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CONGRUENCES MODULO 5 AND 7 FOR 4-COLORED GENERALIZED FROBENIUS PARTITIONS

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Abstract

Let $c\phi_k(n)$ denote the number of k-colored generalized Frobenius partitions of n. Recently, new Ramanujan-type congruences associated with $c\phi_4(n)$ were discovered. In this article, we discuss two approaches in proving such congruences using the theory of modular forms. Our methods allow us to prove congruences such as $c\phi_4(14n+6) \equiv 0 \bmod 7$ and Seller's congruence $c\phi_4(10n+6) \equiv 0 \bmod 5$.

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1. Introduction

A partition π of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that

$$\lambda_1 + \lambda_2 + \cdots + \lambda_r = n.$$

It is known that a partition π can be visualized using a Ferrers diagram by representing the positive integer m of the sth part by m dots on the sth row and an example is given in Figure 1.

Given a Ferrers diagram with d diagonal dots, we first remove the diagonal dots and let $r_{1,j}$ and $r_{2,j}$, $1 \le j \le d$, be the number of dots after the diagonal on the jth row and jth column, respectively. We then obtain a 2 by d matrix $(r_{i,j})_{2\times d}$. The matrix corresponding to the Ferrers diagram in Figure 1 is

$$\begin{pmatrix} 3 & 1 & 0 \\ 3 & 2 & 0 \end{pmatrix}.$$

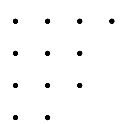

FIGURE 1. Diagram representing $4+3+3+2$.

We observe that, by construction, $r_{1,j}$ and $r_{2,j}$ are strictly decreasing nonnegative integers. A 2 by d matrix with strictly decreasing nonnegative integers $r_{1,j}$ and $r_{2,j}$ is called a Frobenius symbol. It is clear that given a Frobenius symbol, we can construct a Ferrers diagram, which in turn leads to a partition of n given by

$$n = d + \sum_{j=1}^{d}(r_{1,j} + r_{2,j}).$$

In his 1984 AMS Memoir, Andrews [1] introduced a generalized Frobenius symbol with at most k repetitions for each integer by relaxing the 'strictly decreasing' condition and allowing at most k repetitions of each nonnegative integer in each row. In other words, for $i = 1, 2$, the condition that $r_{i,j} > r_{i,j+1}$ for $1 \le j < d$ is replaced by $r_{i,j} \ge r_{i,j+1}$ with at most an unbroken chain of k equalities. From such a symbol, he associated a generalized Frobenius partition of n given by

$$n = d + \sum_{j=1}^{d}(r_{1,j} + r_{2,j})$$

and denoted the number of such partitions of n by $\phi_k(n)$. As an example, we observe that $\phi_2(3) = 5$ and these are given by the following generalized Frobenius partitions with at most two repetitions on each row:

$$\begin{pmatrix}2\\0\end{pmatrix}, \quad \begin{pmatrix}0\\2\end{pmatrix}, \quad \begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}, \quad \begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}.$$

Note that, with this definition,

$$\phi_1(n) = p(n),$$

where $p(n)$ is the number of partitions of n.

In [1], Andrews also defined generalized Frobenius symbols with k colors. To define such a symbol, we first order the given k colors and use $1, 2, \ldots, k$ to represent them. We color the parts using these k colors and order these parts in the following manner:

$$0_1 \prec 0_2 \prec \cdots \prec 0_k \prec 1_1 \prec 1_2 \prec \cdots \prec 1_k \prec 2_1 \prec 2_2 \prec \cdots \prec 2_k \prec \cdots .$$

Here we are using '$\prec$' to differentiate the inequality from the usual inequality '$<$'. A generalized Frobenius symbol with k colors is then defined as a 2 by d matrix with entries

$$r_{i,j} \in \{\ell_c | \ell \text{ and } c \text{ are nonnegative integers with } 1 \le c \le k\}$$

and

$$r_{i,j+1} \prec r_{i,j}, \quad i = 1, 2 \quad \text{and} \quad 1 \le j < d.$$

Once again, given a generalized Frobenius symbol with k colors, he associated a generalized Frobenius partition of n by

$$n = d + \sum_{j=1}^{d}(r_{1,j} + r_{2,j}),$$

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where only the nonnegative integer ℓ is added if $r_{i,j} = \ell_c$. The number of such partitions of n is denoted by $c\phi_k(n)$. As an example, we note that $c\phi_2(2) = 9$ and these are given by the following symbols:

$$\begin{pmatrix}1_1\\0_1\end{pmatrix},\quad \begin{pmatrix}1_1\\0_2\end{pmatrix},\quad \begin{pmatrix}1_2\\0_1\end{pmatrix},\quad \begin{pmatrix}1_2\\0_2\end{pmatrix},\quad \begin{pmatrix}0_1\\1_1\end{pmatrix},$$
$$\begin{pmatrix}0_2\\1_1\end{pmatrix},\quad \begin{pmatrix}0_1\\1_2\end{pmatrix},\quad \begin{pmatrix}0_2\\1_2\end{pmatrix},\quad \begin{pmatrix}0_2 & 0_1\\0_2 & 0_1\end{pmatrix}.$$

Once again, it is clear that $c\phi_1(n) = p(n)$.

The function $p(n)$ satisfies several congruence properties, namely,

$$p(\ell n - \delta_\ell) \equiv 0 \bmod \ell \tag{1.1}$$

when $(\ell, \delta_\ell) = (5, 1), (7, 2), (11, 5)$. These congruences were discovered by Ramanujan when he was examining the table of $p(n)$ computed by MacMahon. In [1], Andrews discovered that the generalized Frobenius partitions also satisfy congruences similar to those of $p(n)$. He showed that [1, Corollary 10.1]

$$\phi_2(5n + 3) \equiv 0 \bmod 5 \quad \text{and} \quad c\phi_2(5n + 3) \equiv 0 \bmod 5.$$

