Can. J. Math., Vol. XLII, No. 1, 1990, pp. 62–79

LAWRENCE POLYTOPES

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1. Introduction. In 1980 Jim Lawrence suggested a construction Λ which assigns to a given rank r oriented matroid M on n points a rank $n+r$ oriented matroid $\Lambda(M)$ on $2n$ points such that the face lattice of $\Lambda(M)$ is polytopal if and only if M is realizable. The Λ-construction generalized a technique used by Perles to construct a nonrational polytope [**10**]. It was used by Lawrence to prove that the class of polytopal lattices is strictly contained in the class of face lattices of oriented matroids (unpublished) and by Billera and Munson to show that the latter class is not closed under polarity. See [**4**] for a discussion of this construction and both of these applications.

Here we are mainly interested in the geometric case, where $\Lambda(M)$ is realized by a suitable $2n \times (n+r)$-matrix $\Lambda(\mathbf{B})$. The corresponding *Lawrence polytopes* are precisely the polytopes having centrally symmetric Gale diagrams. They are universal in the sense that every convex polytope is a quotient of a Lawrence polytope, and consequently the Steinitz problem of recognizing polytopal lattices reduces to the problem of recognizing face lattices of Lawrence polytopes.

It has been observed in [**7**] that the Steinitz problem for Lawrence polytopes over an ordered field K is equivalent to solving arbitrary polynomial equations with integer coefficients. Under the hypothesis that the unsolved rational version of Hilbert's 17th problem (decidability of diophantine equations over $\mathbf{Q}$) has a negative answer, this implies that there is no algorithm to decide whether a given polytope is rational, i.e., combinatorially equivalent to a polytope in $\mathbf{Q}^d$ [**19**].

This paper aims to provide a first systematic study of the geometric, combinatorial and topological properties of Lawrence polytopes. Indeed, it might seem surprising in view of the simple constructions to be discussed in Section 2 that Lawrence polytopes have (to the best of our knowledge) not yet been considered in the polytope literature. One possible explanation is that combinatorial convexity has traditionally emphasized constructions which are natural on the polytope-level (e.g., stellar subdivisions, bistellar operations, prisms, pyramids, and Shemer sewing [**10**], [**12**]) while Lawrence's Λ-construction is natural only on the oriented matroid level: M and M' having isomorphic face lattices does not imply that $\Lambda(M)$ and $\Lambda(M')$ have isomorphic face lattices.

In Sections 3 and 4 we study the combinatorial structure of Lawrence polytopes. There we focus our attention on two important enumerative invariants, namely f-vectors and flag vectors. We prove that both the f-vector and the flag vector of $\Lambda(M)$ depend only the underlying matroid $\underline{M}$ and not on the specific

Received October 6, 1988. The research of both authors was supported by the Institute for Mathematics and Its Applications, Minneapolis. Research of the first author was also supported by a Northeastern University Junior Research Fellowship.

oriented matroid M. In the generic case where M is a uniform oriented matroid, we can express these invariants as functions depending only on n and r. This generalizes a famous result of Zaslavsky for hyperplane arrangements [**24**].

In Section 5 we discuss Lawrence polytopes from the viewpoint of geometric realizability. We give an easy proof for the fact that Steinitz' classical isotopy theorem for 3-polytopes [**18**] does not generalize to higher dimensions. Applying the analogous oriented matroid results of B. Jaggi, P. Mani-Levitska, and N. White [**11**], [**22**], we construct two combinatorially equivalent 19-dimensional Lawrence polytopes P and Q such that neither P and Q nor P and a mirror image of Q can be connected by a continuous path of polytopes of the same type.

A more general solution to the isotopy problem has very recently been given by Mnëv [**16**]. His universality theorem for oriented matroids states that every semi-algebraic variety can be "encoded" into a suitable oriented matroid. In [**16**] the same theorem is given for convex polytopes without giving a complete proof. We remark that our Theorem 5.4 implies a very easy proof for the polytopal part of Mnëv's striking result. See [7, Chapter 6] for details.

2. Basic geometric properties of Lawrence polytopes. In our terminology on convex polytopes we follow Grünbaum [**10**] and Klee & Kleinschmidt [**12**]. We refer to [**23**] for some basics of matroid theory and to [**15**], [**20**] for an introduction to Gale diagrams and their relationship to matroid duality.

Definition 2.1. A polytope is called a *Lawrence polytope* if it has a centrally symmetric Gale diagram.

If $\mathcal{P}$ is a Lawrence polytope of dimension d with $2n$ vertices, then every Gale diagram is of the form $A \cup -A$, where A is a set of n (not necessarily distinct) vectors in $\mathbf{R}^{2n-d-1}$.

Given integers r and n, we denote the set of real $n \times r$-matrices of maximal rank by $M(n \times r, \mathbf{R})$. Given $\mathbf{B} \in M(n \times r, \mathbf{R})$, we write pos($\mathbf{B}$) for the positive hull in $\mathbf{R}^r$ of the rows of $\mathbf{B}$. If the polyhedral cone pos($\mathbf{B}$) is pointed, then it is combinatorially equivalent to the $(r-1)$-dimensional convex polytope $H^{r-1} \cap$ pos($\mathbf{B}$) where H^{r-1} is a suitable affine hyperplane in $\mathbf{R}^r$. Conversely, every $(r-1)$-polytope $P \subset H^{r-1}$ can be written as $P = H^{r-1} \cap$ pos($\mathbf{B}$) for some $\mathbf{B}$, for example, a matrix whose rows are homogeneous coordinates for the vertices of P.

Given any matrix $\mathbf{B} \in M(n \times r, \mathbf{R})$, we define the associated *Lawrence matrix*

$$\Lambda(\mathbf{B}) := \begin{pmatrix} \mathbf{I} & \mathbf{B} \\ \mathbf{I} & \mathbf{O} \end{pmatrix} \in M(2n \times (n+r), \mathbf{R}),$$

where $\mathbf{O}$ denotes the $n \times r$ zero matrix, and $\mathbf{I}$ denotes the $n \times n$ unit matrix.

THEOREM 2.1. *Let $\mathcal{P}$ be a d-polytope with set of $2n$ vertices $\mathcal{V} := \{x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-\} \subset \mathbf{R}^d$. Then the following are equivalent.*

(a) $\mathcal{P}$ *is a Lawrence polytope.*

(b) $\mathcal{F}_i := \operatorname{conv}(\mathcal{V}\setminus\{x_i^+, x_i^-\})$ *is a face of* $\mathcal{P}$ *for all* $i = 1, \ldots, n$.

(c) There exists $\mathbf{B} \in M(n \times r, \mathbf{R})$ *such that after a suitable projective transformation, homogeneous coordinates for* $\mathcal{V}$ *are given by the rows of* $\Lambda(\mathbf{B})$, *where* $r := d + 1 - n \geqq 1$.

Proof. By the above remark, (a) is equivalent to saying that every Gale diagram $\hat{\mathcal{V}} = \{\mathbf{a}_1^+, \ldots, \mathbf{a}_n^+, \mathbf{a}_1^-, \ldots, \mathbf{a}_n^-\} \subset \mathbf{R}^{2n-d-1}$ of $\mathcal{P}$ satisfies $\mathbf{a}_i^- = -\mathbf{a}_i^+$ for $i = 1, \ldots, n$. The equivalence of (a) and (b) follows directly from the basic properties of Gale diagrams [**10**, Theorem 5.4.1]: $\mathcal{F}_i$ is a face of $\mathcal{P}$ if and only if $0 \in \operatorname{relint}\operatorname{conv}\{\mathbf{a}_i^+, \mathbf{a}_i^-\}$. This is equivalent to $\mathbf{a}_i^- = -\mathbf{a}_i^+$.

Note that the distinguished faces $\mathcal{F}_i$ in (b) can either be facets (faces of dimension $d - 1$) or ridges (faces of dimension $d - 2$). In the first case the corresponding vectors $\mathbf{a}_i^+$ and $\mathbf{a}_i^-$ are non-zero, while in the latter (degenerate) case $\mathbf{a}_i^+ = \mathbf{a}_i^- = \mathbf{O}$, and $\mathcal{P}$ is a two-fold pyramid over the $(d-2)$-polytope $\mathcal{F}_i$.

