This is a ``preproof'' accepted article for *Canadian Journal of Mathematics*
This version may be subject to change during the production process.
DOI: 10.4153/S0008414X24001184

Canad. J. Math. Vol. **00** (0), 2024 pp. 1–33
http://dx.doi.org/10.4153/xxxx

Minimum non-chromatic-choosable graphs with given chromatic number*

Jialu Zhu and Xuding Zhu

Abstract. **A graph G is called chromatic-choosable if $\chi(G) = ch(G)$. A natural problem is to determine the minimum number of vertices in a non-chromatic-choosable graph with given chromatic number. It was conjectured by Ohba, and proved by Noel, Reed and Wu that k-chromatic graphs G with $|V(G)| \le 2k+1$ are chromatic-choosable. This upper bound on $|V(G)|$ is tight. It is known that if k is even, then $G = K_{3\star(k/2+1),1\star(k/2-1)}$ and $G = K_{4,2\star(k-1)}$ are non-chromatic-choosable k-chromatic graphs with $|V(G)| = 2k+2$. Some subgraphs of these two graphs are also non-chromatic-choosable. The main result of this paper is that all other k-chromatic graphs G with $|V(G)| = 2k+2$ are chromatic-choosable. In particular, if $\chi(G)$ is odd and $|V(G)| \le 2\chi(G)+2$, then G is chromatic-choosable, which was conjectured by Noel.**

1 Introduction

A *proper colouring* of a graph G is a mapping $\phi : V(G) \to \mathbb{N}$ such that $\phi(u) \neq \phi(v)$ for every edge uv of $E(G)$. A k*-colouring* of G is a proper colouring of G using colours from $[k] = \{1, 2, \dots, k\}$. We say G is k*-colourable* if there is a k-colouring of G. The *chromatic number* $\chi(G)$ of G is the minimum k such that G is k-colourable.

List colouring is a natural generalization of classical graph colouring, introduced independently by Erdős-Rubin-Taylor [4] and Vizing [24] in 1970's. A *list assignment* of G is a mapping L which assigns to each vertex v a set $L(v)$ of permissible colours. An L*-colouring* of G is a proper colouring ϕ of G with $\phi(v) \in L(v)$ for each vertex v. We say that G is L*-colourable* if there exists an L-colouring of G, and G is k*-choosable* if G is L-colourable for any list assignment L of G with $|L(v)| \ge k$ for each vertex v. More generally, for a function $g : V(G) \to \mathbb{N}$, we say G is g*-choosable* if G is L-colourable for every list assignment L with $|L(v)| \ge g(v)$ for all $v \in V(G)$. The *choice number* $\mathrm{ch}(G)$ of G is the minimum k for which G is k-choosable.

A k-colouring of a graph G is a special case of list colouring, where each vertex v has the same list $L(v) = \{1, 2, \dots, k\}$. So k-choosable implies k-colourable. At first glance, one might expect the reverse inequality to hold as well. The smaller intersection between lists would make it easier to assign distinct colours to adjacent vertices. However, the reverse inequality is far from true. It was observed in [4] and [24] that for any integer k, there are bipartite graphs that are not k-choosable. So the difference $ch(G) - \chi(G)$ can be arbitrarily large.

AMS subject classification: 05C15.
Keywords: chromatic-choosable graphs, Ohba conjecture, Noel conjecture, near acceptable L-colouring, extremal graphs.
*The work was supported by NSFC 12371359, U20A2068.

A graph G is called *chromatic-choosable* if $\chi(G) = ch(G)$. Chromatic-choosable graphs have been studied a lot in the literature, and are related to some other difficult problems. For example, the famous Dinitz problem (see e.g. [25]) asks the following question:

"Given an $n \times n$ array of n-sets, is it always possible to choose one from each set, keeping the chosen elements distinct in every row, and distinct in every column?"

This problem can be equivalently stated as whether the line graph of $K_{n,n}$ is chromatic-choosable? This problem was solved by Galvin [5], who proved a more general result: the line graph of any bipartite multigraph is chromatic-choosable. On the other hand, Galvin's result is a special case of a more general conjecture - the list colouring conjecture: line graphs of all multigraphs are chromatic-choosable. The list colouring conjecture was posed independently by many different researchers: Albertson and Collins, Bollobás and Harris, Gupta, and Vizing (see [1, 7, 10]). It has attracted a lot of attention and remains open in general.

Ohba conjecture is another well-known conjecture about chromatic-choosable graphs. It was proved in [18] that for any graph G, $ch(G \vee K_n) = \chi(G \vee K_n)$ for sufficiently large n, where $G \vee H$ is the join of G and H, i.e., the graph obtained from the disjoint union of G and H by adding edges connecting every vertex of G to every vertex of H. This means that graphs G with $|V(G)|$ "close" to $\chi(G)$ are chromatic-choosable. A natural problem is how close should be $|V(G)|$ and $\chi(G)$ to ensure that G be chromatic-choosable. Equivalently, what is the minimum number of vertices in a non-k-choosable k-chromatic graph?

We denote by $K_{k_1\star n_1, k_2\star n_2, \ldots, k_q\star n_q}$ the complete multi-partite graph with n_i parts of size k_i, for $i = 1, 2, \ldots, q$. If $n_j = 1$, then the number n_j is omitted from the notation. It was proved in [3] that if k is an even integer, then $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ are not k-choosable. These two graphs are k-chromatic graphs with $2k + 2$ vertices. Ohba [18] conjectured that for any positive integer k, k-chromatic graphs with at most $2k + 1$ vertices are k-choosable. This conjecture has attracted considerable attention, and many partial results were proved before it was finally confirmed by Noel, Reed and Wu [17].

One approach has been to prove variants of Ohba's conjecture in which $|V(G)| \leq 2k + 1$ is replaced by $|V(G)| \leq \Phi(\chi(G))$ for some function Φ with $\Phi(k) < 2k + 1$. Ohba [18] proved such a variant with $\Phi(k) = k + \sqrt{k}$, and Reed and Sudakov [21] improved the result to $\Phi(k) = \frac{5}{3}k - \frac{4}{3}$. By using a sophisticated probabilistic method, Reed and Sudakov [20] proved that Ohba's conjecture is asymptotically true: if $|V(G)| \leq (2 - o(1))\chi(G)$, then G is chromatic-choosable.

Another approach has been to show the conjecture holds for special families of graphs. He, Li, Shen and Zheng [22] proved Ohba's conjecture for graphs G with independence number $\alpha(G) \leq 3$, by extending a result of Ohba [19] who proved that if $|V(G)| \leq 2\chi(G)$ and $\alpha(G) \leq 3$, then G is chromatic-choosable. Kostochka, Stiebitz and Woodall [13] improved this result and showed that Ohba's Conjecture holds for graphs G with $\alpha(G) \leq 5$. Also Ohba's conjecture were verified for some particular complete multipartite graphs in [9, 22, 23].

In 2015, Ohba's conjecture was finally confirmed by Noel, Reed and Wu [17]:

Theorem* 1.1** (Noel-Reed-Wu Theorem***) *Every k-colourable graph with at most $2k + 1$ vertices is k-choosable.*

Nevertheless, this is not the end of the story. More problems related to Ohba's conjecture are posed and studied. One problem is what would be the choice number of k-chromatic graphs G with $|V(G)|$ slightly bigger than $2k + 1$. This question was addressed in [16]. Another related problem is the online version of Ohba's conjecture, which was posed in [8], and has been studied in a few papers [2, 12, 14]. Some partial cases are verified and the conjecture remains open in general.

This paper explores the tightness of Ohba's conjecture. Although Ohba's conjecture is tight, $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ for even k are the only known k-chormatic graphs with $2k + 2$ vertices that are not k-choosable. In particular, Ohba's conjecture was not known to be tight for odd integer k.

Noel [15] conjectured if k is odd, then all k-chromatic graphs with $2k + 2$ vertices are k-choosable.

Observe that for a k-chromatic graph G, by adding edges between vertices of distinct colour classes, the resulting graph has the same chromatic number, and whose choice number is not decreased. Therefore in the study of minimum non-chromatic choosable graphs, it suffices to consider complete multipartite graphs.

The main result of this paper is that $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ for even k are the only non-k-choosable complete k-partite graphs with $2k + 2$ vertices.

***Theorem* 1.2** *Assume $G = (V, E)$ is a complete k-partite graph with $|V| \leq 2k + 2$, and $G \neq K_{4,2\star(k-1)}, K_{3\star(k/2+1),1\star(k/2-1)}$ when k is even, and L is a k-list assignment of G. Then G is L-colourable.*

As a consequence, Noel's conjecture is confirmed:

***Corollary* 1.3** *If k is odd, then every k-chromatic graph with at most $2k + 2$ vertices is chromatic-choosable.*

For a positive integer k, let

$$\beta(k) = \min\{|V(G)| : \chi(G) = k < ch(G)\}.$$

For an odd integer k, it can be checked that $K_{5,2\star(k-1)}$ is not k-choosable. Thus we have the following corollary.

***Corollary* 1.4** *For the function β defined above,*

$$\beta(k) = \begin{cases} 2k + 2, & \text{if } k \text{ is even}, \\ 2k + 3, & \text{if } k \text{ is odd}. \end{cases}$$

Here is a brief outline of the proof of Theorem 1.2.

Assume G is a complete k-partite graph with $2k + 2$ vertices, $G \neq K_{4,2\star(k-1)}, K_{3\star(k+1)/2,1\star(k-1)/2}$ when k is even, and L is a k-list assignment of G. Let $C_L = \bigcup_{v \in V} L(v)$. The first step is to construct a family $\mathcal{S}$ of independent sets

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

that form a partition of $V(G)$. Let $G/\mathcal{S}$ be the graph obtained from G by identifying each independent set $S \in \mathcal{S}$ into a single vertex v_S. Let $L_{\mathcal{S}}$ be the list assignment of $G/\mathcal{S}$ defined as $L_{\mathcal{S}}(v_S) = \bigcap_{u \in S} L(u)$. Build a bipartite graph $B_{\mathcal{S}}$ with partite sets $V(G/\mathcal{S})$ and C_L, with $\{v_S, c\}$ be an edge if $c \in L_{\mathcal{S}}(v_S)$. If $B_{\mathcal{S}}$ has a matching M that covers $V(G/\mathcal{S})$, then M defines an L-colouring of G, with each $S \in \mathcal{S}$ be coloured with the colour matched to v_S in M.

Assume that there is no such a matching M, and hence by Hall's Theorem, there exists a subset $X_{\mathcal{S}}$ of $V(G/\mathcal{S})$ such that $|Y_{\mathcal{S}}| < |X_{\mathcal{S}}|$, where $Y_{\mathcal{S}} = N_{B_{\mathcal{S}}}(X_{\mathcal{S}})$. By analysing the lists $L(v)$ and independent sets S in $\mathcal{S}$, the inequality $|Y_{\mathcal{S}}| < |X_{\mathcal{S}}|$ may lead to a series of inequalities and eventually lead to a contradiction (which means that no such $X_{\mathcal{S}}$ exists and hence the desired matching M exists).

Assume no contradiction is derived, and hence $X_{\mathcal{S}}$ and $Y_{\mathcal{S}}$ do exist. We choose $X_{\mathcal{S}}$ so that $|X_{\mathcal{S}}| - |Y_{\mathcal{S}}|$ is maximum. By Hall's Theorem, this implies that there is a matching M' in $B_{\mathcal{S}} - (X_{\mathcal{S}} \cup Y_{\mathcal{S}})$ that covers $V(G/\mathcal{S}) - X_{\mathcal{S}}$.

***Definition* 1.1** A *partial L-colouring* of G is an L-colouring of an induced subgraph $G[X]$ of G. Given an L-colouring ϕ of $G[X]$, L^{ϕ} is the list assignment of $G - X$ defined as $L^{\phi}(v) = L(v) - \phi(N_G(v) \cap X)$ for $v \in V(G - X)$. An L-colouring ϕ of $G[X]$ is a *good partial L-colouring* of G if the pair $(G - X, L^{\phi})$ satisfies the condition of Theorem 1.2.

The matching M' constructed above defines a *partial L-colouring ψ of G* that colours vertices in $\bigcup_{S \in V(G/\mathcal{S}) - X_{\mathcal{S}}} S$. One nice property of this partial colouring ψ is that if $\{v\} \in X_{\mathcal{S}}$ is a singleton part of $\mathcal{S}$, then $L^{\psi}(v) = L(v)$ (as $L(v) \subseteq Y_{\mathcal{S}}$). In other words some neighbours of v may have been coloured, and yet v still has the same set of permissible colours.

By using this property, we want to extend ψ to a good partial L-colouring ϕ of G, that colours a subset X of G. If this can be done, then $G - X$ has an L^{ϕ}-colouring θ, and the union $\phi \cup \theta$ would be an L-colouring of G.