Since the discovery of Andrews' congruences, several new congruences satisfied by $\phi_k(n)$ and $c\phi_k(n)$ were found. For example, in [14], Sellers showed, using new identities discovered by Baruah and Sarmah [2], that for all positive integers n,

$$c\phi_4(10n + 6) \equiv 0 \bmod 5. \tag{1.2}$$

Shortly after this discovery, Lin [11] proved that for every positive integer n,

$$c\phi_4(14n + 13) \equiv 0 \bmod 7. \tag{1.3}$$

Lin's proof involved Baruah–Sarmah identities and identities arising from Jacobi's triple-product identity. Motivated by the congruences of Sellers and Lin, the second author and Lin independently observed that

$$c\phi_4(14n + 6) \equiv 0 \bmod 7. \tag{1.4}$$

Congruence (1.4) does not seem to follow from Lin's approach in his proof of (1.3). In this article, we give two proofs of (1.4) using the theory of modular forms and an identity involving the generating function for $c\phi_4(2n)$. Congruence (1.4), together with (1.3), yields the result that for every positive integer n,

$$c\phi_4(7n + 6) \equiv 0 \bmod 7. \tag{1.5}$$

Note that this is an exact analogue of Ramanujan's congruence (1.1) for $\ell = 7$. We remark here that there is no analogue of Ramanujan's congruence for $\ell = 5$ even though (1.2) holds. Recently, Zhang and Wang [20] proved (1.5) using the quintuple-product identity and a new identity involving the generating function for $c\phi_4(n)$ (instead of $c\phi_4(2n)$).

In Section 2, we recall some results in the theory of modular forms that are needed for our proofs. In Section 3, we present proofs of (1.2) and (1.4) based on the structures of certain spaces of modular forms. The method used in this section is similar to that given by Chan and Lewis [6]. In this method, we need to construct new bases for certain spaces of modular forms. We also apply the method to derive an interesting congruence associated with overpartitions.

In Section 4, we prove (1.2) and (1.4) using a method due to Eichhorn and Ono [8].

2. Useful facts from the theory of modular forms

Let $N \geq 1$ be an integer. The congruence subgroup $\Gamma_0(N)$ of $SL_2(\mathbf{Z})$ is defined by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbf{Z}) \;\middle|\; c \equiv 0 \bmod N \right\}.$$

Let

$$\mathbf{H} := \{z \in \mathbf{C} | \mathrm{Im}(z) > 0\}$$

be the complex upper half plane. Let k, N be positive integers and χ be a Dirichlet character modulo N.

Definition 2.1. Suppose that $f(z)$ is holomorphic on $\mathbf{H}$ and at the cusps of $\Gamma_0(N)$. Let $\Gamma_0(N)$ act on $\mathbf{H}$ via the action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}.$$

If $f(z)$ satisfies

$$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z\right) = \chi(d)(cz+d)^k f(z),$$

then we say that $f(z)$ is a modular form of weight k on $\Gamma_0(N)$ with Dirichlet character χ.

The modular forms of weight k on $\Gamma_0(N)$ with Dirichlet character χ form a finite-dimensional vector space over $\mathbf{C}$ (see [10, Ch. III]) denoted by $M_k(\Gamma_0(N), \chi)$. In particular, if χ is the trivial Dirichlet character, we also write $M_k(\Gamma_0(N))$ for $M_k(\Gamma_0(N), \chi)$.

It is known that if $f(z) \in M_k(\Gamma_0(N), \chi)$, then $f(z)$ admits a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n, \quad q = e^{2\pi i z}, \quad z \in \mathbf{H}.$$

For any positive integer m, we define the Hecke operator $T(m)$ as a map which sends a modular form

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n \in M_k(\Gamma_0(N), \chi)$$

to

$$f(z)|_{T(m)} := \sum_{n=0}^{\infty}\Big(\sum_{d|(m,n)} \chi(d)d^{k-1}a\Big(\frac{nm}{d^2}\Big)\Big)q^n.$$

It is known that $f(z)|_{T(m)} \in M_k(\Gamma_0(N), \chi)$. In particular, when $m = p$ is a prime, $p \nmid N$ and χ is trivial,

$$f(z)|_{T(p)} := \sum_{n=0}^{\infty}\Big(a(pn) + p^{k-1}a\Big(\frac{n}{p}\Big)\Big)q^n,$$

where we assume that $a(n/p) = 0$ if $p \nmid n$.

We now state several facts that we need for this article.

Proposition 2.2. *Let k and l be positive integers and $f(z) \in M_k(\Gamma_0(N))$ and $g(z) \in M_l(\Gamma_0(N))$. For any positive integer m:*

(1) $f(z)|_{T(m)} \in M_k(\Gamma_0(N))$;
(2) $f(z)g(z) \in M_{k+l}(\Gamma_0(N))$ *and* $f^m(z) \in M_{mk}(\Gamma_0(N))$;
(3) $f(z) \in M_k(\Gamma_0(mN))$.

Note that (2) and (3) follow immediately from the definition of $M_k(\Gamma_0(N))$. For the proof of (1), see [7, Sections 5.2 and 5.3].

Proposition 2.3. *Suppose that $f(z) \in M_k(\Gamma_0(N))$ with Fourier expansion*

$$f(z) = \sum_{n=0}^{\infty} a(n)q^n.$$

(1) *If d is a positive integer, then*

$$f(z)|_{V(d)} := \sum_{n=0}^{\infty} a(n)q^{dn} \in M_k(\Gamma_0(dN)).$$

(2) *If d is a positive integer and $d|N$, then*

$$f(z)|_{U(d)} := \sum_{n=0}^{\infty} a(dn)q^n \in M_k(\Gamma_0(N)).$$

For a discussion of Proposition 2.3, see [12, Proposition 2.22].

The Dedekind eta-function is defined by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1 - q^n), \quad q = e^{2\pi i z}, \quad z \in \mathbf{H}.$$

It is known that $\eta(z)$ is a weakly holomorphic modular form of weight $1/2$ which does not vanish on $\mathbf{H}$. A function $f(z)$ is called an eta-product if it is of the form

$$f(z) = \prod_{\delta|N} \eta^{r_\delta}(\delta z),$$

where N and δ are positive integers and $r_\delta \in \mathbf{Z}$.