(c) $\Rightarrow$ (b): Suppose $\mathcal{P}$ is given homogeneously as the positive hull in $\mathbf{R}^{d+1}$ of the $2n$ row vectors of the matrix $\Lambda(\mathbf{B}) = (\mathbf{x}_1^+, \ldots, \mathbf{x}_n^+, \mathbf{x}_1^-, \ldots, \mathbf{x}_n^-)^T$. Let $\mathbf{e}_i$ denote the ith unit vector in $\mathbf{R}^{d+1}$, $1 \leqq i \leqq n \leqq d$. Then $\mathbf{e}_i \cdot \mathbf{x}_j^\sigma = 0$ if $j \neq i$, and $\mathbf{e}_i \cdot \mathbf{x}_i^\sigma > 0$ where $\sigma \in \{-, +\}$. This shows that $\mathcal{F}_i$ is a face of $\mathcal{P}$.

(b) $\Rightarrow$ (c): Conversely, let $\mathcal{P}$ be given as the positive hull in $\mathbf{R}^{d+1}$ of the rows of the matrix $\mathbf{X} = (\mathbf{x}_1^+, \ldots, \mathbf{x}_n^+, \mathbf{x}_1^-, \ldots, \mathbf{x}_n^-)^T$, and assume that $\mathcal{F}_i$ is a face of $\mathcal{P}$ for all $i = 1, \ldots, n$. We can choose normal vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n \in \mathbf{R}^{d+1}$ such that $\mathbf{v}_i \cdot \mathbf{x}_j^\sigma = 0$ if $j \neq i$, and $\mathbf{v}_i \cdot \mathbf{x}_i^\sigma > 0$ where $\sigma \in \{-, +\}$. Furthermore, since $\mathbf{x}_1^+$ is a vertex of $\mathcal{P}$, we can choose $\mathbf{v}_0 \in \mathbf{R}^{d+1}$ such that $\mathbf{v}_0 \cdot \mathbf{x}_1^+ = 0, \mathbf{v}_0 \cdot \mathbf{x}_1^- > 0$, and $\mathbf{v}_0 \cdot \mathbf{x}_j^\sigma > 0$, for $j \geqq 2$ and $\sigma \in \{-, +\}$. The vectors $\mathbf{v}_i$ are linearly independent and hence $n+1 \leqq d+1$. Consider the positive definite $2n \times 2n$-diagonal matrix $\mathbf{D} := \operatorname{diag}(\mathbf{v}_1 \cdot \mathbf{x}_1^+, \ldots, \mathbf{v}_n \cdot \mathbf{x}_n^+, \mathbf{v}_1 \cdot \mathbf{x}_1^-, \ldots, \mathbf{v}_n \cdot \mathbf{x}_n^-)$. Pick furthermore linearly independent vectors $\mathbf{w}_1, \ldots, \mathbf{w}_{d+1-n} \in \mathbf{R}^{d+1}$ such that $\mathbf{w}_i \cdot \mathbf{x}_j^- = 0$ for all $i = 1, \ldots, d+1-n, j = 1, \ldots, n$. The $(d+1) \times (d+1)$-matrix $\mathbf{T} := (\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{w}_1, \ldots, \mathbf{w}_{d+1-n})$ is easily seen to be nonsingular. Hence $\mathbf{X} \longmapsto \mathbf{D}^{-1} \cdot \mathbf{X} \cdot \mathbf{T}$ defines a nonsingular projective transformation admissible for $\mathcal{P}$, and we have $\mathbf{D}^{-1} \cdot \mathbf{X} \cdot \mathbf{T} = \Lambda(\mathbf{B})$ for a suitable $\mathbf{B} \in M(n \times r, \mathbf{R})$.

Definition 2.2. A Lawrence d-polytope $\mathcal{P}$ with $2n$ vertices is a *generic Lawrence polytope* provided every Gale diagram of $\mathcal{P}$ is of the form $A \cup -A$ where A is a set of n vectors in linearly general position in $\mathbf{R}^{2n-d-1}$.

The case of generic Lawrence polytopes with $2n - d - 1 = 0$ is special. In this case $A \cup -A$ consists of $2n$ copies of the zero vector. $\mathcal{P}$ is a d-simplex (d odd), and the faces $\mathcal{F}_i$ are ridges. The matrix $\Lambda(\mathbf{B})$ has linearly independent rows; the $n \times n$-submatrix $\mathbf{B}$ is nonsingular, and thus the matroid associated with $\mathbf{B}$ is the rank n uniform matroid on n points.

For $2n - d - 1 > 0$ any vectors in general position in $\mathbf{R}^{2n-d-1}$ are necessarily distinct, and the following characterizations of generic Lawrence polytopes are obtained.

THEOREM 2.2. *Let $\mathcal{P}$ be a d-dimensional Lawrence polytope with $2n > d+1$ vertices. Then the following are equivalent.*

(a′) *$\mathcal{P}$ is a generic Lawrence polytope.*

(b′) *$\mathcal{F}_i$ is a facet of $\mathcal{P}$ for $i = 1, \ldots, n$, and all other facets of $\mathcal{P}$ are $(d-1)$-simplices.*

(c′) *All $r \times r$-subdeterminants of the $n \times r$-matrix $\mathbf{B}$ in* 2. 1 (c) *are non-zero. In other words, the rank r matroid $M(\mathbf{B})$ associated with the matrix $\mathbf{B}$ is uniform.*

Proof. (a′) $\Leftrightarrow$ (b′): (a′) says that every Gale diagram of $\mathcal{P}$ is of the form $A \cup -A \subset \mathbf{R}^{2n-d-1}$ where every $(2n-d-1)$-element subset of A is a basis of $\mathbf{R}^{2n-d-1}$. The faces $\mathcal{F}_i$ being facets means that the sets $\{x_i^+, x_i^-\}$ are minimal cofaces, or equivalently, that $\mathbf{a}_i^- = -\mathbf{a}_i^+$ is non-zero in the Gale diagram $A \cup -A$.

If we assume (a′), then every coface not containing any of the two-element cofaces $\{x_i^+, x_i^-\}$ must have at least $2n-d$ elements. In fact, the minimal such cofaces are in two-to-one correspondence with the minimal dependencies in A. This implies (b′).

This argument is reversible: if every other facet of $\mathcal{P}$ is a $(d-1)$-simplex, then every minimal cofacet not containing any $\{x_i^+, x_i^-\}$ has precisely $2n-d$ elements. Hence there are no linear dependencies among any $2n-d-1$ vectors in A, which means that A is in general position in $\mathbf{R}^{2n-d-1}$.

(a′) $\Leftrightarrow$ (c′): We write $\mathbf{A}$ for the $n \times (2n-d-1)$-matrix whose rows are the vectors in A. The vector configuration $A \cup -A$ being a Gale diagram, we can assume that the column space of the $2n \times (2n-d-1)$-matrix $\binom{\mathbf{A}}{-\mathbf{A}}$ is the orthogonal complement in $\mathbf{R}^{2n}$ of the column space of $\Lambda(\mathbf{B})$. This, however, implies that the column space of $\mathbf{A}$ is the orthogonal complement in $\mathbf{R}^n$ of the column space of $\mathbf{B}$, and hence the Plücker coordinates of these subspaces versus the standard basis of $\mathbf{R}^n$ are equal up to a non-zero scalar. (This well known fact from multilinear algebra is expressed combinatorially in matroid duality by the complementarity of the bases of the primal and dual matroid.) In other words: the $r \times r$-subdeterminants of $\mathbf{B}$ are equal (up to a non-zero factor) to the $(n-r) \times (n-r)$-subdeterminants of $\mathbf{A}$. In particular, all these subdeterminants are non-zero if and only if the vector configuration A is in general position in $\mathbf{R}^{n-r} = \mathbf{R}^{2n-d-1}$.

COROLLARY 2.3. *Every convex $(r-1)$-polytope Q with n vertices is the quotient of an $(n+r-1)$-dimensional Lawrence polytope $\mathcal{P}$ with $2n$ vertices. If Q is simplicial, then $\mathcal{P}$ can be assumed to be generic.*

Proof. Let Q be an $(r-1)$-polytope with n vertices, and let $A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\} \subset \mathbf{R}^{n-r}$ be a Gale diagram of Q. Then the set

$$A \cup -A = \{\mathbf{a}_1, \ldots, \mathbf{a}_n, -\mathbf{a}_1, \ldots, -\mathbf{a}_n\} \subset \mathbf{R}^{n-r}$$

is the Gale diagram of an $(n+r-1)$-dimensional Lawrence polytope $\mathcal{P}$ with vertex set $\{x_1^+, \ldots, x_n^+, x_1^-, \ldots, x_n^-\} \subset \mathbf{R}^{n+r-1}$. By construction, $F := \operatorname{conv}\{x_1^+, \ldots, x_n^+\}$ is a face of $\mathcal{P}$ and $Q = \mathcal{P}/F$.