For the plan above to work, the choice of the partition $\mathcal{S}$ of $V(G)$ in the first step is crucial. Indeed, Theorem 1.2 is equivalent to saying that there is a choice of $\mathcal{S}$ such that $B_{\mathcal{S}}$ has a matching M that covers $V(G/\mathcal{S})$. We usually start with a proper colouring f of G, which is not necessarily an L-colouring, but "close" to an L-colouring, and let $\mathcal{S}$ be the colour classes of f. In particular, the colouring f uses colours from C_L, and if $f(v) = c \notin L(v)$, then $f^{-1}(c) = \{v\}$ and c is contained in many lists. The concept of "near acceptable" L-colouring is defined to capture the required properties needed for the plan above to work. Near acceptable L-colouring was first used in [17]. The definitions of near acceptable L-colourings for the proofs of Noel-Reed-Wu Theorem and Theorem 1.2 are slightly different. The slight difference makes it more difficult to construct a near acceptable L-colouring of G for the proof of Theorem 1.2, while the proof of Noel-Reed-Wu Theorem is already complicated. For the proof of Theorem 1.2, before constructing a near acceptable L-colouring of G, a pseudo-L-colouring of G is constructed as an intermediate step. In many cases, we need to repeatedly modify a pseudo L-colouring until we obtain a near acceptable L-colouring.

In Section 2, we prove a sufficient condition for a complete multipartite graph G with all parts of size at most 3 to be g-choosable for a given function $g : V(G) \to \mathbb{N}$.

This will be used in later proofs. In Section 3, we fix some notation and present some basic properties of a minimum counterexample. In Section 4, we prove Theorem 1.2 for complete k-partite graphs with most parts of size at most 3. These graphs are special as there is little difference between these graphs and the critical graphs $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1(k/2-1)}$ (for even k). In Section 5, we introduce the concept of pseudo-L-colouring of G and prove some properties of such colourings. In Section 6, we define the concept of near-acceptable L-colouring and show that the existence of a near-acceptable L-colouring of G implies the existence of a proper L-colouring of G. Some sufficient conditions for the existence of near-acceptable L-colourings of G are presented in Sections 7 and 8. A final contradiction is derived in Section 9.

2 Graphs with all parts of sizes at most 3

This section proves the following lemma, which gives a sufficient condition for $g : V(G) \to \mathbb{N}$, so that G is g-choosable when all parts of G have size at most 3. This lemma is analog to Lemma 5 in [14], where a sufficient condition for G to be on-line g-choosable was given. The sufficient condition below is almost the same as that in Lemma 5 of [14], except that for two vertices u, v in a 3-part of G, the upper bounds for the sum $g(u) + g(v)$ in the two lemmas are different, and which is needed in later applications.

***Lemma* 2.1** *Let G be a complete multipartite graph with parts of size at most 3. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be a partition of the parts of G into classes such that $\mathcal{A}$ and $\mathcal{D}$ contain only parts of size 1, $\mathcal{B}$ contains all parts of size 2 and $\mathcal{C}$ contains all parts of size 3. Let k_1, k_2, k_3, d denote the cardinalities of classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ respectively. Suppose that classes $\mathcal{A}$ and $\mathcal{D}$ are ordered, i.e. $\mathcal{A} = (A_1, \ldots, A_{k_1})$ and $\mathcal{D} = (D_1, \ldots, D_d)$. If $g : V(G) \to \mathbb{N}$ is a function for which the following hold:*

$$g(v) \geq k_2 + k_3 + i, \quad \text{for all } 1 \leq i \leq k_1 \text{ and } v \in A_i \tag{a-1}$$

$$g(v) \geq k_2 + k_3, \quad \text{for all } v \in B \in \mathcal{B} \tag{b-1}$$

$$g(u) + g(v) \geq 3k_3 + 2k_2 + k_1 + d, \quad \text{for all } u, v \in B \in \mathcal{B} \tag{b-2}$$

$$g(v) \geq k_2 + k_3, \quad \text{for all } v \in C \in \mathcal{C} \tag{c-1}$$

$$g(u) + g(v) \geq 2k_3 + 2k_2 + k_1, \quad \text{for all } u, v \in C \in \mathcal{C} \tag{c-2}$$

$$\sum_{v \in C} g(v) \geq 4k_3 + 3k_2 + 2k_1 + d - 1, \quad \text{for all } C \in \mathcal{C} \tag{c-3}$$

$$g(v) \geq 2k_3 + k_2 + k_1 + i, \quad \text{for all } 1 \leq i \leq d \text{ and } v \in D_i \tag{d-1}$$

then G is g-choosable.

Proof Assume the parts of G are partitioned into $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ and g is a function satisfying the inequalities (a-1)-(d-1), and L is a list assignment with $|L(v)| = g(v)$. We shall colour an independent set S of G with a colour $c \in \bigcap_{v \in S} L(v)$. Let $G' = G - S$ and L' be the list assignment of G' defined as $L'(x) = L(x) - \{c\}$ for $x \in V(G')$ and $g'(v) = |L'(v)|$. We shall verify that the pair (G', f') satisfies the condition of Lemma 2.1, and hence G' is L'-colourable by induction hypothesis (if $|V(G)| = 1$, then the result is trivial). Together with the colouring of S with colour c, we obtain an L-colouring of G.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

In the following, we describe the choice of the independent set S. The colour c is always an arbitrary colour in $\bigcap_{v\in S} L(v)$. We describe briefly how to verify the fact that (G', g') satisfies the condition of Lemma 2.1 (the proof of Lemma 5 of [14] is similar, and contains more detailed explanations). The partition $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$ of the parts of G' and the ordering of parts in $\mathcal{A}'$ and $\mathcal{D}'$ are inherited from the partition and the ordering of the parts of G, except that one part may have some vertices coloured and remaining vertices form a part in another class. When a part from $\mathcal{B}$ or $\mathcal{C}$ has some vertices coloured and the remaining vertex form a part in $\mathcal{A}'$ or $\mathcal{D}'$, we also need to put it in a correct order. Denote by k'_1, k'_2, k'_3, d' the cardinalities of $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}'$, respectively. To verify the inequalities, it suffices to show that with g replaced by g', k_i replaced by k'_i and d replaced by d', the amount reduced on the left hand side is no more than the amount reduced on the right hand side.

The choice of S is determined in 8 cases. For $2 \leq i \leq 8$, Case i is considered only if all cases j with $j \leq i-1$ do not apply.

(1) If there exists $C \in \mathcal{B} \cup \mathcal{C}$ for which $\bigcap_{v\in C} L(v) \neq \emptyset$, then $S = C$.
Verification: For (a-1),(b-1),(c-1), (d-1), the left hand side is reduced by at most 1 (i.e., $g'(v) \geq g(v) - 1$), and the right hand side is reduced by at least 1. (For example, consider (a-1): $k'_2 + k'_3 + i = k_2 + k_3 + i - 1$). For (b-2), (c-2), the left hand side is reduced by at most 2 (i.e., $g'(u) + g'(v) \geq g(u) + g(v) - 2$), and the right hand side is reduced by at least 2. For (c-3), the left hand side is reduced by at most 3 (i.e., $\sum_{v\in C} g'(v) \geq \sum_{v\in C} g(v) - 3$), and the right hand side is reduced by at least 3.

(2) If there exist $C = \{u, v, w\} \in \mathcal{C}$ with $g(u)+g(v) = 2k_3+2k_2+k_1$, and $L(u)\cap L(v) \neq \emptyset$, then $S = \{u, v\}$.
Verification: The part $\{w\}$ of G' is the last member of $\mathcal{D}'$. Thus $k'_3 = k_3 - 1$ and $d' = d + 1$. Note that $g'(w) = g(w) \geq 4k_3 + 3k_2 + 2k_1 + d - 1 - (2k_3 + 2k_2 + k_1) = 2k_3 + k_2 + k_1 + d - 1 = 2k'_3 + k'_2 + k'_1 + d'$. The other inequalities are verified as in Case 1.

(3) If there exists $C = \{v, u, w\} \in \mathcal{C}$, $g(v) = k_2 + k_3$, $L(v) \cap L(u) \neq \emptyset$, then $S = \{u, v\}$.
Verification: The part $\{w\}$ of G' is the last member of $\mathcal{A}'$. Thus $k'_3 = k_3 - 1$ and $k'_1 = k_1 + 1$. Note that $g'(w) = g(w) \geq 2k_3+2k_2+k_1-(k_3+k_2) = k_3+k_2+k_1 = k'_3+k'_2+k'_1$. For $u, v \in C \in \mathcal{C}$, either $g(u)+g(v) \geq 2k_3+2k_2+k_1+1$ or $g'(u)+g'(v) \geq g(u)+g(v)-1$ (as Case 2 does not apply). Hence (c-2) holds for (G', g'). As Case 1 does not apply, the left hand side of (c-3) reduces by at most 2, and the right hand side is reduced by 2. Hence (c-3) holds for (G', g') as Case 1 does not apply. The other inequalities are verified as in Case 1.

(4) If there exists $C = \{v, u, w\} \in \mathcal{C}$, $g(v) = k_2 + k_3$, $L(v) \cap (L(u) \cup L(w)) = \emptyset$, then $S = \{v\}$.
Verification: In the remaining graph $G' = G - v$, the two vertices u, w are identified into a single vertex u^* with $L'(u^*) = L(u) \cap L(w)$. The set $\{u^*\}$ is the last member of $\mathcal{A}'$. So $k'_3 = k_3 - 1$, $k'_1 = k_1 + 1$. Note that

$$g(u) + g(w) \geq (4k_3 + 3k_2 + 2k_1 + d - 1) - (k_3 + k_2) = 3k_3 + 2k_2 + 2k_1 + d - 1.$$

On the other hand the total number of colours is at most $|V| - 1 = 3k_3 + 2k_2 + k_1 + d - 1$. As $L(v)$ is disjoint with $L(u) \cup L(w)$, we have $|L(u) \cup L(w)| \leq 2k_3 + k_2 + k_1 + d - 1$. Hence

$$|L'(u^*)| = |L(u) \cap L(w)| \geq k_3 + k_2 + k_1 = k'_3 + k'_2 + k'_1.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Note that for $C \in \mathcal{C}$, $\sum_{v\in C} g'(v) \geq \sum_{v\in C} g(v) - 2$, as Case 1 does not apply. Hence (c-3) holds for (G', g'). The other inequalities are verified as in Case 3.

(5) If there exists $B = \{u, v\} \in \mathcal{B}$, $g(v) = k_2 + k_3$, then $S = \{v\}$.
Verification: The part $\{u\}$ of G' is the last member of $\mathcal{D}'$. Thus $k_2' = k_2 - 1$ and $d' = d + 1$. Note that $g'(u) = g(u) \geq 3k_3 + 2k_2 + k_1 + d - (k_3 + k_2) = 2k_3 + k_2 + k_1 + d = 2k_3' + k_2' + k_1' + d'$. For $B' = \{x, y\} \in \mathcal{B}$, since Case 1 does not apply, $g'(x) + g'(y) \geq g(x) + g(y) - 1$. So (b-2) holds for (G', g'). The other inequalities are verified as in Case 4.

(6) If $k_1 \neq 0$ and $A_1 = \{v\}$, then $S = \{v\}$.
Verification: In this case, $k_1' = k_1 - 1$. As Cases 2,3,4 do not apply, (b-1), (c-1) and (c-2) were not tight for g, and hence they hold for (G', g'). Also for (a-1), the index of each member reduces by 1, and hence the right hand side reduces by 1, so it holds for (G', g'). The other inequalities are verified as in Case 5.

(7) Assume $k_3 \neq 0$ and $C = \{u, v, w\} \in \mathcal{C}$. As $|C_L| \leq |V| - 1 = 3k_3 + 2k_2 + k_1 + d - 1$, So $g(u) + g(v) + g(w) \geq 4k_3 + 3k_2 + 2k_1 + d - 1 > |C_L|$ and there is a colour c which appears in two of the three colour sets $L(u), L(v), L(w)$, say $c \in L(u) \cap L(v)$. Let $S = \{u, v\}$.
Verification: Let $\{w\}$ be the only member of $\mathcal{A}'$. Then $k_3' = k_3 - 1$ and $k_1' = k_1 + 1$, $g'(w) = g(w) \geq k_2 + k_3 = k_2' + k_3' + 1 = k_2' + k_3' + k_1'$. The other inequalities are verified as in Case 6.

(8) If $d > 0$ and $D_1 = \{v\}$, then $S = \{v\}$.
Verification: In this case, $k_3 = k_1 = 0$ and $d' = d - 1$. (b-1) is not tight for g (as Case 5 does not apply), and hence holds for (G', g'). (b-2) holds for (G', g') as the left-hand size reduces by at most 1, and the right hand side reduces by 1. For other member of $\mathcal{D}'$, its index is recued by 1, and hence (d-1) holds for (G', g'). Note that $k_1, k_3 = 0$, so the other inequalities are vacant.

