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NOTE ON THE SUPPORT OF SOBOLEV FUNCTIONS

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ABSTRACT. We prove a topological restriction on the support of Sobolev functions.

THEOREM. *Let k and n be integers such that* $0 < k < n$*, and suppose that* $p > 1$ *and* $p > k - 1$*. Then the only distribution in the Sobolev space* $W_{1,p}(\mathbf{R}^n)$ *which is supported by a k-cell is the zero distribution.*

Here the Sobolev space $W_{1,p}(\mathbf{R}^n)$ is the set of all distributions u on $\mathbf{R}^n$ such that u and each of its first-order partial derivatives are represented by functions in $L_p(\mathbf{R}^n)$. A *k-cell* is a homeomorph of the closed unit ball of $\mathbf{R}^k$.

In case $p > n$ this theorem is well known; in fact, in this case the Sobolev imbedding theorem states that any distribution in $W_{1,p}(\mathbf{R}^n)$ is represented by a continuous function on $\mathbf{R}^n$ (and hence can be supported by a nowhere-dense compact set only if it is zero). On the other hand, if $1 < p \leq n$, Polking [P, Theorem 4] has shown that there is a nonzero element of $W_{1,p}(\mathbf{R}^n)$ whose support is a nowhere-dense compact set. It is possible to characterize the compact subsets of $\mathbf{R}^n$ which support nonzero Sobolev functions, by adapting certain concepts from classical potential theory (see [AH, Theorem 11.3.2] and the references given there); however, our proof of the theorem above makes no use of such results.

Our theorem is related to the theory of harmonic approximation developed by Keldysh, Deny, and Havin, and discussed in the paper of Hedberg [H]. From [H, Theorem 11.9] and the classical Runge property for harmonic functions proved by Walsh [W, p. 541] [GH, Théorème 2.1.4] we deduce that for a compact set $K \subset \mathbf{R}^n$ whose complement $\mathbf{R}^n \setminus K$ is connected, the following properties are equivalent:

(a) the only distribution $u \in W_{1,2}(\mathbf{R}^n)$ supported by K is the zero distribution.

(b) every continuous function on K can be uniformly approximated by functions harmonic on $\mathbf{R}^n$.

From this equivalence, and the theorem above when $p = 2$, we obtain a uniform approximation theorem: *if* $k = 1$ *or* $k = 2$*, and* $k < n$*, then every continuous function on a k-cell in* $\mathbf{R}^n$ *can be uniformly approximated by functions harmonic on* $\mathbf{R}^n$. In the case $k = 1$ this is included in the main theorem of [BCG]; and in the case $k = 2$, $n = 3$ this was proved by Keldysh and Lavrent'ev, and is included in the book of Landkof

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[L, Theorem 5.20]. The present paper gives in particular a simple proof of the Keldysh-Lavrent'ev theorem, but we do not know if the analogous approximation theorem holds for k-cells with $k \geq 3$.

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We turn now to the preliminaries for the proof. We will use the notation λ_d for d-dimensional Lebesgue measure, and we assume that $1 < p < \infty$. We set $\mathbf{B}_r^d(a) = \{x \in \mathbf{R}^d : \|x - a\| < r\}$ if $a \in \mathbf{R}^d$ and $r > 0$, and we define $\mathbf{B}^d = \mathbf{B}_1^d(0)$. We let $\varphi \in C_0^\infty(\mathbf{B}^n)$ be a fixed nonnegative function with $\int \varphi \, d\lambda_n = 1$.

If $u \in W_{1,p}(\mathbf{R}^n)$ has compact support and $\varepsilon > 0$, we define $u_\varepsilon = u * \varphi_\varepsilon$, where $\varphi_\varepsilon(x) \equiv \varepsilon^{-n}\varphi(x/\varepsilon)$. It is well known [EG, Section 4.2, Theorem 1] that $u_\varepsilon \in C^\infty(\mathbf{R}^n)$ for each $\varepsilon > 0$; and that the function $x \longmapsto \lim_{\varepsilon \to 0} u_\varepsilon(x)$ is defined for λ_n-a.e. point $x \in \mathbf{R}^n$, is in $L_p(\mathbf{R}^n)$, and represents the distribution u. Moreover, the arguments used to prove [GZ, Theorem 3.2] yield the following result (see also [M, Theorem 5]).