Proposition 2.4. *Suppose that*

$$f(z)=\prod_{\delta|N}\eta^{r_\delta}(\delta z)$$

is an eta-product satisfying the following conditions:

$$k=\frac{1}{2}\sum_{\delta|N}r_\delta\in\mathbf{Z},$$
$$\sum_{\delta|N}\delta r_\delta\equiv 0 \bmod 24,$$
$$\sum_{\delta|N}\frac{N}{\delta}r_\delta\equiv 0 \bmod 24.$$

Then, for each $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in\Gamma_0(N)$,

$$f\Big(\frac{az+b}{cz+d}\Big)=\chi(d)(cz+d)^k f(z),$$

where χ *is defined by*

$$\chi(d):=\Big(\frac{(-1)^k s}{d}\Big),$$
$$s:=\prod_{\delta|N}\delta^{r_\delta}$$

and (m/n) *is the Kronecker symbol.*

The orders of an eta-product at the cusps of $\Gamma_0(N)$ can be determined by the following proposition.

Proposition 2.5. *Let* c, d *and* N *be positive integers with* $d|N$ *and* $(c,d)=1$. *If* $f(z)$ *is an eta-product satisfying the conditions in Proposition 2.4, then the order of vanishing of* $f(z)$ *at the cusp* c/d *is*

$$\frac{N}{24}\sum_{\delta|N}\frac{(d,\delta)^2 r_\delta}{\big(d,\frac{N}{d}\big)d\delta}.$$

Propositions 2.4 and 2.5 can be found in [12, page 18].

From Proposition 2.5, we conclude that if

$$f(z)=\prod_{\delta|N}\eta^{r_\delta}(\delta z)$$

is an eta-product satisfying the conditions in Proposition 2.4 and, in addition, for any divisor d of N,

$$\sum_{\delta|N}\frac{(d,\delta)^2 r_\delta}{\delta}\geq 0, \tag{2.1}$$

then $f(z)\in M_k(\Gamma_0(N),\chi)$.

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3. First approach to the congruences

We begin our section by computing the dimension of $M_k(\Gamma_0(16))$.

LEMMA 3.1. *Let $k \geq 2$ be an even integer. Then*

$$\dim M_k(\Gamma_0(16)) = 2k + 1.$$

PROOF. Let $[x]$ denote the integer part of x. From [7, Theorem 3.5.1], we know that

$$\dim M_k(\Gamma_0(N)) = (k-1)(g-1) + \left[\frac{k}{4}\right]\varepsilon_2 + \left[\frac{k}{3}\right]\varepsilon_3 + \frac{k}{2}\varepsilon_\infty, \tag{3.1}$$

where g, ε_2, ε_3 and ε_∞ depend only on N and can be computed explicitly. Now let $N = 16$ in (3.1). From the formulas given in [7, page 107], we deduce that

$$\varepsilon_2 = 0, \quad \varepsilon_3 = 0, \quad \varepsilon_\infty = 6, \quad g = 0.$$

Hence,

$$\dim M_k(\Gamma_0(16)) = 2k + 1. \qquad \square$$

Our next task is to define functions which are needed to produce a basis for $M_k(\Gamma_0(16))$ for $k = 12$ and 24. Let $z \in \mathbf{H}$ and $q = e^{2\pi i z}$. Following [4, pages 27–28], we define

$$\vartheta_F(z) := \sum_{j=-\infty}^{\infty} q^{(j+(1/2))^2},$$

$$\vartheta(z) := \sum_{j=-\infty}^{\infty} q^{j^2},$$

$$\vartheta_M(z) := \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}.$$

The theta functions defined above differ from the classical definitions of Jacobi's theta functions (see [19, Ch. 21]) $\vartheta_j(z)$, $j = 2, 3, 4$, and their relations are given by

$$\vartheta(z/2) = \vartheta_3(z), \quad \vartheta_M(z/2) = \vartheta_4(z) \quad \text{and} \quad \vartheta_F(z/2) = \vartheta_2(z).$$

Next, let

$$\begin{aligned}
f_1(z) &:= \vartheta^4(2z),\\
f_2(z) &:= \vartheta_F^2(2z)\vartheta_M^2(2z),\\
f_3(z) &:= \vartheta^4(z) + 4\vartheta^4(4z),\\
g_1(z) &:= \vartheta^4(z) - 4\vartheta^4(4z),\\
g_2(z) &:= \vartheta_M^4(z) - 4\vartheta_F^4(4z).
\end{aligned}$$

Then the following result holds.

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THEOREM 3.2. *Let*

$$A_i(z) := f_1^{i-1}(z) f_2^{7-i}(z), \quad 1 \le i \le 7,$$
$$U_j(z) := f_3(z) f_1^{6-j}(z) f_2^{j-1}(z), \quad 1 \le j \le 6,$$
$$V_j(z) := g_1(z) f_1^{6-j}(z) f_2^{j-1}(z), \quad 1 \le j \le 6$$

and

$$W_j(z) := g_2(z) f_1^{6-j}(z) f_2^{j-1}(z), \quad 1 \le j \le 6.$$

Then $A_i(z)$ $(1 \le i \le 7)$, $U_j(z)$, $V_j(z)$ *and* $W_j(z)$ $(1 \le j \le 6)$ *form a basis of* $M_{12}(\Gamma_0(16))$.

PROOF. By Jacobi's triple-product identity (see [3, Theorem 1.3.3])

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = \prod_{n\ge1} (1-q^{2n})(1+zq^{2n-1})(1+z^{-1}q^{2n-1}),$$

we find that

$$\vartheta_F(z) = 2\frac{\eta^2(4z)}{\eta(2z)}, \quad \vartheta(z) = \frac{\eta^5(2z)}{\eta^2(z)\eta^2(4z)}, \quad \vartheta_M(z) = \frac{\eta^2(z)}{\eta(2z)}.$$

Since

$$\vartheta_F^2(2z)\vartheta_M^2(2z) = 4\frac{\eta^4(2z)\eta^4(8z)}{\eta^4(4z)},$$

by Propositions 2.4 and 2.5, we conclude that

$$f_2(z) = \vartheta_F^2(2z)\vartheta_M^2(2z) \in M_2(\Gamma_0(16)).$$

Similarly, we can show that

$$\vartheta^4(z) \in M_2(\Gamma_0(4)),$$
$$\vartheta_M^4(z) \in M_2(\Gamma_0(16))$$

and

$$\vartheta_F^4(4z) \in M_2(\Gamma_0(16)).$$

By Propositions 2.2 and 2.3, we deduce that f_1, f_3, g_1 and g_2 belong to $M_2(\Gamma_0(16))$. This implies that $A_i(z)$ $(1 \le i \le 7)$, $U_j(z)$, $V_j(z)$ and $W_j(z)$ $(1 \le j \le 6)$ all belong to $M_{12}(\Gamma_0(16))$.