Assume all the cases above do not apply. Then $G = K_{2\star k_2}$, i.e., G consists of k_2 parts of size 2, and $g(v) \geq k_2$ for each vertex v. It is well-known [4] that in this case, G is g-choosable. ■

3 Some notation and basic properties for a minimum counterexample

By a counterexample of Theorem 1.2, we mean a pair (G, L) such that G is a complete multipartite graph and L is a list assignment of G that satisfy the condition of Theorem 1.2, and G is not L-colourable. We say (G, L) is a minimal counterexample to Theorem 1.2 if (G, L) is a counterexample to Theorem 1.2 with

(1) $|V(G)|$ minimum,
(2) subject to (1), with $|C_L|$ minimum (recall that $C_L = \bigcup_{v\in V} L(v)$),

It is well-known [11] that $|C_L| < |V(G)|$. Let

$$\lambda = |V| - |C_L| > 0. \tag{3.1}$$

In the remainder of this paper, we assume that (G, L) is a minimum counterexample to Theorem 1.2. Assume G is a complete k-partite graph. By Noel-Reed-Wu Theorem, we know that k-chromatic graphs with at most $2k + 1$ vertices are k-choosable and

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

hence G has exactly $2k+2$ vertices, and

$$|C_L| \leq 2k+1. \tag{3.2}$$

A part of G of size i (respectively, at least i or at most i) is called a *i-part* (respectively, *i^+-part*, or *i^--part*). Let

$$T = \{v : \{v\} \text{ is a singleton part of } G\}.$$

Let p_i, p_i^+ and p_i^- be the number of i-parts, i^+-parts and i^--parts, respectively.

For a subset X of $V(G)$, let

$$L(X) = \bigcup_{v\in X} L(v).$$

For three vertices x, y, z of G, let

$$L(x \vee y) = L(x) \cup L(y), L(x \wedge y) = L(x) \cap L(y),$$

$$L((x \wedge y) \vee z) = (L(x) \cap L(y)) \cup L(z).$$

For $c \in C_L$ and $C' \subseteq C_L$, let

$$L^{-1}(c) = \{v : c \in L(v)\},\ L^{-1}(C') = \bigcup_{c\in C'} L^{-1}(c).$$

For a part P of G and integer i, let

$$\begin{aligned} C_{P,i} &= \{c \in C : |L^{-1}(c) \cap P| = i\}, \\ \Lambda_{P,i} &= \max\{|\bigcap_{v\in S} L(v)| : S \subseteq P, |S| = i\}. \end{aligned}$$

.

Assume $\mathcal{S}$ is a partition of $V(G)$ into a family of independent sets. Each $S \in \mathcal{S}$ is called an $\mathcal{S}$ *part*. Recall that $G/\mathcal{S}$ is the graph obtained from G by identifying each part $S \in \mathcal{S}$ into a single vertex v_S, and $L_{\mathcal{S}}$ is the list assignment of $G/\mathcal{S}$ defined as $L_{\mathcal{S}}(v_S) = \bigcap_{v\in S} L(v)$. If $S = \{v\} \in \mathcal{S}$ consists of a single vertex of G, then we denote v_S by v. In this case, $L_{\mathcal{S}}(v) = L(v)$. For the partitions $\mathcal{S}$ constructed in this paper, most parts of $\mathcal{S}$ are singletons. To define $\mathcal{S}$, it suffices to list its non-singleton parts.

Recall that $B_{\mathcal{S}}$ is the bipartite graph with partite sets $V(G/\mathcal{S})$ and C_L, in which $\{v_S, c\}$ is an edge if and only if $c \in L_{\mathcal{S}}(v_S)$. A matching M in $B_{\mathcal{S}}$ covering $V(G/\mathcal{S})$ induces an $L_{\mathcal{S}}$-colouring of $G/\mathcal{S}$, which in turn induces an L-colouring of G. Since G is not L-colourable, no such matching M exists. By Hall's Theorem, there is a subset $X_{\mathcal{S}}$ of $V(G/\mathcal{S})$ such that $|X_{\mathcal{S}}| > |N_{B_{\mathcal{S}}}(X_{\mathcal{S}})|$.

We denote by $X_{\mathcal{S}}$ a subset of $V(G/\mathcal{S})$ for which $|X_{\mathcal{S}}| - |N_{B_{\mathcal{S}}}(X_{\mathcal{S}})|$ is maximum. Let

$$Y_{\mathcal{S}} = N_{B_{\mathcal{S}}}(X_{\mathcal{S}}) = \bigcup_{v_S\in X_{\mathcal{S}}} L_{\mathcal{S}}(v_S).$$

The choice of $X_{\mathcal{S}}$ implies that there is a matching $M_{\mathcal{S}}$ in $B_{\mathcal{S}} - (X_{\mathcal{S}} \cup Y_{\mathcal{S}})$ that covers all vertices in $V(G/\mathcal{S}) - X_{\mathcal{S}}$. The matching $M_{\mathcal{S}}$ defines a partial colouring $\psi_{\mathcal{S}}$ of $G[\bigcup_{S\in\mathcal{S}-X_{\mathcal{S}}} S]$ with colours from $C_L - Y_{\mathcal{S}}$.

These notation will be used throughout the whole paper.

***Observation* 3.1** *The following easy facts will be used often in the argument.*

(1) *There is an injective mapping $\phi : C \longrightarrow V$ such that $c \in L(\phi(c))$.*
(2) *If f is a proper colouring of G, then there is a surjective proper colouring $g : V \longrightarrow C$ such that for every vertex v, $g(v) \in L(v)$ or $g(v) = f(v)$.*
(3) *No two vertices in the same part of G have the same list, and no colour is contained only in the lists of vertices in a same part.*
(4) *$G \neq K_{4,2\star(k-1)}$ for any k and $|T| \geq 1$.*

Proof (1) is well-known (Corollary 1.8 in [17]) and also easy to verify (use the minimality of $|C_L|$).

(2) was proved in Proposition 1.13 in [17].

(3) If u, v are in the same part and $L(u) = L(v)$, then By Noel-Reed-Wu Theorem, there is a proper L-colouring f of $G - u$, which extends to a proper L-colouring of G by letting $f(u) = f(v)$.

If there is a colour c such that $L^{-1}(c) \subseteq P_i$ for some part P_i of G, then by Noel-reed-Wu Theorem, $G - L^{-1}(c)$ has an L-colouring f, which extends to an L-colouring of G by colouring vertices in $L^{-1}(c)$ with colour c.

(4) It was proved in [3] that $K_{4,2\star(k-1)}$ is not k-choosable if and only if k is even. By our assumption, $G \neq K_{4,2\star(k-1)}$ for even k. Thus $G \neq K_{4,2\star(k-1)}$ for any k. It was proved in [6] that $G = K_{3\star 2,2\star(k-2)}$ is k-choosable. Using the fact that $|V(G)| = 2k+2$, it is easy to see that $|T| \geq 1$. ■

***Lemma* 3.2** *If P is a 2^+-part of G, then $\bigcap_{v \in P} L(v) = \emptyset$. Consequently for each colour $c \in C$, $|L^{-1}(c)| \leq k + p_1 + 2$.*

Proof Assume the lemma is not true. We choose such a part P of maximum size, and colour vertices in P by a common colour c. Let $L'(v) = L(v) - \{c\}$ for $v \in V(G) - P$. If $|P| \geq 3$, then L' and $G - P$ satisfies the condition of Noel-Reed-Wu Theorem and hence $G - P$ has an L'-colouring.

Assume $|P| = 2$. By (4) of Observation 3.1, $G - P \neq K_{4,2\star(k-2)}$. If $G - P \neq K_{3\star(q+1),1\star(q-1)}$, then by the minimality of G, $G - P$ has an L'-colouring. If $G - P = K_{3\star(q+1),1\star(q-1)}$, then since each 3-part P has at most two vertices v for which $c \in L(v)$, it is straightforward to verify that $G - P$ and L' satisfy the condition of Lemma 2.1. Hence $G - P$ has an L'-colouring.

For any colour $c \in C$, each 2^+-part contains a vertex $v \notin L^{-1}(c)$. So

$$|L^{-1}(c)| \leq |V(G)| - p_2^+ = 2k + 2 - (k - p_1) = k + p_1 + 2.$$

This completes the proof of Lemma 3.2. ■

4 Graphs with most parts of size at most 3

In this section, we consider complete k-partite graphs whose most parts are 3^--parts.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Let

$$\mathcal{G}_1 = \{K_{5,3\star(q-1),2\star(k-2q),1\star q} : k \geq 2q \geq 2\},$$
$$\mathcal{G}_2 = \{K_{4\star a,3\star(q-a),2\star b,1\star(k-q-b)} : a \leq 2, a \leq q, b \geq 0, q+b \leq k, a+2q+b = k+2.\}$$

Theorem 4.1 *$G \notin \mathcal{G}_1 \cup \mathcal{G}_2$.*

We may assume that $k \geq 8$, as for $k \leq 7$, we can check directly the graphs in $\mathcal{G}_1, \mathcal{G}_2$ are k-choosable.

Assume $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. We order the parts of G as $P_1, P_2, \dots, P_k$ so that

- if $G \in \mathcal{G}_1$, then P_1 is the 5-part and $P_2, P_3, \dots, P_q$ are 3-parts with $\Lambda_{P_2,2} \geq \Lambda_{P_3,2} \geq \dots \geq \Lambda_{P_q,2}$;
- if $G \in \mathcal{G}_2$, then the first a parts are the 4-parts of G, and $P_{a+1}, P_{a+2}, \dots, P_q$ are 3-parts with $\Lambda_{P_2,2} \geq \Lambda_{P_3,2} \geq \dots \geq \Lambda_{P_q,2}$. If $a = 2$, then order P_1, P_2 so that $\Lambda_{P_1,3} \geq \Lambda_{P_2,3}$.

Let

$$i_0 = \max\{j : \Lambda_{P_j,2} \geq j\}.$$

For a 3-part P of G, we have $3k \leq \sum_{v\in P} |L(v)| \leq |C_L| + |C_{P,2}| \leq 2k+1+|C_{P,2}|$. So $|C_{P,2}| \geq k-1$. As P has three 2-subsets, we have $\Lambda_{P,2} \geq (k-1)/3 \geq 2$.

Claim 4.2 *If $G \in \mathcal{G}_1$, then $C_{P_1,4} = \emptyset$ and $C_{P_1,3} \neq \emptyset$.*

Proof If $c \in C_{P_1,4}$, then we colour vertices in $L^{-1}(c) \cap P_1$ with colour c, and let $L'(v) = L(v) - \{c\}$ for $v \in G - (L^{-1}(c) \cap P_1)$. It is easy to verify that $G' = G - (L^{-1}(c) \cap P_1)$ and L' satisfy the condition of Lemma 2.1 (with $P_1 - L^{-1}(c)$ being the last part in $\mathcal{A}$, and with $\mathcal{D} = \emptyset$), and hence G' is L'-colourable, and G is L-colourable, a contradiction.

If $C_{P_1,3} = \emptyset$, then each colour $c \in C_L$ is contained in $L(v)$ for at most two vertices $v \in P_1$. Hence $2(2k+1) \geq 2|C_L| \geq \sum_{v\in P_1} |L(v)| = 5k$, which implies that $k \leq 2$, a contradiction. ∎

Claim 4.3 *$G \neq K_{5,2\star(k-2),1}$.*

Proof If $G = K_{5,2\star(k-2),1}$, then fix a 3-subset S_1 of P_1 with $\bigcap_{v\in S_1} L(v) \neq \emptyset$. Let $\mathcal{S}$ be the partition of $0V(G)$ with one non-singleton part S_1. Then $|V(G/\mathcal{S})| = 2k$ and hence $|X_{\mathcal{S}}| \leq 2k$ and $|Y_{\mathcal{S}}| \leq 2k-1$. By Lemma 3.2, $|X_{\mathcal{S}} \cap P| \leq 1$ for any 2-part P. So $|X_{\mathcal{S}}| \leq k+2$ and $|Y_{\mathcal{S}}| \leq k+1$. On the other hand, $|X_{\mathcal{S}}| \geq 2$ and hence $v \in X_{\mathcal{S}}$ for some vertex v with $|L_{\mathcal{S}}(v)| \geq k$ and hence $|Y_{\mathcal{S}}| \geq k$ and $|X_{\mathcal{S}}| \geq k+1$, and hence $|X_{\mathcal{S}} \cap P'_1| \geq 2$. This in turn implies that $|Y_{\mathcal{S}}| = k+1$ and hence $|X_{\mathcal{S}}| = k+2$. Then $P'_1 \subseteq X_{\mathcal{S}}$ and $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(P'_1)| \geq k+2 = |X_{\mathcal{S}}|$ (by Claim 4.2), a contradiction. ∎

It follows from Observation 3.1 that $G \neq K_{4,2\star(k-1)}$ for any k. As $G \neq K_{5,2\star(k-2),1}$, G has at least two 3^+-parts. Therefore

$$i_0 \geq 2.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

For $i = 1, 2, \dots, i_0$, we shall choose a subset S_i of P_i of size 2 or 3, and let $\mathcal{S}$ be the partition of $V(G)$ with non-singleton parts $\{S_1, S_2, \dots, S_{i_0}\}$. The rules for choosing the sets S_i will be given later.

For simplicity, in the graph $G/\mathcal{S}$, for $i = 1, 2, \dots, i_0$, we denote v_{S_i} by z_i, and let

$$Z = \{z_1, z_2, \dots, z_{i_0}\}.$$

We denote by P'_i the part of $G/\mathcal{S}$, where for $1 \leq i \leq i_0$, P'_i is obtained from the part P_i by identifying S_i into a new vertex z_i, and for $i_0 + 1 \leq i \leq k$, $P'_i = P_i$.