LEMMA 1. *If $u \in W_{1,p}(\mathbf{R}^n)$ has compact support, then there is a sequence $\varepsilon_j \downarrow 0$ with the following property. If M is any subspace of $\mathbf{R}^n$ of dimension ℓ, where $0 < \ell < n$ and $\ell < p$, and $M^\perp$ is the orthogonal complement of M in $\mathbf{R}^n$, then for $\lambda_{n-\ell}$-a.e. point $a \in M^\perp$, the sequence $\{u_{\varepsilon_j}\}$ is uniformly Cauchy on the ℓ-plane through a and parallel to M.*

The following lemma is analogous to a well-known result of Moore [Mo]. For each integer $k \geq 2$ we define $Y_k = P \cup Q$, where

$$P = \{(x_1, \ldots, x_k) \in \mathbf{R}^k : x_1^2 + \cdots + x_{k-1}^2 \leq 1, x_k = 0\},$$
$$Q = \{(x_1, \ldots, x_k) \in \mathbf{R}^k : x_1 = \cdots = x_{k-1} = 0, 0 \leq x_k \leq 1\}.$$

LEMMA 2. *Fix $k \geq 2$. Suppose that for each j in an uncountable set J we have a continuous injection $f_j : Y_k \longrightarrow \mathbf{R}^k$. Then the images $f_j(Y_k)$ cannot be mutually disjoint.*

To prove Lemma 2 we let

$$bP = \{(x_1, \ldots, x_k) \in \mathbf{R}^k : x_1^2 + \cdots + x_{k-1}^2 = 1, x_k = 0\},$$
$$1_k = \{(x_1, \ldots, x_k) \in \mathbf{R}^k : x_1 = \cdots = x_{k-1} = 0, x_k = 1\}.$$

For each index $j \in J$ the distance d_j from $f_j(Q)$ to $f_j(bP)$ and the distance e_j from $f_j(1_k)$ to $f_j(P)$ are strictly positive. Since J is uncountable, we may assume without loss of generality that there is a positive number ρ such that $d_j \geq \rho$ for all $j \in J$, and in fact it is no loss of generality to assume that $\rho = 2$. With this convention, we may also assume that for each j the diameter of $f_j(Q)$ is less than 1/3 (this can be achieved by replacing f_j by a function g_j, where $g_j = f_j$ on P, and $g_j(t) = f_j(T_j t)$ for t in Q, with T_j an appropriate number in $[0, 1]$). It follows that $e_j \leq 1/3$ for each j. Using again the fact that J is uncountable, we may assume that there is a positive number $\sigma \leq 1/3$ such that $e_j \geq \sigma$ for all $j \in J$. Using once more the fact that J is uncountable, we may assume that all points $f_j(1_k)$ lie in an open ball G of radius $\sigma/3$, and in fact it is no loss of generality to assume that G is centered at the origin of $\mathbf{R}^k$. Note that for each j we have $f_j(0) \in f_j(Q) \subset \mathbf{B}^k$, but the set $f_j(bP)$ does not intersect $\mathbf{B}^k$.

For each j we let W_j be the component of $\mathbf{B}^k \cap f_j(P)$ which contains $f_j(0)$. From [I, Chapter V, Theorem 4.6, p. 274] it follows that the set $\mathbf{B}^k \setminus W_j$ is not connected, and we refer to its components as the *complementary components* of W_j. We see from the preceding paragraph that the distance from the ball G to the set $f_j(P)$ is at least $\sigma/3$, and hence one of the complementary components C_j of W_j contains the ball G. We let D_j be a complementary component of W_j other than C_j.

To complete the proof of the lemma we now suppose that the images $f_j(Y_k)$ are mutually disjoint.