If Q is simplicial, then its vertices can be assumed to be in general position. In that case A is a configuration of distinct vectors in general position, and the Lawrence polytope $\mathcal{P}$ with Gale diagram $A \cup -A$ is generic.

3. On the combinatorics of Lawrence polytopes. In this section we use the cone representation pos($\Lambda(\mathbf{B})$) to study the combinatorial structure of the Lawrence polytope $\mathcal{P}$ associated with a matrix $\mathbf{B}$. The face lattice of $\mathcal{P}$ will be related to the central hyperplane arrangement $\mathcal{H}$ defined by $\mathbf{B}$.

According to Zaslavsky's celebrated theorem [**24**], the f-vector of any hyperplane arrangement depends only on the incidence relations among the hyperplanes (i.e., on the associated matroid). We will show that the same is true for the f-vector and flag vector of Lawrence polytopes. In this section we state our results, explain the crucial interplay between the matroid lattice $\mathcal{L}(\mathcal{H})$, the face semilattice $\mathcal{K}(\mathcal{H})$, and the cover lattice $\mathcal{U}(\mathcal{H})$ of the arrangement $\mathcal{H}$, and show how these lattices relate to the polar arrangement of the Lawrence polytope pos($\Lambda(\mathbf{B})$). Complete proofs will be given in the more general setting of oriented matroids in Section 4. At the end of this section we give an explicit formula for the f-vector of a generic Lawrence polytope.

Let $\mathbf{B} \in M(n \times r, \mathbf{R})$ and write $\mathbf{b}_i$ for the ith row of $\mathbf{B}$. For ease of exposition we assume that all vectors $\mathbf{b}_i$ are non-zero. Then $\mathbf{b}_i$ defines a hyperplane through the origin $H_i := \{\mathbf{x} \in \mathbf{R}^r : \mathbf{b}_i \cdot \mathbf{x} = 0\}$. The set of these hyperplanes forms a central arrangement $\mathcal{H}$ in $\mathbf{R}^r$.

In the study of hyperplane arrangements various partially ordered sets have been useful. We define $\mathcal{L}(\mathcal{H})$ to be the lattice of subspaces of $\mathbf{R}^r$ obtained as intersections of hyperplanes, ordered by inclusion. We call $\mathcal{L}(\mathcal{H})$ the *matroid lattice* of the arrangement $\mathcal{H}$ because it is anti-isomorphic to the geometric lattice of flats in the rank r matroid associated with $\mathbf{B}$.

The set $\mathbf{R}^r \setminus \cup_{i=1}^n H_i$ is the disjoint union of open polyhedral cones, called the *regions* of $\mathcal{H}$. A *face* of the arrangement is a face of the closure of a region of $\mathcal{H}$. Faces can be identified by their location with respect to the open halfspaces $H_i^+ := \{\mathbf{x} \in \mathbf{R}^r : \mathbf{b}_i \cdot \mathbf{x} > 0\}, H_i^- := \{\mathbf{x} \in \mathbf{R}^r : \mathbf{b}_i \cdot \mathbf{x} < 0\}$, and their closures $\overline{H}_i^+$ and $\overline{H}_i^-$, respectively.

The set of all faces of $\mathcal{H}$, ordered by inclusion, forms a ranked semilattice $\mathcal{K}(\mathcal{H})$, called the *face semilattice* of $\mathcal{H}$. The f-vector of $\mathcal{H}$ is the r-tuple $(f_0, f_1, \ldots, f_r)$, where f_i is the number of i-dimensional faces of $\mathcal{H}$. Recall the following lovely and surprising result.

THEOREM 3.1. (Zaslavsky [**24**]) *The f-vector of $\mathcal{H}$ is a function of the matroid lattice $\mathcal{L}(\mathcal{H})$.*

Next we consider the central arrangement $\mathcal{H}_\Lambda$ of $2n$ hyperplanes in $\mathbf{R}^{n+r}$ that is defined by the Lawrence matrix $\Lambda(\mathbf{B}) \in M(2n \times (n+r), \mathbf{R})$. As in the previous section, the positive hull of the rows of $\Lambda(\mathbf{B})$ is a cone over the Lawrence polytope. Its polar cone is the closed region of $\mathcal{H}_\Lambda$ given by $\{\mathbf{x} \in \mathbf{R}^{n+r} : \Lambda(\mathbf{B}) \cdot \mathbf{x} \geqq 0\}$. We write $\mathcal{K}^+(\mathcal{H}_\Lambda)$ for the face lattice of this specific region.

We wish to study the f-vector of the Lawrence polytope, or equivalently, the f-vector of the lattice $\mathcal{K}^+(\mathcal{H}_\Lambda)$. Note that, a priori, Theorem 3.1 applies only to the face semilattice $\mathcal{K}(\mathcal{H}_\Lambda)$ of the entire arrangement and not to any individual region of $\mathcal{H}_\Lambda$. However, we will see that for arrangements of the special form $\mathcal{H}_\Lambda$, Zaslavsky's Theorem has an analogue for the distinguished region $\mathcal{K}^+(\mathcal{H}_\Lambda)$.

To begin with we show how to obtain the lattice $\mathcal{K}^+(\mathcal{H}_\Lambda)$ directly from the "small" arrangement $\mathcal{H}$. For any subset $I \subseteq [n] := \{1, 2, \ldots, n\}$ let $\mathcal{H}\,|_I$ be the central arrangement in $\mathbf{R}^r$ formed from the hyperplanes $H_i, i \in I$. (Note that the hyperplane arrangement $\mathcal{H}\,|_\emptyset$ has a single region $\mathbf{R}^r$; hence the corresponding matroid lattice $\mathcal{L}(\mathcal{H}\,|_\emptyset)$ and face semilattice $\mathcal{K}(\mathcal{H}\,|_\emptyset)$ each consist of the one element $\mathbf{R}^r$).

We form a poset on the disjoint union of the posets $\mathcal{K}(\mathcal{H}\,|_I)$ as follows.

Definition 3.1. The *cover poset* $\mathcal{U}(\mathcal{H})$ of a hyperplane arrangement $\mathcal{H}$ is the set of pairs (F, I) where $F \in \mathcal{K}(\mathcal{H}\,|_I)$, together with the relation

$$(F, I) \preceq (G, J) \text{ if and only if } I \supseteq J \text{ and } F \subseteq G \text{ (as subsets of } \mathbf{R}^r).$$

THEOREM 3.2. *Let $\mathcal{H}$ be the central hyperplane arrangement determined by the matrix* $\mathbf{B}$. *Then there is an order-preserving isomorphism between its cover lattice $\mathcal{U}(\mathcal{H})$ and the lattice $\mathcal{K}^+(\mathcal{H}_\Lambda)$. Thus $\mathcal{U}(\mathcal{H})$ is anti-isomorphic to the face lattice of the Lawrence polytope associated with* $\mathbf{B}$.

This theorem is a special case of Theorem 4.2. For concreteness we specify the anti-isomorphism between the cover lattice and the face lattice of the Lawrence polytope $\mathcal{P}$ guaranteed by Theorem 3.2. We suppose that the vertices $x_1^+, x_2^+, \ldots, x_n^+, x_1^-, x_2^-, \ldots, x_n^-$ of $\mathcal{P}$ are labelled as in Theorem 2.1. Then $(F, I) \in \mathcal{U}(\mathcal{H})$ corresponds to the face

$$\operatorname{conv}(\{x_i^+ : i \in I \text{ and } F \subseteq \overline{H_i^-}\} \cup \{x_i^- : i \in I \text{ and } F \subseteq \overline{H_i^+}\})$$

of the Lawrence polytope $\mathcal{P}$.

The cover poset $\mathcal{U}(\mathcal{H})$ may be of independent interest in the study of hyperplane arrangements. For example, Theorem 3.2 implies the non-trivial observation that the cover lattice $\mathcal{U}(\mathcal{H})$ of any central hyperplane arrangement $\mathcal{H}$ is a polytopal lattice. Hence it is an Eulerian poset in the sense of Stanley [**17**].