Now suppose that

$$\sum_{i=1}^{7} a_i A_i(z) + \sum_{j=1}^{6} (b_j U_j(z) + c_j V_j(z) + d_j W_j(z)) = 0.$$

Comparing the coefficients of q^k $(0 \le k \le 24)$ on both sides, we get a system of linear equations of a_i $(1 \le i \le 7)$, b_j, c_j and d_j $(1 \le j \le 6)$. Solving these equations, we deduce that

$$a_i = 0 \quad (1 \le i \le 7), \quad b_j = c_j = d_j = 0 \quad (1 \le j \le 6).$$

Thus, $A_i(z)$ $(1 \le i \le 7)$, $U_j(z)$, $V_j(z)$ and $W_j(z)$ $(1 \le j \le 6)$ are linearly independent modular forms in $M_{12}(\Gamma_0(16))$. Since $\dim M_{12}(\Gamma_0(16)) = 25$ (see Lemma 3.1), these functions form a basis of $M_{12}(\Gamma_0(16))$. □

Remark. We now describe the construction of the basis presented in Theorem 3.2. Consider the Hilbert–Poincaré series

$$H(x) := \sum_{k=0}^{\infty} \dim M_k(\Gamma_0(16)) x^k$$

associated to the graded ring

$$M = \bigoplus_{k=0}^{\infty} M_k(\Gamma_0(16))$$

of modular forms on $\Gamma_0(16)$. From Lemma 3.1, we see that

$$H(x) = \sum_{m=0}^{\infty} (4m+1) x^{2m} = \frac{1+3x^2}{(1-x^2)^2}.$$

This suggests that there are five modular forms $h_1, \dots, h_5$ of weight 2 on $\Gamma_0(16)$ such that the graded ring M is generated by $h_1, \dots, h_5$. In fact, since the $(2n)$th coefficient of $1/(1-x^2)^2 = 1 + 2x^2 + 3x^4 + \cdots$ is precisely the number of possible monomials of degree n in two indeterminates, the identity also suggests that with a suitable choice of $h_1, \dots, h_5$, a basis for $M_k(\Gamma_0(16))$ can be chosen to be

$$\{h_1^i h_2^{k/2-i} : i = 0, \dots, k/2\} \cup \{h_1^i h_2^{k/2-1-i} h_j : i = 0, \dots, k/2-1, j = 3, 4, 5\}. \tag{3.2}$$

In order to find h_j with the required property, we shall analyze the space $M_k(\Gamma_0(16))$ more carefully.

Observe that the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 16 & 0 \end{smallmatrix}\right)$ normalizes $\Gamma_0(16)$ and hence induces a linear transformation W on $M_k(\Gamma_0(16))$, called an Atkin–Lehner involution, defined by

$$f(z)|_W := \frac{4^k}{(16z)^k} f(-1/16z).$$

Note that W^2 is the identity map. Thus, $M_k(\Gamma_0(16))$ decomposes into the direct sum of two eigenspaces with eigenvalues 1 and -1. Let $M_k(\Gamma_0(16))^+$ and $M_k(\Gamma_0(16))^-$ denote these two eigenspaces, respectively. To obtain dimension formulas for these two spaces, we note that the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 16 & 0 \end{smallmatrix}\right)$ also induces an involution on the modular curve $X_0(16) := (\mathbf{H} \cup \mathbf{Q} \cup \{\infty\})/\Gamma_0(16)$. Let $X_0(16)/W$ denote the quotient curve of $X_0(16)$ by the action of this involution. Since $\left(\begin{smallmatrix} 0 & -1 \\ 16 & 0 \end{smallmatrix}\right)$ does not fix any cusps of $X_0(16)$, the ramified points in the double cover $X_0(16) \to X_0(16)/W$ must be elliptic points of order 2 on $X_0(16)/W$. Since both $X_0(16)$ and $X_0(16)/W$ have genus 0, the Riemann–Hurwitz formula implies that $X_0(16)/W$ has two elliptic points of order 2.Therefore,

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from [7, page 107], we deduce that

$$\dim M_k(\Gamma_0(16))^+ = 1 - k + 2\left[\frac{k}{4}\right] + \frac{3k}{2} = 1 + \frac{k}{2} + 2\left[\frac{k}{4}\right]$$

and

$$\dim M_k(\Gamma_0(16))^- = \dim M_k(\Gamma_0(16)) - \dim M_k(\Gamma_0(16))^+ = \frac{3k}{2} - 2\left[\frac{k}{4}\right]$$

for even integers $k \geq 0$. In particular, there are modular forms f_1, f_2, f_3, g_1, g_2 of weights 2 such that $\{g_1, g_2\}$ is a basis of $M_2(\Gamma_0(16))^+$ and $\{f_1, f_2, f_3\}$ is a basis of $M_2(\Gamma_0(16))^-$.

Now recall that our goal is to find $h_1, \ldots, h_5$ such that (3.2) is a basis for $M_k(\Gamma_0(16))$. If we choose $h_1 = g_1$, $h_2 = g_2$ and $h_j = f_{j-2}$ for $j = 3, 4, 5$, we will find that the number of modular forms in (3.2) that lie in $M_k(\Gamma_0(16))^+$ is

$$1 + \frac{k}{2},$$

which cannot be a basis of $M_k(\Gamma_0(16))^+$. It turns out that if we choose $h_1 = f_1$, $h_2 = f_2$, $h_3 = f_3$, $h_4 = g_1$ and $h_5 = g_2$, then the number of modular forms in (3.2) that lie in $M_k(\Gamma_0(16))^+$ coincides with the dimension of $M_k(\Gamma_0(16))^+$ and this leads to a set of functions which could be used to construct a basis for $M_k(\Gamma_0(16))$.