As $i_0 \geq 2$, we have $|V(G/\mathcal{S})| \leq 2k$, and hence

$$|X_{\mathcal{S}}| \leq 2k, \;\; |Y_{\mathcal{S}}| \leq 2k - 1. \tag{4.1}$$

We shall prove further upper and lower bounds for $|X_{\mathcal{S}}|$ and $|Y_{\mathcal{S}}|$ that eventually lead to a contradiction.

The details are delicate and a little complicated, which is perhaps unavoidable, as $K_{4,2\star(k-1)}$ and $K_{3\star(k/2+1),1\star(k/2-1)}$ (for even integer k) are very close to graphs in $\mathcal{G}_1 \cup \mathcal{G}_2$, and they are not k-choosable. We divide the proofs for $G \notin \mathcal{G}_1$ and $G \notin \mathcal{G}_2$ into two subsections.

4.1 $G \notin \mathcal{G}_1$

Assume to the contrary that $G \in \mathcal{G}_1$.

The subsets S_i for $i = 1, 2, \dots, i_0$ are chosen as follows:

(1) S_1 is a 3-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,3}$.
(2) For $2 \leq i \leq i_0$, S_i is a 2-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,2}$.

Assume for $i = 2, 3, \dots, i_0$, $P_i = \{u_i, v_i, w_i\}$ and $S_i = \{u_i, v_i\}$.

Since $|P_1 - S_1| = 2$, by (3) of Observation 3.1, $|L(P_1 - S_1)| \geq k+1$. As $(\bigcap_{v \in S_1} L(v)) \cap L(P_1 - S_1) = \emptyset$, we know that

$$|L_{\mathcal{S}}(P'_1)| \geq k + 2. \tag{4.2}$$

It follows from the definition of $\mathcal{S}$ that for $i = 1, 2, \dots, i_0$, $|L_{\mathcal{S}}(z_i)| \geq i_0$.

If $X_{\mathcal{S}} \subseteq Z$ and $z_i \in X_{\mathcal{S}}$ for some $i \leq i_0$, then we have $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_i)| \geq i_0 \geq |X_{\mathcal{S}}|$, a contradiction. Thus $X_{\mathcal{S}} - Z \neq \emptyset$. Let $v \in X_{\mathcal{S}} - Z$. Then

$$|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(v)| = |L(v)| \geq k, \;\; |X_{\mathcal{S}}| \geq k + 1.$$

This implies that $|X_{\mathcal{S}} \cap P'_i| \geq 2$ for some i. As $|L_{\mathcal{S}}(A)| \geq k + 1$ for any 2-subset A of P'_i (for any i), we have

$$|Y_{\mathcal{S}}| \geq k + 1, \;\; |X_{\mathcal{S}}| \geq k + 2. \tag{4.3}$$

***Claim* 4.4** *$|Y_{\mathcal{S}}| \geq k + i_0$ and hence $|X_{\mathcal{S}}| \geq k + i_0 + 1$.*

Proof If there is an index $i_0 + 1 \leq i \leq q$ such that $u, v \in X_{\mathcal{S}} \cap P'_i$, then $|Y_{\mathcal{S}}| \geq |L(u \vee v)| \geq 2k - |L(u \wedge v)| \geq 2k - i_0 > k + i_0$ (as $i_0 \leq q - 1 < k/2$) and we are done.

Assume $|X_{\mathcal{S}} \cap P'_i| \leq 1$ for any $i_0 + 1 \leq i \leq q$. If $\{z_i, w_i\} \subseteq X_{\mathcal{S}}$ for some $i \geq 2$, then $|Y_{\mathcal{S}}| \geq |L(w_i)| + |L_S(z_i)| + \geq k + i_0$, and we are done. Otherwise, $|X_{\mathcal{S}}| \geq k + 2$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

(by (4.3)) implies that $P_1' \subseteq X_{\mathcal{S}}$ and $|X_{\mathcal{S}}| = k+2$. By (4.2), $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(P_1')| \geq k+2$, a contradiction. ■

Claim 4.5 *If* $|Y_{\mathcal{S}}| = k + i_0$*, then* $\Lambda_{P_i,2} = i_0$ *for* $i = 2, 3, \ldots, i_0$ *and there is an index* $2 \leq i \leq i_0$ *such that* P_i *has a 2-subset* S *with* $|\bigcap_{v\in S} L(v)| \geq 2$ *and* $\bigcap_{v\in S} L(v) \cup L(P_i - S) \not\subseteq Y_{\mathcal{S}}$.

Proof Assume $|Y_{\mathcal{S}}| = k + i_0$. Then $|X_{\mathcal{S}}| \geq k + i_0 + 1$.

By the argument in the proof of Claim 4.4, for any index $i > i_0$, $|X_{\mathcal{S}} \cap P_i| \leq 1$. This implies that $|X_{\mathcal{S}}| \leq k + i_0 + 1$, and hence $|X_{\mathcal{S}}| = k + i_0 + 1$ and $P_i' \subseteq X_{\mathcal{S}}$ for $i = 1, 2, \ldots, i_0$. As $|L_{\mathcal{S}}(P_i')| \geq k + i_0$ for $2 \leq i \leq i_0$, we conclude that for $2 \leq i \leq i_0$, $Y_{\mathcal{S}} = L_{\mathcal{S}}(P_i')$ and $\Lambda_{P_i,2} = i_0$.

We shall find an index $2 \leq i \leq i_0$, a 2-subset S of P_i with $|\bigcap_{v\in S} L(v)| \geq 2$ and $\bigcap_{v\in S} L(v) \cup L(P_i - S) \not\subseteq Y_{\mathcal{S}}$.

Assume first that there is an index $2 \leq i \leq i_0$ such that $L(P_i) \not\subseteq Y_{\mathcal{S}}$.

As $L(w_i) \subseteq Y_{\mathcal{S}}$, we may assume that there is a colour $c \in L(u_i) - Y_{\mathcal{S}}$. If $|L(v_i \wedge w_i)| \geq 2$, then let $S = \{v_i, w_i\}$, we are done.

Assume $|L(v_i \wedge w_i)| \leq 1$. This implies that $|L(v_i \vee w_i)| \geq 2k - 1 > k + i_0$. So there is a colour $c' \in L(v_i) - Y_{\mathcal{S}}$. If $|L(u_i \wedge w_i)| \geq 2$, then let $S = \{u_i, w_i\}$, we are done. Assume $|L(u_i \wedge w_i)| \leq 1$. Hence

$$2+i_0 \geq |L(u_i \wedge w_i)| + |L(v_i \wedge w_i)| + |L(u_i \wedge v_i)| = |C_{P_i,2}| \geq 3k - |L(P_i)| \geq 3k - (2k+1).$$

This implies that $k - 3 \leq i_0 \leq q \leq k/2$, contrary to our assumption that $k \geq 8$.

Assume next that $L(P_i) = Y_{\mathcal{S}}$ for $2 \leq i \leq i_0$. As each colour in $L(P_i)$ is contained in at most two lists of vertices of P_i, we have $2(k + i_0) \geq 3k$, i.e., $i_0 \geq k/2$. Hence $i_0 = k/2 = q$ and $G = K_{5,3\star(q-1),1\star q}$.

For each singleton part $\{v\}$ of G, we have $v \in X_{\mathcal{S}}$ and hence $L(v) \subseteq Y_{\mathcal{S}}$ for each singleton part $\{v\}$. Thus $L(\bigcup_{i=2}^{k} P_i) = Y_{\mathcal{S}}$.

Since $C_{P_1,4} = \emptyset$, we have $|L(P_1)| \geq 5k/3 > k + i_0 = |Y_{\mathcal{S}}|$. Let $c \in L(P_1) - Y_{\mathcal{S}}$. Then c is contained in the lists of vertices in P_1 only, in contradiction to Observation 3.1. ■

If $|Y_{\mathcal{S}}| = k + i_0$, then as $\Lambda_{P_i,2} = i_0$ for $2 \leq i \leq i_0$, we may assume that $S_2' = \{u_2, w_2\}$ is a 2-subset of P_2 for which $|\bigcap_{v\in S_2'} L(v)| \geq 2$ and $\bigcap_{v\in S_2'} L(v) \cup L(P_2 - S_2') \not\subseteq Y_{\mathcal{S}}$.

We let $\mathcal{S}'$ be the partition of $V(G)$ whose non-singleton parts are obtained from that of $\mathcal{S}$ by replacing S_2 with S_2', i.e., $\mathcal{S}' = \{S_1, S_2', S_3, \ldots, S_{i_0}\}$.

Instead of $G/\mathcal{S}$, we consider the graph $G/\mathcal{S}'$. We still have (4.3), i.e.,

$$|Y_{\mathcal{S}'}| \geq k+1, \quad |X_{\mathcal{S}'}| \geq k+2.$$

Then analog to the proof of Claim 4.4, we can show that

$$|Y_{\mathcal{S}'}| \geq k + i_0 + 1, \quad |X_{\mathcal{S}'}| \geq k + i_0 + 2.$$

Let $\mathcal{S}'' = \mathcal{S}$ if $|Y_{\mathcal{S}}| \geq k + i_0 + 1$, and $\mathcal{S}'' = \mathcal{S}'$ if $|Y_{\mathcal{S}}| = k + i_0$. Then

$$|Y_{\mathcal{S}''}| \geq k + i_0 + 1, \quad |X_{\mathcal{S}''}| \geq k + i_0 + 2.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

For simplicity, we assume that $\mathcal{S}'' = \mathcal{S}$. Then $|X_\mathcal{S}| \geq k + i_0 + 2$ implies that $|X_\mathcal{S} \cap P_i| \geq 2$ for some $i \geq i_0 + 1$. Assume $\{u, v\} \subseteq X \cap P_i$ for some $i \geq i_0 + 1$. Then

$$|Y_\mathcal{S}| \geq |L(u \vee v)| = 2k - |L(u \wedge v)| \geq 2k - i_0. \tag{4.4}$$

Since $X_\mathcal{S}$ contains at most one vertex of any 2-part, we have

$$|X_\mathcal{S}| \leq k + 2q + 1 - i_0.$$

If for some $i \geq i_0 + 1$, $P_i = \{u_i, v_i, w_i\} \subseteq X_\mathcal{S}$, then

$$\begin{aligned} |Y_\mathcal{S}| \geq |L(P_i)| &= |L(u_i)| + |L(v_i)| + |L(w_i)| \\ &\quad - (|L(u_i \wedge v_i)| + |L(u_i) \cap L(w_i)| + |L(v_i) \cap L(w_i)|) \\ &\geq 3k - 3i_0. \end{aligned}$$

Hence $k + 2q + 1 - i_0 \geq |X_\mathcal{S}| \geq 3k - 3i_0 + 1$, which implies that $k \leq q + i_0 \leq 2q - 1$, in contrary to $k \geq 2q$.

Thus $|X_\mathcal{S} \cap P'_i| \leq 2$ for $i \geq i_0 + 1$. This implies that $|X_\mathcal{S}| \leq k + q + 1$.

On the other hand, $|Y_\mathcal{S}| \geq 2k - i_0$ (by (4.4)) implies that $|X_\mathcal{S}| \geq 2k - i_0 + 1$. Hence $k + q + 1 \geq |X_\mathcal{S}| \geq 2k - i_0 + 1$, which implies that $k \leq i_0 + q \leq 2q - 1$, in contrary to $k \geq 2q$.

This completes the proof that $G \notin \mathcal{G}_1$.

4.2 $G \notin \mathcal{G}_2$

Assume to the contrary that $G \in \mathcal{G}_2$.

***Claim* 4.6** *Assume P is a 4-part of G and $\Lambda_{P,3} \leq 1$. Then $\Lambda_{P,2} \geq 2$. If $|\Lambda_{P,2}| \geq 3$, then for any 2-subset S of P with $|\bigcap_{v \in S} L(v)| = \Lambda_{P,2}$, for any $x \in P - S$,*

$$|\bigcap_{v \in S} L(v) \cup L(x)| \geq k + 2.$$

If $\Lambda_{P,2} = 2$, then there exists a 2-subset S of P such that $|\bigcap_{v \in S} L(v) \cap C_{P,2}| = 2$, and hence for any $x \in P - S$, $|\bigcap_{v \in S} L(v) \cup L(x)| \geq k + 2$.