The rank function $r_{\mathcal{U}}$ of $\mathcal{U}(\mathcal{H})$ is given by

$$r_{\mathcal{U}}((F, I)) = \dim(F) + n - |I|.$$

Hence the number of rank k elements in $\mathcal{U}(\mathcal{H})$ equals

$$\begin{aligned} f_k(\mathcal{U}) &= \#\{(F, I) : F \in \mathcal{K}(\mathcal{H}\,|_I) \quad \text{and} \quad \dim(F) = |I| + k - n\} \\ &= \sum_{i=n-k}^{n} \sum_{\substack{I \subseteq [n] \\ |I| = i}} f_{i+k-n}(\mathcal{H}\,|_I). \end{aligned}$$

By Zaslavsky's theorem, the f-vector of the arrangement $\mathcal{H}|_I$ is a function of $L(\mathcal{H}|_I)$. But this matroid lattice is a sublattice of $L(\mathcal{H})$, and therefore the numbers $f_{i+k-n}(\mathcal{H}|_I)$ in the above expansion are determined by the matroid lattice $L(\mathcal{H})$. By Theorem 3.2 the f-vector of the Lawrence polytope associated with $\mathbf{B}$ is determined by the rank level numbers of $\mathcal{U}(\mathcal{H})$. This implies

THEOREM 3.3. *The f-vector of the Lawrence polytope associated with the matrix $\mathbf{B}$ is a function of the matroid lattice $L(\mathcal{H})$ of the hyperplane arrangement $\mathcal{H}$ determined by $\mathbf{B}$.*

One of the central unsolved problems in combinatorial convexity is to find a characterization for f-vectors of nonsimplicial polytopes. (Note that all nontrivial Lawrence polytopes are nonsimplicial.) An important tool in attacking this difficult question has been the flag vector, a combinatorial invariant finer than the f-vector.

A chain of faces $\emptyset \subset F_1 \subset F_2 \cdots \subset F_s \subset P$ of a d-polytope P is called an S-*flag* of P if $S = \{\dim F_j : 1 \leqq j \leqq s\}$. Given $S \subseteq \{0, 1, \ldots, d-1\}$, we write $f_S(P)$ for the number of S-flags of P. The *flag vector* of P is then the 2^d-tuple $(f_S(P))_{S\subseteq\{0,1,\ldots,d-1\}}$. The flag vector of any ranked poset is defined analogously. See [**2**] and [**3**] for details on flag vectors. We have the following generalizations of Theorems 3.1 and 3.3.

THEOREM 3.4. *The flag vector of $\mathcal{K}(\mathcal{H})$ is a function of $L(\mathcal{H})$.*

THEOREM 3.5. *The flag vector of the Lawrence polytope is a function of $L(\mathcal{H})$.*

The region $\mathcal{K}^+(\mathcal{H}_\Lambda)$ is not unique in having the special relationship with the arrangement $\mathcal{H}$. Indeed we have the following

PROPOSITION 3.6. *The arrangement $\mathcal{H}_\Lambda$ has 2^n regions combinatorially equivalent to $\mathcal{K}^+(\mathcal{H}_\Lambda)$.*

The combinatorially equivalent regions are those obtained by choosing $T \subseteq [n]$ and reversing those inequalities of $\Lambda(\mathbf{B}) \cdot \mathbf{x} \geqq 0$ indexed by T and by $T + n := \{j + n : j \in T\}$.

We close this section with a formula for the f-vector of a generic Lawrence $\mathcal{P}$ of dimension d with $2n$ vertices. The polytope $\mathcal{P}$ has as its polar cone the region $\mathcal{K}^+(\mathcal{H}_{\Lambda(\mathbf{B})})$, where $\mathbf{B}$ defines a set of hyperplanes in general position through the origin. A formula for the f-vector of a general position through the origin. A formula for the f-vector of a general position central hyperplane arrangement is well-known (see, for example, [**10**]). Using Theorem 3.2, it enables us to derive a formula for the f-vector of the generic Lawrence polytope $\mathcal{P}$. Given

$r := d + 1 - n$ and $0 \leqq k \leqq d - 1$, we obtain

$$\begin{aligned} f_k(\mathcal{P}) = 2\sum_{m=1}^{n} \binom{n}{m} \binom{m}{k+1-m} \sum_{i=0}^{m+r-k-2} \binom{2m-k-2}{i} \\ + \chi(k \geqq 2r-1) \binom{n}{k+1-r} \\ + \chi(k \leqq 2r-3)\chi(k \text{ odd}) \binom{n}{\frac{k+1}{2}}. \end{aligned}$$

(Here $\chi(Q) := 1$ if the statement Q is true, and $\chi(Q) := 0$ if Q is false.)

4. On the combinatorics of Lawrence oriented matroids. In this section we prove the results of Section 3 in the more general setting of oriented matroids. Of the many equivalent axiomatizations of oriented matroids we find Mandel's [**14**] most useful for the purposes of this paper. For other definitions consult [**5**], [**8**], [**21**].

A *signed vector* on a finite set E is a vector indexed by E with components from the set $\{-, 0, +\}$. We use the notation $X^\sigma = \{e \in E : X_e = \sigma\}$, where $\sigma \in \{-, 0, +\}$, for the sign sets of a signed vector X. The *support* of X is the union $\underline{X} = X^+ \cup X^-$. The signed vector with all components 0 is written $\mathbf{0}$, and $-X$ denotes the componentwise negation of a signed vector X.

The *product* $X \cdot Y$ of signed vectors X and Y is given by $(X \cdot Y)_e = X_e$ if $X_e \neq 0$, and $(X \cdot Y)_e = Y_e$ otherwise. An element $e \in E$ *separates* X and Y if $X_e = -Y_e \neq 0$. If no element of E separates them, then X and Y are *conformal*. For two conformal signed vectors the product is commutative and is often written as union.

Definition 4.1. An *oriented matroid* is a pair $M = (E, \mathcal{K})$, where E is a finite set and $\mathcal{K}$ is a set of signed vectors on E satisfying

(1) $\mathbf{0} \in \mathcal{K}$;

(2) if $X \in \mathcal{K}$ then $-X \in \mathcal{K}$;

(3) if $X, Y \in \mathcal{K}$ then $X \cdot Y \in \mathcal{K}$; and

(4) if $X, Y \in \mathcal{K}$ and $e \in E$ separates X and Y, then there exists $Z \in \mathcal{K}$ such that $Z_e = 0$ and for every $f \in E$ that does not separate X and Y, $Z_f = (X \cdot Y)_f = (Y \cdot X)_f$.

In oriented matroid theory, the set $\mathcal{K}$ is known as the *signed cocircuit span* of M. The zero sets X^0 of the signed vectors $X \in \mathcal{K}$ of an oriented matroid M are the closed flats of the matroid $\underline{M}$ underlying M. We write ρ_M for the rank function of the matroid $\underline{M}$.

The signed cocircuit span $\mathcal{K}$ of an oriented matroid M has a natural partial order: $X \subseteq Y$ if and only if $X^+ \subseteq Y^+$ and $X^- \subseteq Y^-$. With this order $\mathcal{K}$ is a ranked poset whose rank function $r_{\mathcal{K}}$ is given by $r_{\mathcal{K}}(X) = \rho_M(E) - \rho_M(E \backslash \underline{X})$

for all elements $X \in \mathcal{K}$. (In general we will use the letter r for poset ranks and ρ for matroid ranks.) The poset $\mathcal{K}$ is generated by its elements of rank 1; these are called *cocircuits*, and their set is denoted $O^{\perp}$. A signed vector $X \in \mathcal{K}$ is the product (union) of all cocircuits Y such that $Y \subseteq X$.

Minors of oriented matroids are defined as follows. Fix a subset $I \subseteq E$. The *restriction* of a signed vector X to I is the signed vector $X|I$ on I defined by $(X|I)_e := X_e$ for all $e \in I$. The set of restrictions $\{X|I : X \in \mathcal{K}(M)\}$ is the cocircuit span of an oriented matroid $M|_I$, called the *restriction* of M to I. The underlying matroid of $M|_I$, is, of course, the ordinary matroid restriction $(\underline{M})|_I$, whose closed sets are the intersections of closed sets of $\underline{M}$ with I. The set $\{X \in \mathcal{K}(M) : \underline{X} \cap I = \emptyset\}$ is the cocircuit span of another oriented matroid M/I, called the *contraction* of M by I. The underlying matroid of M/I is the ordinary matroid contraction $(\underline{M})/I$, whose closed sets are $\{F \backslash I : F$ is a closed set of $\underline{M}$ and $I \subseteq F\}$.

Next we describe Lawrence's Λ-construction of an oriented matroid on a $2n$-element set from an oriented matroid on n elements. Let M be an oriented matroid on a set $E = \{e_1, e_2, \ldots, e_n\}$. Introduce n new elements $E^* = \{e_1^*, e_2^*, \ldots, e_n^*\}$. For any subset $A \subseteq E$ write $A^* = \{e^* : e \in A\}$; for $A \subseteq E^*$ write $A_* = \{e : e^* \in A\}$. For Y a cocircuit of M and $A \subseteq \underline{Y}$, let Y^A be the signed vector on $E \cup E^*$ with $(Y^A)^+ = (Y^+ \backslash A) \cup (Y^- \cap A)^*$ and $(Y^A)^- = (Y^- \backslash A) \cup (Y^+ \cap A)^*$. For $e \in E$ we write $D(e)$ for the signed vector with $D(e)_e = D(e)_{e^*} = +$ and $D(e)_f = 0$ for $f \notin \{e, e^*\}$.