Now, to construct f_1, f_2, f_3 and g_1, g_2, we recall that the Dedekind eta-function satisfies [7, Proposition 1.2.5]

$$\eta\left(-\frac{1}{z}\right) = \sqrt{\frac{z}{i}}\,\eta(z).$$

Using this, we verify that, up to some root of unity, the action of the Atkin–Lehner involution W fixes $\vartheta(2z)$ and interchanges $\vartheta_F(2z)$ and $\vartheta_M(2z)$. In fact, we find that

$$\vartheta^4(2z) \quad \text{and} \quad \vartheta_F^2(2z)\vartheta_M^2(2z) \in M_2(\Gamma_0(16))^-.$$

Also,

$$\vartheta^4(z)|_W = -4\vartheta^4(4z), \qquad \vartheta_M^4(z)|_W = -4\vartheta_F^4(4z), \qquad \vartheta_F^4(z)|_W = -\tfrac{1}{4}\vartheta_M^4(4z)$$

and, hence,

$$\vartheta^4(z) - 4\vartheta^4(4z), \vartheta_M^4(z) - 4\vartheta_F^4(4z) \in M_2(\Gamma_0(16))^+$$

and

$$\vartheta^4(z) + 4\vartheta^4(4z) \in M_2(\Gamma_0(16))^-.$$

These are the modular forms that we use to construct a basis for $M_{12}(\Gamma_0(16))$ in Theorem 3.2.

We are now ready to give a proof of Seller's congruence (1.2). In [2], Baruah and Sarmah showed that if

$$G(z) = \frac{\eta^{29}(2z)}{\eta^{20}(z)\eta^{10}(4z)} + 48\frac{\eta^5(2z)\eta^6(4z)}{\eta^{12}(z)}, \tag{3.3}$$

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then

$$q^{1/12}G(z) = \sum_{n=0}^{\infty} c\phi_4(2n)q^n. \tag{3.4}$$

Our first step is to write

$$h(z) = \eta^{25}(2z)G(z) = h_1(z) + 48h_2(z), \tag{3.5}$$

where

$$h_1(z) := \frac{\eta^{54}(2z)}{\eta^{20}(z)\eta^{10}(4z)}, \quad h_2(z) := \frac{\eta^{30}(2z)\eta^{6}(4z)}{\eta^{12}(z)}.$$

By Propositions 2.4 and 2.5, we deduce that both $h_1(z)$ and $h_2(z)$ belong to $M_{12}(\Gamma_0(16))$. By Proposition 2.2, we know that

$$h_1(z)|_{T(5)} \in M_{12}(\Gamma_0(16)) \quad \text{and} \quad h_2(z)|_{T(5)} \in M_{12}(\Gamma_0(16)).$$

By Theorem 3.2, we deduce that

$$\begin{aligned} h_1(z)|_{T(5)} = 0 \cdot A_1 &+ \frac{17145}{8}A_2 + \frac{84225}{32}A_3 + \frac{3315}{32}A_4 + \frac{47475}{32}A_5 + 110A_6 \\ &+ 0 \cdot A_7 + 0 \cdot U_1 + 0 \cdot U_2 - \frac{15825}{32}U_3 + \frac{14145}{32}U_4 - \frac{28075}{32}U_5 - 55U_6 \\ &+ 0 \cdot V_1 - 55V_2 + \frac{15825}{64}V_3 - \frac{45365}{64}V_4 + \frac{28075}{64}V_5 - \frac{55}{2}V_6 \\ &+ 0 \cdot W_1 - 55W_2 - \frac{15825}{64}W_3 - \frac{73655}{64}W_4 - \frac{28075}{64}W_5 + \frac{55}{2}W_6 \end{aligned}$$

and

$$\begin{aligned} h_2(z)|_{T(5)} = 0 \cdot A_1 &+ \frac{525}{4}A_2 + \frac{25485}{128}A_3 + \frac{1585}{64}A_4 + 90A_5 + \frac{15}{2}A_6 + 0 \cdot A_7 \\ &+ 0 \cdot U_1 + 0 \cdot U_2 - 30U_3 + \frac{1055}{32}U_4 - \frac{8495}{128}U_5 - \frac{15}{4}U_6 \\ &+ 0 \cdot V_1 - \frac{15}{4}V_2 + 15V_3 - \frac{6005}{128}V_4 + \frac{8495}{256}V_5 - \frac{15}{8}V_6 \\ &+ 0 \cdot W_1 - \frac{15}{4}W_2 - 15W_3 - \frac{10225}{128}W_4 - \frac{8495}{256}W_5 + \frac{15}{8}W_6. \end{aligned}$$

Hence,

$$h_1(z)|_{T(5)} \equiv 0 \bmod 5 \quad \text{and} \quad h_2(z)|_{T(5)} \equiv 0 \bmod 5$$

and we conclude that

$$h(z)|_{T(5)} \equiv 0 \bmod 5. \tag{3.6}$$

Next, we know that if

$$h(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

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then

$$h(z)|_{T(5)} = \sum_{n=0}^{\infty} (a(5n) + 5^{11} a(n/5)) q^n.$$

From (3.6), we deduce that

$$a(5n) \equiv 0 \bmod 5. \tag{3.7}$$

Now, by (3.5), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} a(n) q^n = \eta^{25}(2z) G(z) &= (q^2; q^2)_{\infty}^{25} \sum_{n=0}^{\infty} c\phi_4(2n) q^{n+2} \\ &\equiv (q^{50}; q^{50})_{\infty} \sum_{n=2}^{\infty} c\phi_4(2n-4) q^n \bmod 5,\end{aligned}$$

where we have used the notation

$$(a; q)_{\infty} = \prod_{k=1}^{\infty} (1 - a q^{k-1}).$$

Note that

$$\sum_{n=2}^{\infty} c\phi_4(2n-4) q^n \equiv \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p(k) a(n) q^{50k+n} \bmod 5.$$

Therefore, by (3.7),

$$\sum_{n=2}^{\infty} c\phi_4(10n-4) q^{5n} \equiv \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} p(k) a(5n) q^{50k+5n} \equiv 0 \bmod 5$$

and the proof of (1.2) is complete.