Proof Assume P is a 4-part of G and $\Lambda_{P,3} \leq 1$. Assume $\Lambda_{P,2} \geq 3$ and S is a 2-subset of P with $|\bigcap_{v \in S} L(v)| = \Lambda_{P,2}$. Then for any $x \in P - S$, since $|\bigcap_{v \in S} L(v) \cap L(x)| \leq \Lambda_{P,3} \leq 1$, we have

$$|\bigcap_{v \in S} L(v) \cup L(x)| = |\bigcap_{v \in S} L(v)| + |L(x)| - |\bigcap_{v \in S} L(v) \cap L(x)| \geq \Lambda_{P,2} + k - 1 \geq k + 2.$$

Assume $\Lambda_{P,2} \leq 2$. As P has four 3-subsets, we have $|C_{P,3}| \leq 4$. As $\sum_{i=1}^{3} i|C_{P,i}| = \sum_{v \in P} |L(v)| \geq 4k$ and $\sum_{i=1}^{3} |C_{P,i}| \leq |C_L| \leq 2k + 1$, it follows that $|C_{P,2}| \geq 2k - 9 \geq 7$ (as $k \geq 8$). Since P has six 2-subsets, there exists a 2-subset S of P such that $|\bigcap_{v \in S} L(v) \cap C_{P,2}| \geq 2$. Hence $\Lambda_{P,2} \geq 2$ and therefore $\Lambda_{P,2} = 2$. Moreover, there exists a 2-subset S of P such that $|\bigcap_{v \in S} L(v) \cap C_{P,2}| = 2$. For any $x \in P - S$,

$$|\bigcap_{v \in S} L(v) \cup L(x)| \geq |\bigcap_{v \in S} L(v) \cap C_{P,2}| + |L(x)| \geq 2 + k.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

■

***Definition* 4.1** For $i = 1, 2, \ldots, i_0$, we choose a subset S_i of P_i of size 2 or 3 as follows:

(1) For $a + 1 \le i \le i_0$, S_i is a 2-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,2}$.
(2) If $a = 1$ and $\Lambda_{P_1,3} > 0$, then let S_1 be a 3-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,3}$. Otherwise, let S_1 be a 2-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,2}$.
(3) Assume $a = 2$.

(i) If $\Lambda_{P_2,3} \ge 2$, then for $i = 1, 2$, let S_i be a 3-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,3}$.
(ii) If $\Lambda_{P_1,3} > 0$ and $\Lambda_{P_2,3} \le 1$, then let S_1 be a 3-subset of P_1 with $|\bigcap_{v \in S_1} L(v)| = \Lambda_{P_1,3}$, and let S_2 be a 2-subset of P_2 such that

(A) $|\bigcap_{v \in S_2} L(v)| = \Lambda_{P,2}$,
(B) $|\bigcap_{v \in S_2} L(v) \cup L(x)| \ge k + 2$ for any $x \in P_2 - S_2$,
(C) Subject to (A) and (B), $|L_{\mathcal{S}}(P'_1) \bigcup L(P_2 - S_2)|$ is maximum.

(iii) If $\Lambda_{P_1,3} = 0$, then for $i = 1, 2$, let S_i be a 2-subset of P_i with $|\bigcap_{v \in S_i} L(v)| = \Lambda_{P_i,2}$, such that $|\bigcap_{v \in S_i} L(v) \cup L(x)| \ge k + 2$ for any $x \in P_i - S_i$ and subject to this condition, $|L_{\mathcal{S}}(P'_1) \bigcup L_{\mathcal{S}}(P'_2)|$ is maximum.

The existence of the 2-subset S in (ii) and (iii) has been proved in Claim 4.6.
It follows from the definition of $\mathcal{S}$ that for $i = 1, 2, \ldots, i_0$, $|L_{\mathcal{S}}(z_i)| \ge i$.
The same argument as in the previous subsection shows that

$$|Y_{\mathcal{S}}| \ge k + 1, \quad |X_{\mathcal{S}}| \ge k + 2. \tag{4.5}$$

***Claim* 4.7** *If $|P_i| = 4$, then $|X_{\mathcal{S}} \cap P'_i| \le 2$.*

Proof Assume $P_i = \{u_i, v_i, x_i, y_i\}$. Then $2 \le |P'_i| \le 3$. If $|P'_i| = 2$, then the conclusion is trivial.

Assume $|P'_i| = 3$ and assume to the contrary of the claim that $P'_i = \{z_i, x_i, y_i\} \subseteq X_{\mathcal{S}}$, where z_i is the identification of u_i and v_i. In this case, $L_{\mathcal{S}}(z_i) = L(u_i \wedge v_i)$ and $L_{\mathcal{S}}(x_i) = L(x_i)$, $L_{\mathcal{S}}(y_i) = L(y_i)$.

If $\Lambda_{P_i,3} = 0$, then $L_{\mathcal{S}}(z_i) \cap L(x_i \vee y_i) = \emptyset$. By the choice of S_i, $|L(x_i \wedge y_i)| \le |L_{\mathcal{S}}(z_i)|$ and hence $|L(x_i \vee y_i)| \ge 2k - |L_{\mathcal{S}}(z_i)|$. Therefore $|Y_{\mathcal{S}}| \ge |L_{\mathcal{S}}(z_i)| + |L(x_i \vee y_i)| \ge 2k$, in contrary to (4.1).

If $\Lambda_{P_i,3} > 0$, then by the choice of S_i, we know that $i = a = 2$, $\Lambda_{P_2,3} = 1$ and $|S_1| = 3$, $|P'_1| = 2$. Therefore $|X_{\mathcal{S}}| \le |V(G/\mathcal{S})| \le 2k - 1$, and $|Y_{\mathcal{S}}| \le 2k - 2$.

Assume $S_2 = \{u_2, v_2\}$. By the choice of S_2 (see Claim 4.6), $|L_{\mathcal{S}}(z_2)| \ge |L(x_i \wedge y_i)|$ and $|L_{\mathcal{S}}(z_i) \cap L(x_i \vee y_i)| \le |L_{\mathcal{S}}(z_i) \cap L(x_i)| + |L_{\mathcal{S}}(z_i) \cap L(y_i)| \le 2\Lambda_{P_i,3} = 2$. Hence $|Y_{\mathcal{S}}| \ge |L_{\mathcal{S}}(z_i)| + |L(x_i \vee y_i)| - 2 \ge 2k - 2$. So $|X_{\mathcal{S}}| = 2k - 1$ and $|Y_{\mathcal{S}}| = 2k - 2$, and hence $X_{\mathcal{S}} = V(G/\mathcal{S})$. This implies that $i_0 = 2$.

By Lemma 3.2, G has no 2-part. Assume $P_3 = \{u_3, v_3, w_3\}$. Then since $\Lambda_{P_3,2} \le 2$, and P_3 has three 2-subsets, we know that $|C_{P_3,2}| \le 6$. Therefore

$$3k \le |L(u_3)| + |L(v_3)| + |L(w_3)| = 2|C_{P_3,2}| + |C_{P_3,1}| \le |C_L| + |C_{P_3,2}| \le 2k + 6,$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

a contradiction (as $k \geq 8$). ■

Since $3p_3^+ + 2p_2 + p_1 \leq 2k + 2 = 2(p_1 + p_2 + p_3^+) + 2$ and $G \neq K_{3\star(k/2+1),1\star(k/2-1)}$ (i.e., $k \neq 2q - 2$), we have

$$G \in \{K_{4,3\star(q-1),1\star(q-1)}, K_{3\star q,2,1\star(q-2)}\} \text{ or } k \geq 2q.$$

Note that $X_{\mathcal{S}}$ contains at most one vertex of any 2-part. Combining with Claim 4.7, we have

$$|X_{\mathcal{S}}| \leq k + 2q - i_0.$$

***Claim* 4.8** *For any $i \geq i_0 + 1$, $|X_{\mathcal{S}} \cap P_i| \leq 1$.*

Proof If $i \geq q+1$, then P_i is 2^--part and hence $|P_i \cap X_{\mathcal{S}}| \leq 1$ (by Lemma 3.2 and (4.1)). Assume $i_0 + 1 \leq i \leq q$.

First we prove that $|X_{\mathcal{S}} \cap P_i| \leq 2$. Assume to the contrary that $|X_{\mathcal{S}} \cap P_i| = 3$ for some $i \geq i_0 + 1$. Assume $P_i = \{u_i, v_i, w_i\}$. Then

$$\begin{aligned} |Y_{\mathcal{S}}| \geq |L(P_i)| &= |L(u_i)| + |L(v_i)| + |L(w_i)| \\ &\quad - (|L(u_i \wedge v_i)| + |L(u_i) \cap L(w_i)| + |L(v_i) \cap L(w_i)|) \\ &\geq 3k - 3i_0. \end{aligned}$$

Hence $k + 2q - i_0 \geq |X_{\mathcal{S}}| \geq 3k - 3i_0 + 1$, which implies that $2k + 1 \leq 2q + 2i_0 \leq 4q$. As $k \geq 2q - 1$, we have $k = 2q - 1$. Hence $q = i_0$, in contrary to $i_0 + 1 \leq i \leq q$.

Since $|X_{\mathcal{S}} \cap P_i| \leq 2$ for all $i \geq i_0 + 1$, we know that $|X_{\mathcal{S}}| \leq k + q$ (by Claim 4.7).

If $|X_{\mathcal{S}} \cap P_i| = 2$ for some $q \geq i \geq i_0 + 1$, then $|Y_{\mathcal{S}}| \geq 2k - i_0$. Hence $k + q \geq |X_{\mathcal{S}}| \geq 2k - i_0 + 1$, which implies that $k = 2q - 1$ and $i_0 = q$, again in contrary to $i_0 + 1 \leq i \leq q$. ■

It follows from Claim 4.7 and Claim 4.8 that $|X_{\mathcal{S}}| \leq k + i_0$ and hence $|Y_{\mathcal{S}}| \leq k + i_0 - 1$.

Thus $|X_{\mathcal{S}} \cap P_i'| \leq 1$ for any $a + 1 \leq i \leq i_0$. Combining with Claim 4.8, we know that $|X_{\mathcal{S}} \cap P_i'| \leq 1$ for any $i \geq a + 1$. Since $|X_{\mathcal{S}}| \geq k + 2$ (by (4.5)), it follows from Claim 4.7 that $a = 2$ and $|X_{\mathcal{S}} \cap P_i'| = 2$ for $i = 1, 2$, and

$$|X_{\mathcal{S}}| = k + 2, |Y_{\mathcal{S}}| = k + 1 \text{ and } Y_{\mathcal{S}} = L_{\mathcal{S}}(X_{\mathcal{S}} \cap P_1') = L_{\mathcal{S}}(X_{\mathcal{S}} \cap P_2'). \tag{4.6}$$

For $i = 1, 2$, assume $P_i = \{u_i, v_i, x_i, y_i\}$.

If $\Lambda_{P_2,3} \geq 2$, then $|S_2| = 3$, say $S_2 = \{u_2, v_2, x_2\}$. Then $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_2)| + |L(P_2 - S_2)| \geq k + 2$, a contradiction.

Assume $\Lambda_{P_2,3} \leq 1$. Then (ii) or (iii) holds, and $|S_2| = 2$, say $S_2 = \{u_2, v_2\}$. If $z_2 \in X_{\mathcal{S}}$, say $P_2' \cap X_{\mathcal{S}} = \{z_2, x_2\}$, then $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(z_2) \cup L(x_2)| \geq k + 2$ (by Claim 4.6), contrary to (4.6).

Assume $z_2 \notin X_{\mathcal{S}}$. Then $x_2, y_2 \in X_{\mathcal{S}}$. Now $|L(x_2 \vee y_2)| \leq |Y_{\mathcal{S}}| = k + 1$ implies that $|L(x_2 \vee y_2)| = k + 1$ and $|L(x_2 \wedge y_2)| = k - 1$. This implies that $\Lambda_{P_2,2} = k - 1$ and hence $|L(u_2 \wedge v_2)| = k - 1$. As $k \geq 8$, i.e., $\Lambda_{P_2,2} = k - 1 \geq 7$, it follows from Claim 4.6 that $|L(x_2 \wedge y_2)| = \Lambda_{P,2} \geq 2$ and $|L(x_2 \wedge y_2) \cup L(v)| \geq k + 2$ for any $v \in P_2 - \{x_2, y_2\}$.

If (ii) holds, say $S_1 = \{u_1, v_1, x_1\}$, then $L(u_2 \vee v_2) = L_{\mathcal{S}}(z_1) \cup L(y_1)$. This implies that $L(x_2 \vee y_2) = L_{\mathcal{S}}(z_1) \cup L(y_1)$, for otherwise, by see (ii), we should have chosen

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

$S_2 = \{x_2, y_2\}$. So $|L(P_2)| = k + 1$. Hence

$$\begin{aligned} 2k - 2 &= |L(x_2 \wedge y_2)| + |L(u_2 \wedge v_2)| \\ &= |L(x_2 \wedge y_2) \cap L(u_2 \wedge v_2)| + |L(x_2 \wedge y_2) \cup L(u_2 \wedge v_2)| \\ &\leq |L(x_2 \wedge y_2) \cap L(u_2 \wedge v_2)| + k + 1. \end{aligned}$$

This implies that $L(x_2 \wedge y_2) \cap L(u_2 \wedge v_2) \neq \emptyset$, in contrary to Lemma 3.2.

Assume (iii) holds, and for $i = 1, 2$, $P_i = \{u_i, v_i, x_i, y_i\}$ and $S_i = \{u_i, v_i\}$. If $z_i \in X_{\mathcal{S}}$ for some $i = 1, 2$, then by Claim 4.6, $|L_{\mathcal{S}}(z_i)| \geq 2$ and hence $|Y_{\mathcal{S}}| \geq |L_{\mathcal{S}}(P'_i)| \geq k + 2$, contrary to (4.6).

Assume $z_1, z_2 \notin X_{\mathcal{S}}$. Then again by the choice of S_2, we have $L(u_2 \vee v_2) = L(x_1 \vee y_1) = L(x_2 \vee y_2)$, $|L(x_2 \wedge y_2)| = |L(u_2 \wedge v_2)| = k - 1$, and $|L(P_2)| = k + 1$. This leads to the same contradiction. This completes the proof of Theorem 4.1.