The following theorem is due to Lawrence (unpublished); a proof can be found in [**4**].

THEOREM 4.1. *Let M be an oriented matroid with set of cocircuits $O^{\perp}$. There exists an oriented matroid $\Lambda(M)$ with cocircuits $O^{\perp}(\Lambda(M)) = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} := \{D(e), -D(e) : \{e\}$ is not a cocircuit of $M\}$, and $\mathcal{B} = \{Y^A : Y \in O^{\perp}(M)$ and $A \subseteq \underline{Y}\}$.*

The oriented matroid $\Lambda(M)$ is called the *Lawrence oriented matroid* associated with M. Its rank is $|E|$ greater than the rank of M. The cocircuit span $\mathcal{K}(\Lambda(M))$ of $\Lambda(M)$ is the closure of $O^{\perp}(\Lambda(M))$ under successive conformal union. The subset of all positive signed vectors in $\mathcal{K}(\Lambda(M))$ is denoted $\mathcal{K}^+(\Lambda(M))$ and is called the *positive cocircuit span* of $\Lambda(M)$. It follows from the definition of $\Lambda(M)$ that each element of $E \cup E^*$ is contained in a positive cocircuit. Therefore $\Lambda(M)$ is acyclic, and $\mathcal{K}^+(\Lambda(M))$ (ordered by inclusion) is a poset with a unique maximal element, namely the signed vector which is positive on all of $E \cup E^*$. In general, Mandel [**14**] proved that the positive cocircuit span of any oriented matroid is the face lattice of a PL-sphere; when the oriented matroid is $\Lambda(M)$ we call the sphere the *Lawrence sphere* associated to M.

The archetype of a rank r oriented matroid on $E = [n]$ is the column space $\mathrm{Im}(\mathbf{B})$ of a matrix $\mathbf{B} \in M(n \times r, \mathbf{R})$. More precisely, we write $\mathrm{sgn}(v)$ for the vector of signs of the components of a real vector v, and we define the oriented matroid $M(\mathbf{B})$ associated with $\mathbf{B}$ by its cocircuit span

$$\mathcal{K}(\mathbf{B}) := \{\mathrm{sgn}(\mathbf{v}) \in \{-,0,+\}^n | \mathbf{v} \in \mathrm{Im}(\mathbf{B})\},$$

and $\mathcal{K}^+(\mathbf{B}) := \mathcal{K}(\mathbf{B}) \cap \{0,+\}^n$. As is well known, the cocircuit span $\mathcal{K}(\mathbf{B})$ of $M(\mathbf{B})$ is canonically isomorphic to the face semilattice $\mathcal{K}(\mathcal{H})$ of the arrangement $\mathcal{H}$ defined by $\mathbf{B}$ as in Section 3. To see this observe that each face of $\mathcal{H}$ consists of all vectors $\mathbf{x}$ in $\mathbf{R}^r$ whose images $\mathbf{v} = \mathbf{Bx}$ have a given signed support. The underlying matroid $\underline{M(\mathbf{B})}$ has as its closed flats the linear spans, and as its rank function the linear dimensions of row sets of $\mathbf{B}$. Thus there is a natural order-reversing bijection between the matroid lattice $L(\mathcal{H})$ defined in Section 3 and the geometric lattice of closed flats in $\underline{M(\mathbf{B})}$. Finally, we remark that the Lawrence oriented matroid $\Lambda(M(\mathbf{B}))$ associated with the matroid $M(\mathbf{B})$ equals the oriented matroid $M(\Lambda(\mathbf{B}))$ of the Lawrence matrix $\Lambda(\mathbf{B})$. The face lattice of the Lawrence polytope of $\mathbf{B}$ (as in Theorem 2.1) is the polar of the Lawrence sphere $\mathcal{K}^+(\Lambda(M(\mathbf{B})))$. In order to prove the results stated in Section 3 it remains to generalize cover posets to oriented matroids.

Definition 4.3. The *cover poset* $\mathcal{U}(M)$ of an oriented matroid $M = (E, \mathcal{K})$ is the set of pairs (F,I) where $F \in \mathcal{K}(M|_I)$, together with the relation

$$(F,I) \preceq (G,J) \quad \text{if and only if} \quad I \supseteq J, F^+ \cap J \subseteq G^+, \\ \text{and} \quad F^- \cap J \subseteq G^-.$$

THEOREM 4.2. *For any oriented matroid M the map*

$$\phi : (F,I) \longmapsto (F^+ \cup I^c) \cup (F^- \cup I^c)^*$$

is an order-preserving isomorphism from $\mathcal{U}(M)$ to $\mathcal{K}^+(\Lambda(M))$. Thus $\mathcal{U}(M)$ is isomorphic to the face lattice of the Lawrence sphere associated with M.

Proof. We first show that $\phi(F,I)$ is an element of $\mathcal{K}^+(\Lambda(M))$ for any $F \in \mathcal{K}(M|_I)$. We can write $F = \cup_{j=1}^k F_j$ for some cocircuits F_j of $M|_I$ with $F^+ = \cup_{j=1}^k F_j^+$ and $F^- = \cup_{j=1}^k F_j^-$. There are cocircuits G_j of M such that $G_j^+ \cap I = F_j^+$ and $G_j^- \cap I = F_j^-$. By Theorem 4.1, the signed sets D_j defined by $D_j^+ := G_j^+ \cup (G_j^-)^*, D_j^- := \emptyset$ are cocircuits of $\Lambda(M)$. Clearly, $D_j^+ \cap (I \cup I^*) = F_j^+ \cup (F_j^-)^*$. This implies

$$\phi(F,I) = (F^+ \cup I^c) \cup (F^- \cup I^c)^* = [\cup_{j=1}^k D_j] \cup [\cup_{e \in I^c} \{e, e^*\}],$$

where each D_j is a positive cocircuit of $\Lambda(M)$, and $\{e, e^*\}$ is either a positive cocircuit or the union of two positive cocircuits of $\Lambda(M)$ for each $e \in I^c$. Thus $\phi(F,I) \in \mathcal{K}^+(\Lambda(M))$. It is easy to check that ϕ is order-preserving.

We now construct an inverse for ϕ. Given any $C \in \mathcal{K}^+(\Lambda(M))$, we write $S := C \cap E$ and $T := C \cap E^*$, and we define a signed vector F on E by

$F^+ := S \backslash T_*, F^- := T_* \backslash S$. We will show that F is in the cocircuit span of $M|_I$ where $I := (S \cap T_*)^c$. Consider a decomposition $C = \cup_{j=1}^k C_j$ of C into positive cocircuits C_j of $\Lambda(M)$, and partition each cocircuit $C_j = S_j \cup T_j$ as before.

Suppose first that $S \cap T_* = \emptyset$. By Theorem 4.1, the signed vectors F_j with $F_j^+ = S_j, F_j^- = (T_j)_*$, are cocircuits of M. Since the F_j are conformal, $F^+ := \cup_{j=1}^k S_j = S, F^- := \cup_{j=1}^k (T_j)_* = T_*$ defines an element of $\mathcal{K}(M)$, which equals $\mathcal{K}(M|_I)$.

Now suppose $S \cap T_* \neq \emptyset$. Then the signed vectors D_j defined by $D_j^+ = (S_j \backslash T_*) \cup (T_j \backslash S^*), D_j^- = \emptyset$ are positive cocircuits of $\Lambda(M|_I)$. Applying the argument above to $D = \cup_{j=1}^k D_j = C \backslash [(S \cap T_*) \cup (S^* \cap T)]$ (with $(D \cap E) \cap (D \cap E^*)_* = \emptyset$) shows that the signed vector F given by $F^+ = D \cap E, F^- = (D \cap E^*)_*$ is in $\mathcal{K}(M|_I)$.

It is easy to check that the map from $\mathcal{K}^+(\Lambda(M))$ to $\mathcal{U}(M)$ defined by $C \longmapsto (F, I)$ is the inverse of ϕ, and thus ψ is a lattice isomorphism from $\mathcal{K}^+(\Lambda(M))$ to $\mathcal{U}(M)$.