The method used in the proof of (1.2) can be adapted to yield proofs of (1.3) and (1.4). Since the proofs of these two congruences are similar, it suffices to prove (1.4). We simply state the following result needed for the proof.

Theorem 3.3. *Let*

$$\begin{aligned}B_i(z) &:= f_1^{13-i}(z) f_2^{i-1}(z), \quad 1 \le i \le 13, \\ E_j(z) &:= f_3(z) f_1^{12-j}(z) f_2^{j-1}(z), \quad 1 \le j \le 12, \\ F_j(z) &:= g_1(z) f_1^{12-j}(z) f_2^{j-1}(z), \quad 1 \le j \le 12, \\ G_j(z) &:= g_2(z) f_1^{12-j}(z) f_2^{j-1}(z), \quad 1 \le j \le 12.\end{aligned}$$

Then $B_i(z)$ $(1 \le i \le 13)$, $E_j(z)$, $F_j(z)$ and $G_j(z)$ $(1 \le j \le 12)$ form a basis of $M_{24}(\Gamma_0(16))$.

The proof of Theorem 3.3 is similar to Theorem 3.2.

To prove (1.4), we replace (3.5) by

$$H(z) = \eta^{49}(2z) G(z) = H_1(z) + 48 H_2(z),$$

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where

$$H_1(z) = \frac{\eta^{78}(2z)}{\eta^{20}(z)\eta^{10}(4z)} \quad \text{and} \quad H_2(z) = \frac{\eta^{54}(2z)\eta^{6}(4z)}{\eta^{12}(z)}.$$

Using Propositions 2.4 and 2.5, we conclude that $H_1(z), H_2(z) \in M_{24}(\Gamma_0(16))$. This implies that $H(z) \in M_{24}(\Gamma_0(16))$. Hence,

$$H(z)|_{T(7)} \in M_{24}(\Gamma_0(16))$$

and

$$\begin{aligned}
H(z)|_{T(7)} = {}& 0\cdot B_1 + 266B_2 + \frac{1366995}{4}B_3 - \frac{217306383}{8}B_4 + \frac{4034706291}{128}B_5 \\
& - \frac{3377131961}{128}B_6 + \frac{6386194899}{16}B_7 + \frac{112095809819}{128}B_8 \\
& - \frac{57905721111}{128}B_9 - \frac{58634742285}{128}B_{10} - \frac{93182859}{16}B_{11} \\
& + 456064B_{12} + 0\cdot B_{13} + 0\cdot E_1 + 0\cdot E_2 - \frac{455665}{4}E_3 \\
& + \frac{18805745}{4}E_4 - \frac{1344902097}{128}E_5 - \frac{530116111}{128}E_6 \\
& - \frac{2128731633}{16}E_7 - \frac{3241721959}{128}E_8 + \frac{19301907037}{128}E_9 \\
& - \frac{5698594195}{128}E_{10} + \frac{31060953}{16}E_{11} - 133E_{12} \\
& + 0\cdot F_1 - 133F_2 + \frac{455665}{8}F_3 - \frac{4721493}{16}F_4 \\
& + \frac{1344902097}{256}F_5 + \frac{11965914359}{256}F_6 + \frac{2128731633}{32}F_7 \\
& + \frac{1957710559}{256}F_8 - \frac{19301907037}{256}F_9 - \frac{7301480949}{256}F_{10} \\
& - \frac{31060953}{32}F_{11} + 0\cdot F_{12} + 0\cdot G_1 - 133G_2 \\
& - \frac{455665}{8}G_3 - \frac{79944473}{16}G_4 - \frac{1344902097}{256}G_5 + \frac{13026146581}{256}G_6 \\
& - \frac{2128731633}{32}G_7 + \frac{8441154477}{256}G_8 + \frac{19301907037}{256}G_9 \\
& + \frac{4095707441}{256}G_{10} + \frac{31060953}{32}G_{11} + \frac{133}{2}G_{12}.
\end{aligned}$$

Since all 49 coefficients are congruent to 0 modulo 7, we conclude that

$$H(z)|_{T(7)} \equiv 0 \bmod 7.$$

As in the proof of (1.2), we conclude that if

$$H(z) = \sum_{n=0}^{\infty} A(n)q^n,$$

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then

$$A(7n) \equiv 0 \bmod 7.$$

Since $H(z) = \eta^{49}(2z)G(z)$, we deduce that

$$c\phi_4(14n+6) \equiv 0 \bmod 7$$

and the proof of (1.4) is complete.

The method illustrated in this section can be applied to congruences satisfied by other types of partition functions. We illustrate this by giving a new proof of a congruence satisfied by the overpartition function. We recall that an overpartition of n is a nonincreasing sequence of natural numbers whose sum is n in which the first occurrence of a number may be overlined. For example, the overpartitions of 2 are

$$2, \quad \overline{2}, \quad 1+1, \quad \overline{1}+1.$$

We denote the number of overpartitions of n by $\overline{p}(n)$. In our example above, we find that $\overline{p}(2) = 4$. Using some elementary methods, Wang [17] proved that

$$\overline{p}(5n) \equiv (-1)^n r_3(n) \bmod 5, \tag{3.8}$$

where $r_3(n)$ denotes the number of representations of n as a sum of three squares. This result was first discovered by Treneer [16, (5.14)] using modular forms. Congruence (3.8) gave a satisfactory explanation of

$$\overline{p}(5(8n+7)) \equiv 0 \bmod 40, \tag{3.9}$$

which is a congruence discovered by Hirschhorn and Sellers [9]. More precisely, the appearance of $8n+7$ in (3.9) is a consequence of (3.8) and the fact that any positive integer N of the form $8n+7$ is not a sum of three squares.