It was proved in [23] that $K_{6,2\star(k-3),1\star2}$ is k-choosable. Combining with Theorem 4.1, we conclude that

$$p_1 \geq 3, p_3^+ \leq p_1 - 1, 3p_3^+ + 2p_2 + p_1 \leq |V| - 3. \tag{4.7}$$

5 Pseudo-L-colouring

As described in Section 1, our strategy for proving Theorem 1.2 is to partition $V(G)$ into a family $\mathcal{S}$ of independent sets, so that either there is a matching $M_{\mathcal{S}}$ in the bipartite graph $B_{\mathcal{S}}$ that covers $V(G/\mathcal{S})$ and hence produce an L-colouring of G, or using Hall's Theorem to produce a good partial L-colouring of G that leads to an L-colouring of G by using induction. The partition $\mathcal{S}$ is obtained by constructing a proper colouring f of G, and the parts in $\mathcal{S}$ are the colour classes of f. For this strategy to succeed, the colouring f needs to have some nice property. In this section, we define the concept of pseudo-L-colouring of G, and study properties of the partition $\mathcal{S}$ of $V(G)$ induced by such colourings.

***Definition* 5.1** A *pseudo L-colouring* of G is a proper colouring f of G such that $f(v) \in C_L$ for every vertex v, and if $f(v) = c \notin L(v)$, then $f^{-1}(c) = \{v\}$ is a singleton f-class.

In a pseudo L-colouring f of G, if $f(v) \notin L(v)$, then we say v is *badly f-coloured* (or *badly coloured* if f is clear from the context).

By Observation 3.1, if f is a pseudo-L-colouring of G, then there is a pseudo-L-colouring g of G such that $g(G) = C_L$ and for every badly g-coloured vertex v of G, $g(v) = f(v)$. In the following, we may assume that all the pseudo-L-colourings f satisfy $f(G) = C_L$. However, when we construct a pseudo-L-colouring f of G, we do not need to verify that $f(G) = C_L$ (because if $f(G) \neq C_L$, then we change it to the pseudo-L-colouring g described above).

***Definition* 5.2** Assume f is a pseudo L-colouring of G. Let $\mathcal{S}_f$ be the family of f-classes, which is a partition of $V(G)$, i.e., $\mathcal{S}_f = \{f^{-1}(c) : c \in C_L\}$ where $f^{-1}(c)$ is the set of all vertices coloured by c under f. We denote $G/\mathcal{S}_f$, $L_{\mathcal{S}_f}$, $B_{\mathcal{S}_f}$, $X_{\mathcal{S}_f}$ and $Y_{\mathcal{S}_f}$ by G_f, L_f, B_f, X_f and by Y_f, respectively.

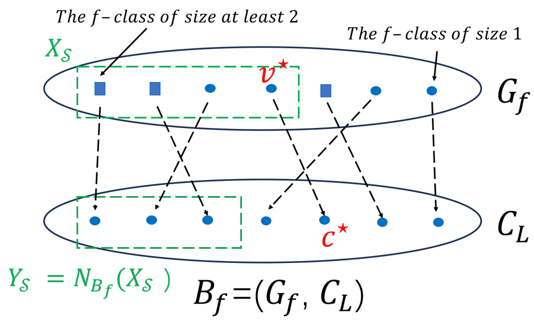

Figure 1: The bipartite graph B_f with partite sets G_f and C_L. Vertices in G_f are f-classes, some of them are singleton classes represented by solid circles, and other are 2^+-classes, represented by solid squares. The broken arrowed line indicate the colouring f. The edges of B_f are not drawn, and $Y_{\mathcal{S}} = N_{B_{\mathcal{S}}}(X_{\mathcal{S}})$. Vertex v^* is contained in $X_{\mathcal{S}}$ but $f(v^*) = c^* \notin Y_{\mathcal{S}}$. So $v^{\star}$ is a badly f-coloured vertex..

In the remainder of this section, assume f is a pseudo L-colouring of G. In the graph G_f, $f^{-1}(c)$ is identified into a single vertex. For simplicity, we denote this vertex by $f^{-1}(c)$. So $f^{-1}(c)$ is both a subset of $V(G)$ and a vertex of G_f. It will be clear from the context which one it is.

Since $|X_f| > |Y_f|$, there is a colour class $f^{-1}(c) \in X_f$ such that $c \notin Y_f$. Hence $f^{-1}(c)$ is a singleton f-class $\{v\}$ and v is badly coloured by f.

For a subset Q of $V(G_f)$, let $V(Q)$ be the subset of $V(G)$ defined as

$$V(Q) = \bigcup_{f^{-1}(c) \in Q} f^{-1}(c).$$

Let ℓ be the number of f-classes $f^{-1}(c)$ of size $|f^{-1}(c)| \geq 2$. As $|V| > |C_L|$, $\ell \geq 1$. On the other hand, $\lambda = |V| - |C_L| \geq \ell$ and equality holds if and only if $f(G) = C_L$ and each f-class has size at most 2.

Recall that there is a matching $M_{\mathcal{S}_f}$ in $B_f - (X_f \cup Y_f)$ that covers all vertices in $V(G_f) - X_f$. The matching $M_{\mathcal{S}_f}$ defines a partial L-colouring of $G[\bigcup_{f^{-1}(c) \notin X_f} f^{-1}(c)]$ that colours vertices in $f^{-1}(c)$ with c', where $\{c', f^{-1}(c)\}$ is an edge in $M_{\mathcal{S}_f}$. We denote this partial L-colouring of G by ψ_f. The matching $M_{\mathcal{S}_f}$ maybe not unique. In this case, we let $M_{\mathcal{S}_f}$ be an arbitrary matching that covers $V(G_f) - X_f$.

We extend ψ_f to a partial L-colouring ϕ_f of G by colouring each f-classes $f^{-1}(c) \in X_f$ of size at least 2 by colour c. By definition of pseudo-L-colouring, for such an f-class $f^{-1}(c)$, $c \in L_{\mathcal{S}}(f^{-1}(c))$. So ϕ_f is a proper L-colouring of G. Denote by X the set of vertices of G coloured by ϕ_f. Note that only those f-classes $f^{-1}(c)$ of size at least 2 contained in X_f are coloured by colours from Y_f. So

$$|\phi_f(X) \cap Y_f| \leq \ell.$$

If $G - X$ has an L^{ϕ_f}-colouring θ, then $\phi_f \cup \theta$ would be an L-colouring of G. Thus $G - X$ is not L^{ϕ_f}-colourable.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

***Lemma* 5.1** *$V_f - X_f$ contains at most $\lambda - 1$ singletons of G. Moreover, if $V_f - X_f$ contains $\lambda - 1 \geq 1$ singletons of G, then $\ell = \lambda$ and the following hold:*

(1) All f-classes have size 2 or 1, and there are exactly ℓ f-classes of size 2.
(2) All the ℓ f-classes of size 2 are contained in X_f.
(3) For each non-singleton part P of G, there is a singleton f-class $\{v\} \in X_f$ such that $v \in P$.
(4) If f has exactly one badly coloured vertex, then $|Y_f| \geq k+1$.

Proof It follows from the definition of ϕ_f that for each vertex v of $G - X$, $\{v\} \in X_f$ is a singleton f-class, and $L(v) \subseteq Y_f$. As $|L^{\phi_f}(X) \cap Y_f| \leq \ell$,

$$|L^{\phi_f}(v)| \geq k - \ell, \forall v \in V(G - X).$$

If $G_f - X_f$ contains ℓ singletons of G, then

$$\chi(G - X) \leq k - \ell \text{ and } |V(G - X)| \leq 2k + 2 - 2\ell - \lambda \leq 2(k - \ell) + 1.$$

By Noel-Reed-Wu Theorem, $G - X$ is L^{ϕ_f}-colourable, a contradiction.

So $G_f - X_f$ contains at most $\ell - 1$ singletons of G.

Assume $G_f - X_f$ contains $\lambda - 1$ singletons of G. Since $\ell \leq \lambda$, we have $\ell = \lambda$ and hence each f-class has size at most 2, and there are exactly ℓ f-classes of size 2, i.e., (1) holds. We shall prove that (2)-(4) hold.

(2): Assume to the contrary that there is an f-class of size 2 not in X_f. Then at most $\ell - 1$ f-classes are coloured by colours from $Y_{\mathcal{S}}$. Hence

$$|L^{\phi_f}(v)| \geq k - (\ell - 1), \forall v \in V(G - X).$$

As

$$|V(G - X)| \leq 2k + 2 - 2\ell = 2(k - \ell) + 2 = 2(k - \ell + 1) \text{ and } \chi(G - X) \leq k - \ell + 1,$$

$G - X$ with list assignment L^{ϕ_f} satisfy the condition of Noel-Reed-Wu Theorem, and hence $G - X$ has an L^{ϕ}-colouring, a contradiction.

(3): If (3) does not hold, then there is a non-singletone part P of G such that all vertices of P are colourd, i.e., P is a non-singleton part of G contained in X, and hence $\chi(G - X) \leq k - \lambda = k - \ell$. We still have $|V(G - X)| \leq 2k + 2 - 2\ell - (\lambda - 1) \leq 2(k - \ell) + 1$. Hence by Noel-Reed-Wu Theorem, $G - X$ has an L^{ϕ_f}-colouring, a contradiction.

(4): Assume v^* is the only badly coloured vertex. Then $\{v^*\}$ is an f-class of size 1 in X_f. This implies that $|Y_f| \geq |L(v^*)| \geq k$. Assume to the contrary that $|Y_f| = k$. This implies that for all singleton f-classes $\{v\} \in X_f$, $L(v) = Y_f$.

Assume $f^{-1}(c) \in X_f$ is an f-class of size at least 2, and P_i is the part of G containing $f^{-1}(c)$. As the size of $f^{-1}(c)$ is at least 2, P_i is not singleton-part and hence it follows from (3) that there is an f-class $\{v\} \in X_f$ such that $v \in P_i$. Thus, $L(v) = Y_f$, $c \in L(v)$ and we can colour v with colour c, and colour v^* with $f(v)$. The resulting colouring is a pseudo L-colouring of G with no badly coloured vertices, i.e., an L-colouring of G, a contradiction.

This completes the proof of Lemma 5.1. ∎

***Lemma* 5.2** *Assume $\lambda \geq 2$ and G has at most $\lambda - 1$ singletons. Then $G_f - X_f$ contains at most $\lambda - 2$ singletons of G.*

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Proof If G has at most $\lambda - 2$ singletons, then the conclusion is trivial. Assume G has exactly $\lambda - 1$ singletons (i.e., $p_1 = \lambda - 1$), and assume to the contrary that all the $\lambda - 1$ singletons of G are contained in $G_f - X_f$. By (3) of Lemma 5.1, for each of the $k - \lambda + 1$ 2^+-parts P of G, X_f has a singleton f-class $\{v\}$ with $v \in P$. By Lemma 5.1, we have $\ell = \lambda$. By (2) of Lemma 5.1, all the ℓ f-classes of size 2 are contained in X_f. Thus

$$|V(X_f)| \geq 2\ell + k - \lambda + 1 = \lambda + k + 1. \tag{5.1}$$

If a 2-part P of G is contained in $V(X_f)$, then $L(P) \subseteq Y_f$. By Lemma 3.2,

$$2k \leq |L(P)| \leq |Y_f|.$$

This contradicts to the fact that $|Y_f| < |C_L| = |V| - \lambda \leq 2k$.

Thus for each 2-part P of G, $|P \cap V(G_f - X_f)| \geq 1$. (Note that a 2-part has no common colour in the lists of its vertices, so P is not an f-class). Hence

$$|V(G_f - X_f)| \geq \lambda - 1 + p_2. \tag{5.2}$$

As $p_1 = \lambda - 1$, it follows from (5.1) and (5.2) that

$$2k+2 = |V| = |V(X_f)|+|V(G_f-X_f)| \geq (\lambda+k+1)+(\lambda-1+p_2) = 2\lambda+k+p_2 = 2p_1+2+k+p_2.$$

So

$$p_3^+ + p_2 + p_1 = k \geq 2p_1 + p_2,$$

which implies that $p_3^+ \geq p_1$, in contrary to (4.7).

This completes the proof of Lemma 5.2. ∎

6 Near acceptable colourings

We have shown in the previous section that the partition $\mathcal{S}$ of $V(G)$ induced by a pseudo-L-colouring of G has some nice properties. However, for the proof of Theorem 1.2, one more restriction need to be added to a pseudo-L-colouring. In this section, we define the concept of near acceptable L-colouring of G, and prove that the partition $\mathcal{S}$ of G induced by a near acceptable L-colouring of G enables us to construct a proper L-colouring.

***Definition* 6.1** A colour c is called *frequent* if one of the following holds:

(1) $|L^{-1}(c)| \geq k + 2$.
(2) $|L^{-1}(c) \cap T| \geq \lambda$.
(3) $|T| = \lambda - 1 \geq 1$ and $T \subseteq L^{-1}(c)$.

***Definition* 6.2** A pseudo L-colouring f of G is *near acceptable* if each badly coloured vertex is coloured by a frequent colour.