The next result is on the f-vector of the Lawrence oriented matroid. Because our motivation is from the geometric case, we will assign to each element of $\mathcal{K}^+(\Lambda(M))$ and to each element of each $\mathcal{K}(M|_I)$ a dimension, which is one less than its rank in the appropriate poset. Thus $f_i(\Lambda)$ (respectively, $f_i(M|_I)$) stands for the number of rank $i+1$ elements of $\mathcal{K}^+(\Lambda(M))$ (respectively, $\mathcal{K}(M|_I)$), and the f-vector of $\Lambda(M)$ is defined as $f(\Lambda) = (f_0(\Lambda), f_1(\Lambda), \ldots, f_{n+r}(\Lambda))$.

Las Vergnas [**13**] generalized Zaslavsky's Theorem to oriented matroids.

THEOREM 4.3. (Las Vergnas [**13**]) *The f-vector of $\mathcal{K}(M)$ is a function of M.*

THEOREM 4.4. *For any oriented matroid M, the f-vector of $\Lambda(M)$ is a function of M.*

Proof. It suffices to show that for each $I \subseteq E$ each rank level set of $\mathcal{K}(M|_I)$ is mapped by ϕ into a single rank level set of $\mathcal{K}^+(\Lambda(M))$. That implies that each $f_i(\Lambda)$ is the sum of a finite number of terms of the form $f_j(M|_I)$. Theorem 4.3 says that $f_j(M|_I)$ depends only on $\underline{M|_I} = \underline{M}|_I$, which depends only on $\underline{M}$ (and I).

Fix $I \subseteq E$. Write $r_I(F)$ for the rank of F in the poset $\mathcal{K}(M|_I)$. Let $C_F = (F^+ \cup I^c) \cup (F^- \cup I^c)^*$ and write $r_\Lambda(C_F)$ for the rank of C_F in the poset $\mathcal{K}^+(\Lambda(M))$. Let $D_F = F^+ \cup (F^-)^*$ and write $r_{\Lambda|I}(D_F)$ for the rank of D_F in the poset $\mathcal{K}^+(\Lambda(M)|_{I \cup I^*})$, which is the same as $\mathcal{K}^+(\Lambda(M|_I))$. Matroid rank functions are denoted by ρ (ρ_M for $M, \rho_{M|_I}$ for $M|_I, \rho_\Lambda$ for $\Lambda(M)$, and $\rho_{\Lambda|I}$ for $\Lambda(M|_I)$); without an argument ρ stands for the rank of an entire matroid. Note that for $S \subseteq I \cup I^*, \rho_\Lambda(S) = \rho_{\Lambda|I}(S)$. We show that $r_\Lambda(C_F)$ is a function of $r_I(F)$.

As observed earlier the poset rank functions can be written in terms of the

https://doi.org/10.4153/CJM-1990-004-4 Published online by Cambridge University Press

matroid rank functions. For $F \in \mathcal{K}(M|_I)$,

$$\begin{aligned} r_\Lambda(C_F) &= \rho_\Lambda - \rho_\Lambda(E \cup E^* \backslash C_F) \\ &= \rho_\Lambda - \rho_\Lambda(I \cup I^* \backslash D_F) \\ &= \rho_\Lambda - \rho_\Lambda(I \cup I^*) + \rho_\Lambda(I \cup I^*) - \rho_\Lambda(I \cup I^* \backslash D_F) \\ &= \rho_\Lambda - \rho_\Lambda(E \cup E^* \backslash C_\emptyset) + \rho_{\Lambda|I} - \rho_{\Lambda|I}(I \cup I^* \backslash D_F) \\ &= r_\Lambda(C_\emptyset) + r_{\Lambda|I}(D_F). \end{aligned}$$

The rank $r_\Lambda(C_\emptyset)$ is the length of a maximal chain of elements in $\mathcal{K}^+(\Lambda(M))$ ending in $C_\emptyset$; similarly for $r_{\Lambda|I}(D_F)$. We compute these using Theorem 4.1. The signed set $C_\emptyset$ is the union of positive cocircuits of $\Lambda(M)$ of two types: singleton cocircuits $\{e\}$ and $\{e^*\}$, for $\{e\}$ a cocircuit of $M, e \in I^c$, and doubletons $\{e, e^*\}$, for $\{e\}$ not a cocircuit of $M, e \in I^c$. A maximal chain ending in $C_\emptyset$ can be formed from unions of these cocircuits, each union containing one more cocircuit than the previous. Letting $k_I(M)$ denote the number of singleton cocircuits of M in I^c, we get

$$r_\Lambda(C_\emptyset) = 2k_I(M) + |I^c| - k_I(M) = k_I(M) + |E| - |I|.$$

Now D_F contains no pairs of the form $\{e, e^*\}$. By Theorem 4.1 it contains only positive cocircuits of the form Y^A, for Y a cocircuit of $M|_I$ and $A = Y^- \subseteq I$. So D_F is the top of a length j chain of elements of $\mathcal{K}^+(\Lambda(M|_I))$ if and only if F is the top of a length j chain of elements of $\mathcal{K}(M|_I)$. So $r_{\Lambda|I}(D_F) = r_I(F)$. Thus

$$(*) \qquad r_\Lambda(C_F) = k_I(M) + |E| - |I| + r_I(F).$$

So for fixed I, all pairs (F, I) with F of given rank in $\mathcal{K}(M|_I)$ are mapped to elements of $\mathcal{K}^+(\Lambda(M))$ of the same rank.

The definition of flag vector of a polytope generalizes naturally to the flag vector of the cocircuit span or positive cocircuit span of an oriented matroid. We write $f_S(\Lambda)$ for the components of the flag vector of $\mathcal{K}^+(\Lambda(M))$.

THEOREM 4.5. *For any oriented matroid M, the flag vector of the Lawrence sphere $\mathcal{K}^+(\Lambda(M))$ is a function of the underlying matroid $\underline{M}$.*

Proof. Let $S = \{s_1, s_2, \ldots, s_t\}$ with the s_j in increasing order. By Theorem 4.2 and (*) in the proof of Theorem 4.4, $f_S(\Lambda)$ is the number of chains $(F_1, I_1) \prec (F_2, I_2) \prec \cdots \prec (F_t, I_t)$ with $r_j := r_{I_j}(F_j) = s_j - k_{I_j}(M) - |E| + |I_j|$.

We can choose the first pair in the chain by choosing arbitrarily a subset $I_1 \subseteq E$, a closed flat $X_1 \subseteq I_1$ of rank $\rho_{I_1} - r_1$ in the matroid $\underline{M}|_{I_1}$, and an element F_1 of the cocircuit span of $M|_{I_1}$ with support $I_1 \backslash X_1$. Given I_1 and X_1, the allowable F_1 correspond naturally to the maximal elements of the cocircuit span of the oriented matroid $M|_{I_1}/X_1$. These are the elements of $\mathcal{K}(M|_{I_1}/X_1)$ of rank $\rho_1 := \rho(M|_{I_1}/X_1)$, and their number $f_{\rho_1}(\mathcal{K}(M|_{I_1}/X_1))$ is a function of $\underline{M}$.

Suppose an initial pair (F_1, I_1) has been chosen as above. If $(F_2, I_2) \succ (F_1, I_1)$ then $I_2 \subseteq I_1$ and $F_1^+ \cap (I_2 \backslash X_1) = F_1^+ \cap I_2 \subseteq F_2^+$ and $F_1^- \cap (I_2 \backslash X_1) = F_1^- \cap I_2 \subseteq F_2^-$. So the zero set $X_2 \subseteq I_2$ of F_2 is contained in the zero set X_1 of F_1. Thus we can choose the pair (F_2, I_2) by choosing arbitrarily a subset $I_2 \subseteq I_1$, a closed flat $X_2 \subseteq I_2 \cap X_1$ of rank $\rho_{I_2} - r_2$ of the matroid $\underline{M}|_{I_2}$, and an element F_2 of the cocircuit span of $M|_{I_2}$ with support $I_2 \backslash X_2$ and satisfying $F_1^+ \cap I_2 \subseteq F_2^+$ and $F_1^- \cap I_2 \subseteq F_2^-$. Given I_2 and X_2 as above, there is a one-to-one correspondence between the allowable F_2 and elements G of $\mathcal{K}(M|_{I_2 \cap X_1})$ with support $(I_2 \cap X_1) \backslash X_2$. The correspondence is given by $F_2 \longmapsto F_2 | (I_2 \cap X_1)$, with inverse $G \longmapsto (F_1 | I_2) \cdot G'$, where $G' \in \mathcal{K}(M|_{I_2})$ and $G = G' | (I_2 \cap X_1)$ (here the product $(F_1 | I_2) \cdot G'$ does not depend on the choice of G' because $F_1 | I_2$ is nonzero on $I_2 \backslash X_1$). These G correspond naturally to the maximal elements of the cocircuit span of the oriented matroid $M|_{I_2 \cap X_1} / X_2$. These are the elements of $\mathcal{K}(M|_{I_2 \cap X_1} / X_2)$ of rank $\rho_2 := \rho(M|_{I_2 \cap X_1} / X_2)$; their number $f_{\rho_2}(\mathcal{K}(M|_{I_2 \cap X_1} / X_2))$ is a function of $\underline{M}$. Subsequent elements of the S-flag can be chosen in the same way.

