mathbf{Z}$ defined by $\mu(V) = 1$ and for $X > V$ by $\sum_{Y \leq X}\mu(Y) = 0$.

THEOREM 4.3.

$$\mathrm{Poin}(W(\mathscr{A}), t) = \mathrm{Poin}(U(\mathscr{A}), t) = \mathrm{Poin}(AO(\mathscr{A}), t) = \sum_{X \in L(\mathscr{A})} \mu(X)(-t)^{r(X)}.$$

Proof. It suffices to show that $\dim W(\mathscr{A})_X = |\mu(X)|$. It follows from Theorem 4.2 that j induces an isomorphism $j_X : AO(\mathscr{A})_X \to W(\mathscr{A})_X$ for all $X \in L$. Thus it suffices to show that $\dim AO(\mathscr{A})_X = |\mu(X)|$. Since $AO(\mathscr{A})_X = AO(\mathscr{A}_X)_X$, we may assume that $X = \cap \mathscr{A}$ is the maximal element of $L(\mathscr{A})$. Choose coordinates so that $X = \{x_1 = \cdots = x_m = 0\}$. Suppose $\mathscr{B} \subset \mathscr{A}$ is independent and $\cap \mathscr{B} = X$. Then $\mathscr{B} = \{H_1, \ldots, H_m\}$ and $d\alpha_{H_1} \wedge \cdots \wedge d\alpha_{H_m}$ is a constant multiple of $dx_1 \wedge \cdots \wedge dx_m$. Recall the graded $\mathbf{K}$-algebra $R(\mathscr{A})$ generated by 1 and $d\alpha_H/\alpha_H$. It follows that multiplication by $dx_1 \wedge \cdots \wedge dx_m$ induces an isomorphism $AO(\mathscr{A})_X \simeq R(\mathscr{A})^m$. It was shown in [6] (see also [7, 3.129]) that $\dim R(\mathscr{A})^m = |\mu(X)|$. $\square$

COROLLARY 4.4. *We have*

$$\dim W(\mathscr{A}) = \dim U(\mathscr{A}) = \dim AO(\mathscr{A}) = \sum_{X\in L(\mathscr{A})} |\mu(X)|.$$

When $\mathbf{K} = \mathbf{R}$, *this number equals the number of chambers of* $M(\mathscr{A})$.

Proof. The first part is by Theorem 4.3 and the fact that $(-1)^{r(X)}\mu(X) = |\mu(X)|$, see [7, 2.47]. When $\mathbf{K} = \mathbf{R}$, connected components are called chambers. The second part follows from Zaslavsky's theorem [8]. □

5. NBC bases

In this section we construct explicit $\mathbf{K}$-bases for $AO(\mathscr{A})$, $U(\mathscr{A})$, and $W(\mathscr{A})$. These bases are in one-to-one correspondence with the set of NBCs (non-broken circuits).

Fix a total order on $\mathscr{A}$ by $\mathscr{A} = \{H_1, H_2, \ldots, H_n\}$. Recall that a subset C of $\mathscr{A}$ is a circuit if it is a minimally dependent set. A subset C of $\mathscr{A}$ is a *broken circuit* or a *BC* if there exists a hyperplane $K \in \mathscr{A}$ satisfying $K < \min C$ so that the set $C \cup \{K\}$ is a circuit. A subset T of $\mathscr{A}$ is called a *non-broken circuit* or an *NBC* if T contains no broken circuit.

LEMMA 5.1. *If an independent subset* C *of* $\mathscr{A}$ *contains a BC, then* $Q(C)^{-1}$ *is a linear combination of* $\{Q(T)^{-1} \mid \cap T = \cap C,\ T \text{ is an NBC}\}$.

Proof. We may assume that $C = \{H_{i_1}, \ldots, H_{i_m}\}$ itself is a BC. Suppose that $T = \{H_{i_0}\} \cup C$ is a circuit and that $i_0 < i_1 < i_2 < \cdots < i_m$. Let $T_j = T \setminus \{H_{i_j}\}$. This shows that $Q(C)^{-1}$ is a linear combination of $\{Q(T_j)^{-1}\} \mid j = 1, \ldots, m\}$. Note that $\cap C = \cap T = \cap T_j$ and we get the desired result. □

For any subset $C = \{H_{i_1}, \ldots, H_{i_m}\}$ of $\mathscr{A}$, define

$$\text{height}(C) = i_1 + \cdots + i_m.$$

THEOREM 5.2. *Let* $X \in L$. *The set*

$$NBC(\mathscr{A})_X = \{Q(C)^{-1} \mid C \text{ in an NBC and } \cap C = X\}$$

is a $\mathbf{K}$-*basis for* $AO(\mathscr{A})_X$. *Therefore the set* $\{Q(C)^{-1} \mid C \text{ is an NBC}\}$ *is a* $\mathbf{K}$-*basis for* $AO(\mathscr{A})$.

Proof. It follows from [2],[4],[5] that the cardinality of the set

$$\{C \mid C \text{ is an NBC and } \cap C = X\}$$

is $|\mu(X)|$. We showed in the proof of Theorem 4.3 that $\dim AO(\mathscr{A})_X = |\mu(X)|$. Thus it suffices to show that $NBC(\mathscr{A})_X$ spans $AO(\mathscr{A})_X$. If not, there exists C_0 such that:

(1) C_0 is independent,

(2) $\cap C_0 = X$,

(3) $Q(C_0)^{-1}$ is not spanned by elements of $NBC(\mathscr{A})_X$, and

(4) the height of C_0 is minimum among all subsets satisfying (1)-(3). Since C_0 is not an NBC, $Q(C_0)^{-1}$ is a linear combination of

$$\{Q(T)^{-1} \mid T \text{ is an NBC and } \cap\ T = X\}$$

by Lemma 5.1. By condition (4) and (3), this is a contradiction. □

COROLLARY 5.3. (i) *The residue classes of the set* $\{u_C \mid C$ *is an* $NBC\}$ *give a* $\mathbf{K}$-*basis for* $U(\mathscr{A})$ *as a* $\mathbf{K}$-*vector space.*

(ii) *The residue classes of the set* $\{Q(C)^{-1} \mid C$ *is an* $NBC\}$ *give a* $\mathbf{K}$-*basis for* $W(\mathscr{A})$ *as a* $\mathbf{K}$-*vector space.* □

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