The concept of near acceptable L-colouring was first used in [17] for the proof of Noel-Reed-Wu Theorem. For the proof of Theorem 1.2, as G has one more vertex, the definition of frequent colours is different from that in [17]. Thus the near acceptable L-colouring in this paper is different from the one in [17]. The difference makes it more difficult to find a near acceptable L-colouring of G. Nevertheless, we shall show that

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

analog to [17], the existence of a near acceptable L-colouring of G implies the existence of an L-colouring of G.

Lemma 6.1 *G has no near acceptable L-colouring.*

Proof Assume to the contrary that f is a near acceptable L-colouring of G. Since $|X_f| > |Y_f|$, there is a colour class $f^{-1}(c^*) \in X_f$ with $c^* \notin Y_f$. Hence $f^{-1}(c^*) = \{v^*\}$ is a badly coloured singleton f-class.

Since $f^{-1}(c^*) = \{v^*\} \in X_f$, we have $L(v^*) \subseteq Y_f$, and hence

$$k \leq |L(v^*)| \leq |Y_f| < |X_f|.$$

On the other hand, $c^* \notin Y_f$ implies that for each $f^{-1}(c) \in X_f$, there exists $v \in f^{-1}(c)$, such that $c^* \notin L(v)$. Thus

$$|L^{-1}(c^*)| \leq 2k + 2 - |X_f| \leq k + 1.$$

So c^* is not a frequent colour of Type (1).

By Lemma 5.1, $V_f - X_f$ contains at most $\lambda - 1$ singletons of G. Hence

$$|L^{-1}(c^*) \cap T| \leq \lambda - 1.$$

So c^* is not a frequent colour of Type (2).

If $|T| = \lambda - 1 \geq 1$, then by Lemma 5.2, $|L^{-1}(c^*) \cap T| \leq |V(V_f - X_f) \cap T| \leq \lambda - 2$. Hence $T \nsubseteq L^{-1}(c^*)$. So c^* is not a frequent colour of Type (3).

Therefore, c^* is not frequent, a contradiction. ∎

7 Upper bound on the number frequent colours

This section proves that there are at most $k-1$ frequent colours. Assume to the contrary that there is a set F of k frequent colours. We will construct a near acceptable colouring f of G in the following three steps:

(1) Construct a partial L-colouring f_1 of G using colours from $C_L - F$, that colours as many vertices as possible, and subject to this, the coloured vertices are distributed among the parts of G as evenly as possible. Let V_1 be the set of vertices coloured by f_1.

(2) Order the parts of G as $P_1, P_2, \ldots, P_k$ so that $|P_i - V_1| \geq |P_{i+1} - V_1|$ for $i = 1, 2, \ldots, k-1$. Colour greedily in this order the vertices of $P_i - V_1$ by a common permissible colour from F, until this process cannot be carried out any more. This partial L-colouring will be denoted by f_2. Let V_2 be the set of vertices coloured by f_2.

(3) Extend $f_1 \cup f_2$ to a near acceptable L-colouring (for example, if for each remaining part P_i, $P_i - V_1$ contains at most one vertex, then we arbitrarily colour that vertex by a remaining colour from F to obtain a near acceptable L-colouring of G).

The difficult part is to prove that $f_1 \cup f_2$ can be extended to a near acceptable L-colouring. What we really proved is that if this cannot be done, then every part of G is a 3^--part, which is in contrary to Theorem 4.1.

In the proof, we often need to modify a partial L-colouring.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

***Definition* 7.1** Assume f is a partial L-colourings of G. For distinct colours $c_1, c_2, \ldots, c_t \in C_L$, and distinct indices $i_1, i_2, \ldots, i_t \in \{1, 2, \ldots, k\}$, we denote by

$$f(c_1 \to P_{i_1}, c_2 \to P_{i_2}, \ldots, c_t \to P_{i_t})$$

the partial L colouring of G obtained from f by the following operation:

- First, for $j = 1, 2, \ldots, t$, uncolour vertices in $f^{-1}(c_j)$ (it is allowed that $f^{-1}(c_j) = \emptyset$, i.e., c_j is not used by f).
- Second, for $j = 1, 2, \ldots, t$, colour vertices in $L^{-1}(c_j) \cap P_{i_j}$ by colour c_j.

Now we are ready to prove the following lemma.

***Lemma* 7.1** *There are at most $k - 1$ frequent colours.*

Proof Assume to the contrary that there is a set F of k frequent colours. A *valid* partial L-colouring f of G is a partial L-colouring of G using colours from $C_L - F$.

For a valid partial L-colouring f of G, for $i = 1, 2, \ldots, k$, let

$$S_{f,i} = P_i \cap f^{-1}(C_L - F)$$

be the set of coloured vertices in P_i. Let

$$\tau_1(f) = \sum_{i=1}^{k} |S_{f,i}|,$$

$$\tau_2(f) = \sum_{i=1}^{k} |S_{f_1,i}|^2.$$

We choose a valid partial L-colouring f_1 of G such that

$$\tau(f_1) = (\tau_1(f_1), -\tau_2(f_1))$$

is lexicographically maximum, i.e., the number of coloured vertices $\tau(f_1)$ is maximum, and subject to this, $\tau_2(f) = \sum_{i=1}^{k} |S_{f_1,i}|^2$ is minimum, which means that the coloured vertices are distributed among the parts of G as evenly as possible.

Let $V_1 = f_1^{-1}(C_L - F) = \bigcup_{i=1}^{k} S_{f,i}$ be the set of vertices coloured by f_1. By the maximality of $\tau_1(f_1)$, V_1 must have used all the colours in $C_L - F$, and hence $|C_L - F| \leq |V_1|$.

If $|V - V_1| \leq k$, then let $g : V - V_1 \to F$ be an arbitrary injective mapping. The union $f_1 \cup g$ is a near acceptable L-colouring of G, and we are done. Thus we may assume that

$$|V - V_1| \geq k + 1, \text{ and hence } |V_1| \leq k + 1. \tag{7.1}$$

For $i = 1, 2, \ldots, k$, let

$$R_{f_1,i} = P_i - S_{f,i}.$$

For a colour $c \in C_L$, let

$$R_i(c) = |L^{-1}(c) \cap R_{f_1,i}|$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

be the number of vertices in $R_{f_1,i}$ that can be coloured by c, and

$$R_i(C_L - F) = \sum_{c \in C_L - F} R_i(c)$$

be the total number of vertices in $R_{f_1,i}$ that can be coloured by colours from $C_L - F$.

If $c \in C_L - F$, then

$$R_i(c) \le |f_1^{-1}(c)|,$$

for otherwise, $f_1(c \to P_i)$ is a valid partial L-colouring of G which colours more vertices than f_1, in contrary to the choice of f_1.

Definition 7.2 A colour $c \in C_L - F$ is said to be *movable to* P_i if $R_i(c) = |f_1^{-1}(c)|$. ■

Observation 7.2 *The following facts will be used frequently in the argument below.*

(1) If $c \in C_L - F$ is movable to P_i, then $f_1(c \to P_i)$ is a valid partial L-colouring of G with $\tau_1(f_1(c \to P_i)) = \tau_1(f_1)$.
(2) $R_i(C_L - F) \le |V_1 - P_i|$, and if $R_i(C_L - F) = |V_1 - P_i|$, then

every colour $c \in C_L - F$ with $f_1^{-1}(c) \cap P_i = \emptyset$ is movable to P_i. (P1)

(3) If $f_1^{-1}(c)$ is a singleton f_1-class, then c is movable to P_i if and only if $c \in L(R_{f,i})$.
(4) For any choices of distinct colours $c_1, c_2, \ldots, c_t \in C_L$ and indices $i_1, i_2, \ldots, i_t$, $f_1(c_1 \to P_{i_1}, c_2 \to P_{i_2}, \ldots, c_t \to P_{i_t})$ is a partial L-colouring of G.

Proof (1),(3), (4) are trivial.

(2): If $c \in C_L - F$ and $f^{-1}(c) \cap P_i \ne \emptyset$, then $R_i(c) = 0$, for otherwise, we can colour vertices in $\{v \in R_{f_1,i} : c \in L(v)\}$ with colour c. By the fact that $R_i(c) \le |f_1^{-1}(c)|$, we have $R_i(c) \le |f_1^{-1}(c) - P_i|$ for any colour $c \in C_L - F$. Hence $R_i(C_L - F) = \sum_{c \in C_L - F} R_i(c) \le |V_1 - P_i|$, and equality holds only if $R_i(c) = |f_1^{-1}(c)|$ for all $c \in C_L - F$ with $f_1^{-1}(c) \cap P_i = \emptyset$. ■

Claim 7.3 *If $|P_i| = 2$, then $S_{f_1,i} \ne \emptyset$.*

Proof Assume to the contrary that $P_i = \{u, v\}$ and $S_{f_1,i} = \emptyset$. By Lemma 3.2, $L(u \wedge v) = \emptyset$. Hence $|C_L| \ge 2k$ and $|V_1| \ge |C_L - F| \ge k$. So there are at least k f_1-classes. As $|V_1| \le k+1$ (see (7.1)), each f_1-class is a singleton, except at most one f_1-class is of size 2.

Since $S_{f_1,i} = \emptyset$, there is an index j_0 such that $|f_1(S_{f_1,j_0}))| \ge 2$. Assume $c_1, c_2 \in f_1(S_{f_1,j_0})$. At least one of $f_1^{-1}(c_1), f_1^{-1}(c_2)$ is a singleton f_1-class.

If $|C_L| = 2k$, then $L(u \vee v) = C_L$ and by (3) of Observation 7.2, one of c_1, c_2, say c_1, is movable to P_i and $f_1^{-1}(c_1)$ is a singleton f_1-class. If $|C_L| = 2k + 1$, then there are $k + 1$ f_1-classes, and hence each f_1-class is a singleton. So both $f_1^{-1}(c_1), f_1^{-1}(c_2)$ are singleton f_1-classes, and at least one of c_1, c_2 belongs to $L(R_{f_1,i})$ and hence is movable to P_i.

Assume $f_1^{-1}(c_1)$ is a singleton f_1-class and c_1 is movable to P_i.

Then $\tau_1(f_1(c_1 \to P_i)) = \tau_1(f_1)$, $\tau_2(f_1(c_1 \to P_i)) < \tau_2(f_1)$. This is in contrary to our choice of f_1. ■

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

By a re-ordering, if needed, we assume that

$$|R_{f_1,1}| \geq |R_{f_1,2}| \geq \ldots \geq |R_{f_1,k}|. \tag{R1}$$

In the second step, starting from $i = 1$ to k, we do the following: If there is a colour $c \in F$ such that $c \in \bigcap_{v \in R_{f_1,i}} L(v)$ and c is not used by $R_{f_1,j}$ for $j < i$, then we colour $R_{f_1,i}$ with c. The step terminates when such a colour does not exist.

Assume the second step stopped at $i_0 + 1$, and hence $R_{f_1,1}, \ldots, R_{f_1,i_0}$ are coloured in the second step.

Note that in the ordering of $R_{f_1,1}, R_{f_1,2}, \ldots, R_{f_1,k}$, if some of the $R_{f_1,j}$'s has the same cardinality, then we can choose different ordering so that (R1) still holds. Also with a given ordering of $R_{f_1,1}, R_{f_1,2}, \ldots, R_{f_1,k}$, when we colour all the vertices of $R_{f_1,i}$, there may be more than one choice of the colours. We assume that

$$\begin{array}{l}\text{Subject to (R1), the ordering of } R_{f_1,1}, R_{f_1,2}, \ldots, R_{f_1,k} \text{ and} \\ \text{the colouring of the } R_{f_1,i}\text{'s are chosen so that } i_0 \text{ is maximum .}\end{array} \tag{R2}$$

We denote by f_2 the colouring constructed in the second step, and by V_2 the set of vertices coloured in this step, and let $V_3 = V - V_1 - V_2$ be the set of uncoloured vertices after the second step. Let F_1 be the frequent colours used in second step, and let $F_2 = F - F_1$. So $|F_1| = i_0$ and $|F_2| = k - i_0$. Note that it is possible that $i_0 = 0$ and $V_2 = \emptyset$.

If $|R_{f_1,i_0+1}| \leq 1$, then $|V_3| \leq k - i_0 = |F_2|$, and $f_1 \cup f_2$ can be extended to a near acceptable L-colouring of G by colouring V_3 injectively by F_2, and we are done.