Write $\mathcal{S}(\underline{M})$ for the set of closed flats of the matroid $\underline{M}$. We conclude that the number of S-chains in $\mathcal{K}^+(\Lambda(M))$ equals

$$f_S(\Lambda) = \sum_{I_1 \subseteq E} \sum_{\substack{X_1 \in \mathcal{S}(\underline{M}|_{I_1}) \\ \rho_{I_1}(X_1) = \rho_{I_1} - r_1}} f_{\rho_1}(\mathcal{K}(M|_{I_1} / X_1))$$

$$\times \sum_{I_2 \subseteq I_1} \sum_{\substack{X_2 \in \mathcal{S}(\underline{M}|_{I_2}) \\ X_2 \subseteq I_2 \cap X_1 \\ \rho_{I_2}(X_2) = \rho_{I_2} - r_2}} f_{\rho_2}(\mathcal{K}(M|_{I_2 \cap X_1} / X_2))$$

$$\cdots\cdots\cdots \sum_{I_t \subseteq I_{t-1}} \sum_{\substack{X_t \in \mathcal{S}(\underline{M}|_{I_t}) \\ X_t \subseteq I_t \cap X_{t-1} \\ \rho_{I_t}(X_t) = \rho_{I_t} - r_t}} f_{\rho_t}(\mathcal{K}(M|_{I_t \cap X_{t-1}} / X_t)).$$

Every term depends only on the matroid $\underline{M}$, and hence the flag vector of $\mathcal{K}^+(\Lambda(M))$ depends only on the matroid $\underline{M}$.

The proof of Theorem 4.5 gives information about the flag vector $(f_S(M))_{S \subseteq [r]}$ of the oriented matroid M itself. Here $f_S(M)$ denotes the number of S-chains in the poset $\mathcal{K}(M)$. Corollary 4.6 is a generalization of Las Vergnas's generalization [**13**] of Zaslavsky's theorem.

COROLLARY 4.6. *For any oriented matroid M, the flag vector of M is a function of the underlying matroid $\underline{M}$.*

Proof. The S-chains $F_1 \subset F_2 \subset \cdots \subset F_t$ in the cocircuit span $\mathcal{K}(M)$ of M correspond to chains in the cover lattice $\mathcal{U}(M)$ of the form $(F_1, E) \prec (F_2, E) \prec \cdots \prec (F_t, E)$ with $r_E(F_j) = s_j - k_E(M) - |E| + |E| = s_j$. Thus a formula for $f_S(M)$ is obtained from the formula for $f_S(\Lambda)$ by taking the summands where all I_j are chosen to be E.

In order to derive the final result of this section we need one more oriented matroid construction. Given an oriented matroid M on a set E and a subset $A \subseteq E$, the oriented matroid $M_{\overline{A}}$ obtained by *reorientation* of A is defined as follows. The cocircuit span of $M_{\overline{A}}$ consists of the signed vectors of the form $F_{\overline{A}}$ defined by $F^+_{\overline{A}} := (F^+ \backslash A) \cup (F^- \cap A)$ and $F^-_{\overline{A}} := (F^- \backslash A) \cup (F^+ \cap A)$, where $F \in \mathcal{K}(M)$. Clearly, $\underline{M_{\overline{A}}} = \underline{M}$.

PROPOSITION 4.7. *For any oriented matroid on a set E, the cocircuit span of the Lawrence oriented matroid has $2^{|E|}$ intervals isomorphic to the face lattice of the Lawrence sphere.*

Proof. For any $A \subseteq E$, the oriented matroid obtained by reorienting $\Lambda(M)$ on $A \cup A^*$ equals the Lawrence oriented matroid $\Lambda(M_{\overline{A}})$. The maximal element of the positive cocircuit span of $\Lambda(M|_{\overline{A}})$ is of the form $F_{\overline{A \cup A^*}}$, where F is the (maximal) element of $\mathcal{K}(\Lambda(M))$ with $F^+ = E \cup E^* \backslash (A \cup A^*)$ and $F^- = A \cup A^*$. Therefore $\mathcal{K}^+(\Lambda(M_{\overline{A}}))$ is isomorphic to the interval $[\emptyset, F]$ of $\mathcal{K}(\Lambda(M))$. By Theorem 4.2, $\mathcal{K}^+(\Lambda(M_{\overline{A}}))$ is isomorphic to $\mathcal{U}(M_{\overline{A}})$, which is clearly isomorphic to $\mathcal{U}(M)$. So the interval $[\emptyset, F]$ is isomorphic to the poset $\mathcal{K}^+(\Lambda(M))$, the face lattice of the Lawrence sphere.

More generally, if F is any maximal element of $\mathcal{K}(\Lambda(M))$ and t is the number of elements e of E such that e and e^* have the same sign in F, then $\mathcal{K}(\Lambda(M))$ has 2^t intervals isomorphic to $[\emptyset, F]$.

In the representable case the intervals $[\emptyset, F]$ correspond to maximal regions of the hyperplane arrangement $\mathcal{H}_\Lambda$, thus implying Proposition 3.6.

5. On the topology of Lawrence polytopes. In this section we study the topological space of all polytopes combinatorially equivalent to a given Lawrence polytope. As the main result (Theorem 5.4) it is proved that this space is homotopy equivalent to the realization space of the underlying oriented matroid. Using the recent solutions of N. White, B. Jaggi, P. Mani-Levitska and B. Sturmfels **[11]**, **[22]** to the isotopy problem for oriented matroids, we then construct a 19-polytope without the isotopy property. As a byproduct we obtain a new asymptotic bound on the number of combinatorially distinct polytopes with few vertices.

A convex d-polytope P satisfies the *isotopy property* if for any d-polytope Q combinatorially equivalent to P either Q or a mirror image of Q can be connected to P by a continuous path of polytopes of the same type. This definition agrees for the usual topology induced from $\mathbf{R}^{d \cdot |vert\, P|}$ and the Hausdorff topology on convex polytopes. Moreover, the definition of isotopy property for polytopes stays the same if we replace "combinatorially equivalent" by "combinatorially equivalent as labelled polytopes". To see this, suppose that P admits an even affine symmetry, say π. Then the labelled polytopes P and $\pi(P)$ are connected because $SO(d, \mathbf{R})$ is a connected group.

The following classical result was obtained by E. Steinitz in the 1920's.

THEOREM 5.1. (Steinitz [**18**]) *Every* 3-*polytope satisfies the isotopy property.*

In this section we prove that Steinitz' isotopy theorem fails in higher dimensions.

THEOREM 5.2. *There exists a* 19-*dimensional Lawrence polytope* $\mathcal{P}$ *with* 34 *vertices that does not satisfy the isotopy property.*

As before, we write $\mathcal{K}(\mathbf{B})$ for the cocircuit span of the oriented matroid associated with $\mathbf{B} \in M(n \times r, \mathbf{R})$. Its positive cocircuit span is denoted $\mathcal{K}^+(\mathbf{B})$. We define the *realization space* of the oriented matroid $M(\mathbf{B})$ by

$$[\mathbf{B}] := \{\mathbf{C} \in M(n \times r, \mathbf{R}) | \mathcal{K}(\mathbf{B}) = \mathcal{K}(\mathbf{C})\}.$$

Furthermore we consider the space

$$[\mathbf{B}]^+ := \{\mathbf{C} \in M(n \times r, \mathbf{R}) | \mathcal{K}^+(\mathbf{B}) = \mathcal{K}^+(\mathbf{C})\}.$$

The latter space equals the set of matrices $\mathbf{C}$ such that pos($\mathbf{B}$) and pos($\mathbf{C}$) have isomorphic labelled face lattices. Clearly, $[\mathbf{B}] \subseteq [\mathbf{B}]^+$. Both spaces are topologized as subsets of $\mathbf{R}^{rn}$.