To prove (3.8), we recall that the generating function for $\overline{p}(n)$ is

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{1}{\vartheta_M(z)}. \tag{3.10}$$

As in the proofs of (1.2) and (1.4), we first multiply the right-hand side of (3.10) by a suitable modular form to obtain a modular form of positive integer weight. By Propositions 2.4 and 2.5, we conclude that

$$\vartheta_M^{24}(z) = \frac{\eta^{48}(z)}{\eta^{24}(2z)} \in M_{12}(\Gamma_0(2)).$$

Recall that

$$E_{12}^{(\infty)}(q) = 691 + 16\sum_{k=1}^{\infty} \frac{k^{11}(-q)^k}{1-q^k}$$

and

$$E_{12}^{(0)}(q) = \sum_{k=1}^{\infty} \frac{k^{11}q^k}{1-q^{2k}}$$

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are the Eisenstein series of weight 12 on $\Gamma_0(2)$ associated with the cusps infinity and zero, respectively. Moreover, for $m > 1$, let

$$T_{2m}(q) = \sum_{k=1}^{\infty} \frac{k^{2m-1} q^k}{1 - q^{2k}}.$$

From [5, Lemma 2.1], we know that

$$T_4(q)T_8(q) \in M_{12}(\Gamma_0(2)) \quad \text{and} \quad T_6^2(q) \in M_{12}(\Gamma_0(2))$$

and, from [13, Theorem 7.1.4], we know that $\dim M_{12}(\Gamma_0(2)) = 4$. By using the coefficients of the q-expansions of $E_{12}^{(\infty)}(q)$, $E_{12}^{(0)}(q)$, $T_4(q)T_8(q)$ and $T_6^2(q)$, we verified that these functions are linearly independent and, thus, they form a basis of $M_{12}(\Gamma_0(2))$.

Observe that

$$\vartheta_M^{24}(z) = \frac{\vartheta_M^{25}(z)}{\vartheta_M(z)} = \vartheta_M^{25}(z) \sum_{n=0}^{\infty} \overline{p}(n) q^n. \tag{3.11}$$

Applying the Hecke operator $T(5)$ to $\vartheta_M^{24}(z)$ and using the fact that $\vartheta_M^{24}(z)|_{T(5)} \in M_{12}(\Gamma_0(2))$, we find by direct computations that

$$\begin{aligned}\vartheta_M^{24}(z)|_{T(5)} = {} & \frac{296326553600}{2073} T_4(q)T_8(q) + \frac{698055608320}{2073} T_6^2(q) \\ & - \frac{160124160}{691} E_{12}^{(0)}(q) + \frac{48828126}{691} E_{12}^{(\infty)}(q).\end{aligned}$$

Therefore,

$$\begin{aligned}\vartheta_M^{24}(z)|_{T(5)} &\equiv E_{12}^{(\infty)}(q) \bmod 5 \\ &\equiv 1 + 16 \sum_{k=1}^{\infty} \frac{k^3(-q)^k}{1 - q^k} \bmod 5 \\ &\equiv \vartheta_M^8(z) \bmod 5.\end{aligned} \tag{3.12}$$

The last equality is due to Jacobi [3, Theorem 3.5.3].

On the other hand, as applying $T(5)$ is the same as applying $U(5)$ modulo 5, from (3.11),

$$\begin{aligned}\vartheta_M^{24}(z)|_{T(5)} &\equiv \vartheta_M^5(5z) \sum_{n=0}^{\infty} \overline{p}(n) q^n|_{U(5)} \bmod 5 \\ &\equiv \vartheta_M^5(z) \sum_{n=0}^{\infty} \overline{p}(5n) q^n \bmod 5.\end{aligned} \tag{3.13}$$

Comparing (3.12) with (3.13), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}(5n) q^n \equiv \vartheta_M^3(z) \bmod 5$$

and (3.8) follows immediately.

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In a similar manner, we can also derive the congruence [18]

$$\mathrm{pod}(5n+2) \equiv 2(-1)^n r_3(8n+3) \bmod 5, \tag{3.14}$$

where $\mathrm{pod}(n)$ denotes the number of partitions where the odd parts are distinct with generating function

$$\sum_{n=0}^{\infty} \mathrm{pod}(n) q^n = \frac{1}{\psi(-q)},$$

where

$$\psi(q) = \sum_{j=0}^{\infty} q^{j(j+1)/2}.$$

We leave the proof of (3.14) as an exercise.

4. Second approach to the congruences

Let M be a positive integer and

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n$$

be a formal power series in the variable $q = e^{2\pi i z}$ with rational integer coefficients. The M-adic order of f is defined as

$$\mathrm{ord}_M(f) := \inf\{n | a(n) \not\equiv 0 \mod M\}.$$

Our second proof relies on Sturm's criterion [15] for determining whether two modular forms are congruent modulo l, where l is a prime number.

Proposition 4.1 (Sturm's criterion). *Let l be a prime and $f(z), g(z) \in M_k(\Gamma_0(N))$ with rational integer coefficients. If*

$$\mathrm{ord}_l(f(z) - g(z)) > \frac{kN}{12} \prod_{p|N} \left(1 + \frac{1}{p}\right),$$

where the product is over the distinct prime divisors of N, then $\mathrm{ord}_l(f(z) - g(z)) = \infty$ or, equivalently,

$$f(z) \equiv g(z) \bmod l.$$

Proof of (1.2). In view of (3.3) and (3.4), we shall choose appropriate integers a_1, a_2, a_3, a_4 so that both

$$\ell_1(z) = \eta^{a_1}(5z)\eta^{a_2}(10z)\eta^{a_3}(20z) \cdot \left(\frac{\eta^5(z)}{\eta(5z)}\right)^{a_4} \cdot \frac{\eta^{29}(2z)}{\eta^{20}(z)\eta^{10}(4z)}$$

and

$$\ell_2(z) = \eta^{a_1}(5z)\eta^{a_2}(10z)\eta^{a_3}(20z) \cdot \left(\frac{\eta^5(z)}{\eta(5z)}\right)^{a_4} \cdot \frac{\eta^5(2z)\eta^6(4z)}{\eta^{12}(z)}$$

belong to $M_k(\Gamma_0(20))$ for some integer k. By Proposition 2.4, a_j, $1 \le j \le 4$, have to satisfy the following conditions:

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(i) $k = 1/2(a_1 + a_2 + a_3 + 4a_4 - 1) \in \mathbf{Z}$;
(ii) $\sum_{\delta|20} \delta r_\delta \equiv 0 \bmod 24$ or $a_1 + 2a_2 + 4a_3 \equiv 10 \bmod 24$;
(iii) $\sum_{\delta|20} (20/\delta) r_\delta \equiv 0 \bmod 24$ or $4a_1 + 2a_2 + a_3 \equiv 16 \bmod 24$.