Let i and j be any pair of distinct indices in J. We then have $f_i(0) \in C_j$ (for otherwise the connected set $f_i(Q) \subset \mathbf{B}^k$ would contain points in distinct complementary components of W_j, and therefore would intersect W_j). It follows that the connected set W_i lies in C_j, and hence D_j must be one of the components of the set $\mathbf{B}^k \setminus (W_i \cup W_j)$. Similarly, D_i must be one of the components of the set $\mathbf{B}^k \setminus (W_i \cup W_j)$. But the sets D_i and D_j cannot be identical (since $\mathbf{B}^k \cap \partial D_i \subset W_i$ and $\mathbf{B}^k \cap \partial D_j \subset W_j$), and we conclude that D_i and D_j must be disjoint.

According to the preceding paragraph the sets $\{D_j\}_{j \in J}$ form an uncountable mutually-disjoint family of nonempty open subsets of $\mathbf{B}^k$, which is impossible. This completes the proof of Lemma 2.

To prove the theorem, we let $X \subset \mathbf{R}^n$ be the image of the closed unit ball in $\mathbf{R}^k$ under a homeomorphism H, and we let u be any element of $W_{1,p}(\mathbf{R}^n)$ which is supported in X. If $\{\varepsilon_j\}$ is the sequence given in Lemma 1, we let $\tilde{u}(x) \equiv \lim_{j \to \infty} u_{\varepsilon_j}(x)$ wherever this limit exists. Since the support of u is contained in X, and the support of φ_ε is contained in $\mathbf{B}^n_\varepsilon(0)$ for each $\varepsilon > 0$, it is clear that $\tilde{u} \equiv 0$ on $\mathbf{R}^n \setminus X$. We suppose that u is not the zero element of $W_{1,p}(\mathbf{R}^n)$, and hence the set $E = \{x \in \mathbf{R}^n : \tilde{u}(x) \text{ is defined and nonzero}\}$ has positive Lebesgue measure.

We next complete the proof in the case $k = 1$. From Lemma 1 and the hypotheses of the theorem we see that $\tilde{u}$ is continuous on λ_{n-1}-a.e. line parallel to the x_1-axis. Thus we may find a set $E_0 \subset E$ such that $\lambda_n(E \setminus E_0) = 0$, and every point $a \in E_0$ is the center of an open line segment, parallel to the x_1 axis, contained in E. The set E_0 has positive λ_n-measure, so by Fubini's theorem the projection of E_0 on the x_n-axis is uncountable. We thus see that the set $E \subset X$ contains an uncountable family of mutually disjoint homeomorphs of the open unit interval, so $H^{-1}(X) = [-1, 1]$ has the same property, and this is impossible. This completes the proof of the theorem in the case $k = 1$.

To complete the proof in case $k \geq 2$ we introduce the following notation: for each point $a \in \mathbf{R}^n$ we let $P(a)$ be the $(k-1)$-plane which is parallel to the coordinate $x_1 \cdots x_{k-1}$-plane and contains the point a, and we let $Q(a)$ be the line which is parallel to the x_k-axis and contains the point a; if $r > 0$, we set $P_r(a) = P(a) \cap \overline{\mathbf{B}^n_r(a)}$ and $Q_r(a) = Q(a) \cap \overline{\mathbf{B}^n_r(a)}$. From Lemma 1 and the hypotheses of the theorem we see that $\tilde{u}$ is continuous on λ_{n-k+1}-a.e. $(k-1)$-plane parallel to $P(0)$. Thus we may find a set $E_1 \subset E$ such that $\lambda_n(E \setminus E_1) = 0$, and for every point $a \in E_1$ there is some $r > 0$ such that $P_r(a) \subset E$. Moreover, from

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Lemma 1 and the hypotheses of the theorem we see that $\tilde{u}$ is continuous on λ_{n-1}-a.e. line parallel to the x_k axis. Thus we may find a set $E_2 \subset E_1$ such that $\lambda_n(E_1 \setminus E_2) = 0$, and for every point $a \in E_2$ there is some $s > 0$ such that $Q_s(a) \subset E$. The set E_2 has positive λ_n-measure, so by Fubini's theorem the projection of E_2 on the x_n-axis is uncountable. We conclude that the set $E \subset X$ contains an uncountable family of mutually disjoint homeomorphs of the set Y_k, so $H^{-1}(X) = \overline{\mathbf{B}^k}$ has the same property. This contradicts Lemma 2, so the theorem is proved in all cases.

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