Until recently it was a prominent open question, known as the isotopy problem for oriented matroids, whether for every matrix $\mathbf{B}$ the realization space $[\mathbf{B}]$ has precisely two connected components. This problem has been first solved by Neil White.

THEOREM 5.3. (N. White, [**22**]) *There exists a matrix* $\mathbf{V} \in M(42 \times 3, \mathbf{R})$ *such that* $[\mathbf{V}]$ *has four connected components.*

White's result has very recently been improved by Jaggi and Mani-Levitska who gave a smaller uniform counterexample.

THEOREM 5.3$'$. (B. Jaggi & P. Mani-Levitska [**11**]) *There exists a matrix* $\mathbf{W} \in M(17 \times 3, \mathbf{R})$ *such that the oriented matroid* $M(\mathbf{W})$ *is uniform and* $[\mathbf{W}]$ *has four connected components.*

Using these results, we shall prove the following general fact from which Theorem 5.2 can be derived easily.

THEOREM 5.4. *Let* $\mathbf{B}$ *be any matrix in* $M(n \times r, \mathbf{R})$ *and let* $\Lambda(\mathbf{B}) \in M(2n \times (n+r), \mathbf{R})$ *be the Lawrence matrix associated with* $\mathbf{B}$. *Then the spaces* $[\mathbf{B}]$ *and* $[\Lambda(\mathbf{B})]^+$ *are homotopy equivalent.*

Derivation of Theorem 5.2 *from Theorem* 5.4. Theorem 5.3$'$ and Theorem 5.4 imply the existence of a Lawrence matrix $\Lambda(\mathbf{W}) \in M(34 \times 20, \mathbf{R})$ such that $[\Lambda(\mathbf{W})]^+$ has four connected components. Let $\mathcal{P}$ be the 19-dimensional Lawrence polytope associated with $\Lambda(\mathbf{W})$. We assume that $\mathcal{P}$ satisfies the isotopy property, and from this we derive a contradiction to the fact that $[\Lambda(\mathbf{W})]^+$ has four connected components.

Let $\mathbf{X}, \mathbf{Y} \in [\Lambda(\mathbf{W})]^+$, that is, the labelled face lattices of the cones pos($\mathbf{X}$), pos($\mathbf{Y}$) $\subset \mathbf{R}^{20}$ are isomorphic to the labelled face lattice of $\mathcal{P}$. There exists a rotation $\mathbf{T} \in SO(20, \mathbf{R})$ and a half space $H^+ \subset \mathbf{R}^{20}$ such that pos($\mathbf{X}$), pos($\mathbf{Y}'$) $\subset H^+$, where $\mathbf{Y}' := \mathbf{Y} \cdot \mathbf{T}$. The connectedness of the rotation group $SO(20, \mathbf{R})$ implies that $\mathbf{Y}$ and $\mathbf{Y}'$ lie in the same connected component of $[\Lambda(\mathbf{W})]^+$. Choose an affine hyperplane $\bar{H}$ parallel to the boundary of H^+ such that $\mathcal{Q} := \bar{H} \cap$ pos($\mathbf{X}$) and $\mathcal{Q}' := \bar{H} \cap$ pos($\mathbf{Y}'$) are 19-polytopes combinatorially equivalent to $\mathcal{P}$.

Identifying $\bar{H}$ with $\mathbf{R}^{19}$, the hypothesis implies that either $\mathcal{Q}$ and $\mathcal{Q}'$ or $\mathcal{Q}$ and a mirror image of $\mathcal{Q}'$ can be connected by a path of polytopes in $\bar{H}$ of the same type. This path lifts in the obvious manner to a path in $[\Lambda(W)]^+$, and hence either $\mathbf{X}$ and $\mathbf{Y}'$ or $\mathbf{X}$ and a mirror image of $\mathbf{Y}'$ lie in the same connected component. This implies that $[\Lambda(\mathbf{W})]^+$ has only two connected components, and Theorem 5.2 follows by contradiction.

In order to prove Theorem 5.4 we need the following lemma.

LEMMA 5.5. *Let* $\mathbf{B} \in M(n \times r, \mathbf{R})$, *and let* $\Lambda(\mathbf{B}) \in M(2n \times (n+r), \mathbf{R})$ *be the Lawrence matrix associated with* $\mathbf{B}$. *Then* $[\Lambda(\mathbf{B})] = [\Lambda(\mathbf{B})]^+$.

Proof. It is sufficient to show that the face semilattice $\mathcal{K}(\Lambda(\mathbf{B}))$ is uniquely determined by the face lattice $\mathcal{K}^+(\Lambda(\mathbf{B}))$ of the specific region polar to pos($\mathbf{B}$). By Theorem 4.2 and the proof of Corollary 4.6, the face semilattice $\mathcal{K}(\mathbf{B})$ is a subposet of the Lawrence sphere $\mathcal{K}^+(\Lambda(\mathbf{B}))$. On the other hand, $\mathcal{K}(\Lambda(\mathbf{B}))$ is uniquely determined by $\mathcal{K}(\mathbf{B})$, using Theorem 4.1. This implies that every matrix $\mathbf{C} \in [\Lambda(\mathbf{B})]^+$ must also be in $[\Lambda(\mathbf{B})]$.

Now we are ready to complete the proof of Theorem 5.4 and hence Theorem 5.2.

Proof of Theorem 5.4. By Lemma 5.5, it suffices to show that the spaces $[\mathbf{B}]$ and $[\Lambda(\mathbf{B})]$ are homotopy equivalent. By Theorem 2.1, every matrix $\mathbf{X}$ in $[\Lambda(\mathbf{B})]$ is projectively equivalent to some $\Lambda(\mathbf{C})$ where $\mathbf{C} \in [\mathbf{B}]$. More precisely, we consider the continuous map

$$\begin{aligned} \psi : [\Lambda(\mathbf{B})] &\longrightarrow [\Lambda(\mathbf{B})] \\ \mathbf{X} &\longmapsto \Lambda(\mathbf{C}) = \mathbf{D}^{-1} \cdot \mathbf{X} \cdot \mathbf{T} \end{aligned}$$

defined in the proof of Theorem 2.1. In defining ψ it could be assumed that the $(r+n) \times (r+n)$-matrix $\mathbf{T}$ has positive determinant. Recall that $\mathbf{D}^{-1}$ is a positive definite $2n \times 2n$-diagonal matrix, and that both $\mathbf{T}$ and $\mathbf{D}^{-1}$ depend continuously on $\mathbf{X}$. This shows that the map ψ is homotopic to the identity map on $[\Lambda(\mathbf{B})]$.

Since $\psi(\Lambda(\mathbf{C})) = \Lambda(\mathbf{C})$ for all $\mathbf{C} \in [\mathbf{B}]$, it follows that the map

$$[\mathbf{B}] \longrightarrow [\Lambda(\mathbf{B})], \mathbf{C} \longmapsto \Lambda(\mathbf{C})$$

is a homotopy equivalence.

Clearly, the above homotopy equivalence $[\mathbf{B}] \longrightarrow [\Lambda(\mathbf{B})]^+$ is a polynomial map with integer coefficients. This implies the following

COROLLARY 5.6. *Let K be any ordered subfield of the real numbers. Then $[\mathbf{B}]$ contains K-rational points if and only if $[\Lambda(\mathbf{B})]^+$ contains K-rational points. Hence the algorithmic Steinitz problem for K is equivalent to the realizability problem for oriented matroids over K.*

We complete this section with a remark on the number of combinatorial types of polytopes with few vertices. By Lemma 5.5, the number of combinatorial types of $(n+r-1)$-dimensional Lawrence polytopes with $2n$ vertices is bounded below by the number of real realizable rank r oriented matroids on n points. This implies the following lower bound on the number of all convex polytopes with few vertices, both in the labelled and the unlabelled case.

COROLLARY 5.7. *For fixed $\beta > 0$ and $d \geqq \beta$, the number $c(d+\beta, d)$ of combinatorially distinct labelled d-polytopes with $d+\beta$ vertices is bounded below by the number $t(\lfloor \frac{d+\beta}{2} \rfloor, \lfloor \frac{d-\beta}{2} \rfloor)$ of labelled rank $\lfloor \frac{d-\beta}{2} \rfloor$ real oriented matroids on $\lfloor \frac{d+\beta}{2} \rfloor$ points.*

The lower bounds given by Goodman & Pollack [**9**] for the function $t(n,r)$ imply a lower bound for $c(d+\beta,d)$, which is asymptotically equal to Alon's upper bound for $c(d+\beta,d)$ [1].

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https://doi.org/10.4153/CJM-1990-004-4 Published online by Cambridge University Press