We also need to ensure that (2.1) holds. It turns out that the choice

$$(a_1, a_2, a_3, a_4) = (4, -1, 2, 3)$$

meets all these requirements. With this choice of a_j, $1 \le j \le 4$, we define

$$\ell_1(z) := \frac{\eta(5z)\eta^2(20z)\eta^{29}(2z)}{\eta^5(z)\eta^{10}(4z)\eta(10z)} \quad \text{and} \quad \ell_2(z) := \frac{\eta^3(z)\eta^5(2z)\eta^6(4z)\eta(5z)\eta^2(20z)}{\eta(10z)}.$$

By Propositions 2.4 and 2.5, we can verify that both $\ell_1(z)$ and $\ell_2(z)$ belong to $M_8(\Gamma_0(20))$.

Let

$$\ell(z) := \ell_1(z) + 48\ell_2(z).$$

Then

$$\ell(z) = \frac{\eta^4(5z)\eta^2(20z)}{\eta(10z)} \cdot \left(\frac{\eta^5(z)}{\eta(5z)}\right)^3 \cdot \left(\frac{\eta^{29}(2z)}{\eta^{20}(z)\eta^{10}(4z)} + 48\frac{\eta^5(2z)\eta^6(4z)}{\eta^{12}(z)}\right).$$

By the binomial theorem,

$$\eta^5(z) \equiv \eta(5z) \bmod 5$$

and hence

$$\ell(z) \equiv \frac{\eta^4(5z)\eta^2(20z)}{\eta(10z)} \cdot \left(\frac{\eta^{29}(2z)}{\eta^{20}(z)\eta^{10}(4z)} + 48\frac{\eta^5(2z)\eta^6(4z)}{\eta^{12}(z)}\right) \bmod 5. \tag{4.1}$$

This implies that if

$$\ell(z) = \sum_{n=0}^{\infty} a(n)q^n,$$

then, by (4.1) and (3.4), we deduce that

$$\sum_{n=0}^{\infty} a(n)q^n \equiv q^2 \frac{(q^5; q^5)_\infty^4 (q^{20}; q^{20})_\infty^2}{(q^{10}; q^{10})_\infty} \sum_{n=0}^{\infty} c\phi_4(2n)q^n \bmod 5.$$

Thus, to prove that

$$c\phi_4(10n + 6) \equiv 0 \bmod 5,$$

it suffices to show that

$$a(5n) \equiv 0 \bmod 5.$$

By Proposition 2.3, we know that

$$\ell(z)|_{U(5)} \in M_8(\Gamma_0(20)).$$

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Applying Sturm's criterion, it suffices to check that

$$a(5n) \equiv 0 \bmod 5$$

for

$$0 \leq n \leq \frac{8 \cdot 20}{12}\Big(1 + \frac{1}{2}\Big)\Big(1 + \frac{1}{5}\Big) + 1 = 25.$$

This follows directly from the Fourier series expansion of $\ell(z)$ and the proof is complete. □

Proof of (1.4). Using the same idea as in the proof of (1.2), we construct the functions

$$L_1(z) = \frac{\eta^{29}(2z)\eta^3(7z)\eta^9(14z)}{\eta^{13}(z)\eta^{10}(4z)\eta^2(28z)}, \quad L_2(z) = \frac{\eta^5(2z)\eta^6(4z)\eta^3(7z)\eta^9(14z)}{\eta^5(z)\eta^2(28z)}.$$

By Propositions 2.4 and 2.5, we verify that both $L_1(z)$ and $L_2(z)$ belong to $M_8(\Gamma_0(28))$.

Let

$$L(z) := L_1(z) + 48L_2(z) = \sum_{n=0}^{\infty} a(n)q^n.$$

Then

$$L(z) = \frac{\eta^4(7z)\eta^9(14z)}{\eta^2(28z)} \cdot \frac{\eta^7(z)}{\eta(7z)}\bigg(\frac{\eta^{29}(2z)}{\eta^{20}(z)\eta^{10}(4z)} + 48\frac{\eta^5(2z)\eta^6(4z)}{\eta^{12}(z)}\bigg).$$

By the binomial theorem, we deduce that

$$L(z) \equiv \frac{\eta^4(7z)\eta^9(14z)}{\eta^2(28z)}\bigg(\frac{\eta^{29}(2z)}{\eta^{20}(z)\eta^{10}(4z)} + 48\frac{\eta^5(2z)\eta^6(4z)}{\eta^{12}(z)}\bigg) \bmod 7.$$

This implies that

$$\sum_{n\geq 0} a(n)q^n \equiv q^4\frac{(q^7;q^7)_\infty^4(q^{14};q^{14})_\infty^9}{(q^{28};q^{28})_\infty^2}\sum_{n\geq 0} c\phi_4(2n)q^n \bmod 7.$$

Thus, to prove that

$$c\phi_4(14n + 6) \equiv 0 \bmod 7,$$

it suffices to show that

$$a(7n) \equiv 0 \bmod 7.$$

By Proposition 2.3, we know that

$$L(z)|_{U(7)} \in M_8(\Gamma_0(28)).$$

Applying Sturm's criterion, it suffices to check that

$$a(7n) \equiv 0 \bmod 7$$

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for

$$0 \le n \le \frac{8 \cdot 28}{12}\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{7}\right) + 1 = 33.$$

This follows from the Fourier series expansion of $L(z)$ and our proof is complete. □

5. Concluding remarks

In the first method, we use spaces of modular forms of level 16 with different weights to establish the congruences of the form

$$c\phi_4(m\ell n - d_\ell) \equiv 0 \bmod \ell$$

for various odd primes ℓ. In the second method, we use spaces of modular forms of weight 8 with different levels to establish such congruences. It is beneficial to understand both methods when one wishes to prove congruences associated with other partition functions.

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