Therefore the following hold:

$$\begin{aligned} |R_{f_1,i_0+1}| &\geq 2, \\ |V_2| &\geq 2i_0, \\ |V_3| &\geq k - i_0 + 1, \\ |V_1| &= |V| - |V_2| - |V_3| \leq k - i_0 + 1. \end{aligned} \tag{7.2}$$

Observe that for each colour $c \in F_2$,

$$R_{i_0+1}(c) \leq |R_{f_1,i_0+1}| - 1,$$

and for each colour $c \in F_1$,

$$R_{i_0+1}(c) \leq |R_{f_1,i_0+1}|.$$

Hence

$$\begin{aligned} R_{i_0+1}(C_L - F) &= \sum_{c \in C_L - F} R_{i_0+1}(c) \\ &= \sum_{c \in C_L} R_{i_0+1}(c) - \sum_{c \in F_1 \cup F_2} R_{i_0+1}(c) \\ &\geq k|R_{f_1,i_0+1}| - (|R_{f_1,i_0+1}| - 1)(k - i_0) - |R_{f_1,i_0+1}| i_0 = k - i_0, \end{aligned} \tag{7.3}$$

and if the equality holds, then

$$\forall c \in F_2, R_{i_0+1}(c) = |R_{f_1,i_0+1}| - 1, \tag{7.4}$$

$$\forall c \in F_1, R_{i_0+1}(c) = |R_{f_1,i_0+1}|. \tag{7.5}$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Combining (7.2) with (7.3) and by (2) of Observation 7.2, we have

$$k - i_0 + 1 \geq |V_1| \geq |V_1 - P_{i_0+1}| \geq R_{i_0+1}(C_L - F) \geq k - i_0. \tag{7.6}$$

So $|V_1| = k - i_0$ or $|V_1| = k - i_0 + 1$.

Case 1: $|V_1| = k - i_0$.

In this case,

$$\begin{aligned} |V_1| = |V_1 - P_{i_0+1}| = R_{i_0+1}(C_L - F) = k - i_0, \\ S_{f_1,i_0+1} = V_1 \cap P_{i_0+1} = \emptyset, \text{ and } P_{i_0+1} = R_{f_1,i_0+1}. \end{aligned} \tag{7.7}$$

So (P1) and (P2) holds for i_0+1. By Claim 7.3, $|P_{i_0+1}| = |R_{f_1,i_0+1}| \geq 3$. Hence $|V_2| \geq 3i_0$. This implies that

$$k - i_0 = |V_1| \leq k - 2i_0 + 1,$$

and hence $i_0 \leq 1$.

Case 1.1: $i_0 = 1$.

In this case, $|V_1| = k - 1$, $|V_2| \geq 3$ and $|V_3| \geq k$ (by (7.2), i.e. $|V_3| \geq k - i_0 + 1 = k$). Since $|V| = 2k + 2$, we conclude that $|V_2| = |R_{f_1,1}| = 3$ and $|V_3| = k$.

By (7.7), $R_2(C_L - F) = |V_1| = k - 1$. This implies that

$$\sum_{c \in F} R_2(c) = \sum_{c \in C} R_2(c) - \sum_{c \in C-F} R_2(c) = 3k - R_2(C_L - F) = 2k + 1.$$

Hence there is a colour $c_1 \in F$ such that $R_2(C_L - F)(c_1) = |L^{-1}(c_1) \cap R_{f_1,2}| \geq 3 = |R_{f_1,1}| \geq |R_{f_1,2}|$. So $c_1 \in \bigcap_{v \in R_{f_1,2}} L(v)$. On the other hand, by (7.7), $P_2 = R_{f_1,2}$, and by Lemma 3.2, $\bigcap_{v \in P_2} L(v) = \emptyset$, a contradiction.

Case 1.2: $i_0 = 0$.

In this case,

$$|V_1| = R_1(C_L - F) = k, \quad |V_2| = 0, \quad |V_3| = k + 2. \tag{7.8}$$

Combining with $i_0 = 0$ and (P2), for each colour $c \in F$, $R_1(c) = |R_{f_1,1}| - 1$.

Claim 7.4 *$|P_1| = |R_{f_1,1}| = 3$ and $R_1(c) = 2$ for any colour $c \in F$.*

Proof If $|R_{f_1,1}| \geq 4$, then for any colour $c \in F$, $f_1(c \to P_1)$ can be extended to a near acceptable L-colouring of G by colouring the remaining $k - 1$ vertices of V_3 injectively with the remaining $k - 1$ colours of F (note that $|L^{-1}(c) \cap P_1| = |R_{f_1,1}| - 1 \geq 3$).

Thus $|P_1| = |R_{f_1,1}| = 3$ (cf. (7.7)). This implies that $R_1(c) = 2$ for any colour $c \in F$. ■

If there is a colour $c \in F$ such that $R_2(c) \geq 2$, then f_1 can be extended to a near acceptable L-colouring of G by colouring a 2-subset U_1 of of $R_{f_1,2}$ with a colour $c \in \bigcap_{v \in U_1} L(v) \cap F$, colouring a 2-subset U_2 of $R_{f_1,1}$ by a colour from $c' \in \bigcap_{v \in U_2} L(v) \cap (F - \{c\})$, and colouring the remaining $k-2$ vertices of V_3 injectively with the remaining $k - 2$ colours of F.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Thus

$$R_2(c) \le 1, \forall c \in F \text{ and } \sum_{c\in F} R_2(c) \le k. \tag{7.9}$$

This implies that $|R_{f_1,2}| \le 2$, for otherwise interchanging the roles of $R_{f_1,1}$ and $R_{f_1,2}$, we would have $R_2(c) = |R_{f_1,2}| - 1 \ge 2$ for all $c \in F$, in contrary to (7.9).

Claim 7.5 $|R_{f_1,i}| = 1$ *for* $i = 2, 3, \dots, k$.

Proof Assume to the contrary that $|R_{f_1,2}| = 2$ (as $|R_{f_1,2}| \le 2$), then by Observation 7.2, $R_2(C_L - F) \le |V_1 - P_2| = |V_1| - |S_{f_1,2}| = k - |S_{f_1,2}|$ and

$$\sum_{c\in F} R_2(c) = \sum_{c\in C_L} R_2(c) - \sum_{c\in C_L - F} R_2(c) = 2k - R_2(C_L - F) \ge k + |S_{f_1,2}|. \tag{7.10}$$

Combining with (7.9), we have $|S_{f_1,2}| = 0$ and hence $R_{f_1,2} = P_2$, in contrary to Claim 7.3. By Claim 7.3, $|S_{f_1,2}| \ge 1$, in contrary to (7.9).

Therefore $|R_{f_1,2}| = 1$ and hence $|R_{f_1,i}| = 1$ for $i = 2, 3, \dots, k$ (note that $|V_3| = k + 2$). ■

Claim 7.6 $|S_{f_1,j}| \le 2$ *for* $j = 2, 3, \dots, k$.

Proof If $|f_1(S_{f_1,j})| \ge 2$ for some j, say $c_1, c_2 \in f_1(S_{f_1,j})$, then $\tau_1(f_1(c_1 \to P_1)) = \tau_1(f_1)$ (as (P1) holds) and $\tau_2(f_1(c \to P_1)) < \tau_2(f_1)$, because

$$|S_{f_1,1}| = 0 \text{ (by 7.7)}, \ |S_{f_1(c_1\to P_1),1}| = R_1(C_L - F)(c_1),$$

and

$$|S_{f_1(c_1\to P_1),j}| = |S_{f_1,j}| - R_1(C_L - F)(c_1) > 0,$$

since $|f_1(S_{f_1,j})| \ge 2$. This is in contrary to our choice of f_1.

Hence for each $j \in \{2, 3, \dots, k\}$, $|f_1(S_{f_1,j})| \le 1$, and $|S_{f_1,j}| \le |f_1^{-1}(c_j)|$ for some $c_j \in C_L - F$. As (P1) holds, $|f_1^{-1}(c_j)| = R_1(C_L - F)(c_j) \le 2$. So $|S_{f_1,j}| \le 2$. ■

Combining with Claim 7.4, 7.5 and 7.6, we have

$$|R_{f_1,1}| = 3, \ |S_{f_1,1}| = 0, \text{ and for } 2 \le j \le k, \ |R_{f_1,j}| = 1, \ |S_{f_1,j}| \le 2.$$

So each part of G is 3^--part, in contrary to Theorem 4.1.

Case 2: $|V_1| = k - i_0 + 1$**.**

If $P_{i_0+1} = R_{f_1,i_0+1}$, then by Claim 7.3, $|R_{f_1,i_0+1}| \ge 3$ and $|V_2| \ge 3i_0$. By (7.2), $|V_1| = |V| - |V_2| - |V_3| \le 2k + 2 - 3i_0 - (k - i_0 + 1) = k - 2i_0 + 1$, and hence $i_0 = 0$. This implies that

$$|V_1| = k + 1, \ |V_2| = 0, \ |V_3| = k + 1.$$

By Observation 7.2, $R_1(C_L - F) \le |V_1| = k + 1$, we conclude that

$$\sum_{c\in F} R_1(c) \ge \sum_{c\in C_L} R_1(c) - \sum_{c\in C_L - F} R_1(c) \ge 3k - R_1(C_L - F) \ge 2k - 1 \ge k + 1.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

So there is a colour $c \in F$ such that $R_1(c) \geq 2$. We can extend f_1 to a near acceptable L-colouring of G by colouring two vertices of $R_{f_1,1}$ with c, and the remaining $k-1$ vertices of V_3 injectively with the remaining $k-1$ colours of F.

Thus $P_{i_0+1} \neq R_{f_1,i_0+1}$, i.e., $S_{f_1,i_0+1} \neq \emptyset$.

As $S_{f_1,i_0+1} \neq \emptyset$, $|V_1 - P_{i_0+1}| = |V_1| - |S_{f_1,i_0+1}| < |V_1|$ and by (7.6), we have

$$|V_1 - P_{i_0+1}| = k - i_0 = R_{i_0+1}(C_L - F), \ |S_{f_1,i_0+1}| = 1. \tag{7.11}$$

So (P1) and (P2) holds for $i_0 + 1$.

***Claim* 7.7** *For each $1 \leq i \leq i_0+1$, $|R_{f_1,i}| = 2$ and for $j \geq i_0+2$, $|R_{f_1,i}| \leq 2$.*

Proof By (7.2), we have $|V_2| \geq 2i_0$, $|V_3| \geq k - i_0 + 1$. Since $|V_1| + |V_2| + |V_3| = 2k+2$, we conclude that

$$|V_1| = k - i_0 + 1, \ |V_2| = 2i_0, \ |V_3| = k - i_0 + 1.$$

So $\forall j \leq i_0+1, \ |R_{f_1,j}| = 2, \ \text{and} \ \forall j \geq i_0+2, \ |R_{f_1,j}| \leq 2.$ ■

***Claim* 7.8** *For $1 \leq i \leq k$, if $|R_{f_1,i}| = 2$, then $|S_{f_1,i}| = 1$.*

Proof By Claim 7.7, $|R_{f_1,1}| = \ldots = |R_{f_1,i_0+1}|$. As (P2) holds, there are i_0 colours $c \in F_1 \subseteq F$ such that $R_{i_0+1}(c) = |R_{f_1,i_0+1}|$. Therefore, for any index j with $|R_{f_1,j}| = 2$, if we re-order the parts so that $R_{f_1,j}$ and R_{f_1,i_0+1} interchange positions (while the other parts stay at their position), (R1) and (R2) are satisfied. So the conclusions we have obtained for P_{i_0+1} hold for P_j. In particular, for any j with $|R_{f_1,j}| = 2$, we have $|S_{f_1,j}| = 1$. ■

***Claim* 7.9** $|S_{f_1,j}| \leq 2$ *for all* j.

Proof As $(P1)$ holds for i_0+1, $|f_1^{-1}(c)| = R_{i_0+1}(c) \leq |R_{f_1,i_0+1}| = 2$ for any $c \in C_L - F$. If $|S_{f_1,j}| \geq 3$ for some j, then there is a colour $c \in C_L - F$ for which the following holds:

- $|f_1^{-1}(c) \cap P_j| = 1$, or
- $|S_{f_1,j}| \geq 4$, and $|f_1^{-1}(c) \cap P_j| = 2$.

Let

$$f_1' = f_1(c \to P_{i_0+1}).$$

Then f_1' is a valid partial L-colouring of G with $\tau_1(f_1') = \tau_1(f_1)$ (as (P1) holds). By (7.11), $|S_{f_1,i_0}| = 1$. Thus either $|S_{f_1',j}| = |S_{f_1,j}| - 1 \geq 2$ and $|S_{f_1',i_0+1}| = 2$, or $|S_{f_1',j}| = |S_{f_1,j}| - 2 \geq 2$ and $|S_{f_1',i_0+1}| = 3$. Hence $\tau_2(f_1')) < \tau_2(f_1)$, in contrary to our choice of f_1. ■

It follows from Claims 7.8 and 7.9 that each part of G is 3^--part, in contrary to Theorem 4.1.

This completes the proof of Lemma 7.1.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

8 Tighter upper bound for the number of frequent colours

In this section and the next section, we assume that (G, L) is a minimum counterexample to Theorem 1.2 with $\sum_{v \in V(G)} |L(v)|$ maximum.

This section proves that there are at most $k - p_1 - 1$ frequent colours. Assume to the contrary that there are $k - p_1$ frequent colours. We shall construct another k-list assignment L' of G that has k frequent colours. By Lemma 7.1, (G, L') is not a counterexample to Theorem 1.2. Hence there is an L'-colouring f of G. Using this colouring f, we construct a near-acceptable L-colouring of G, which contradicts Lemma 6.1.

Let F be the set of frequent colours, and $F' \subseteq F$ be the set of frequent colours of Type (1).

By Lemma 7.1, we may assume that $|F| \leq k - 1$. If $\lambda = 1$, then for any $v \in T$, all colours in $L(v)$ are frequent of Type (2), a contradiction (note that $p_1 \geq 3$, so $T \neq \emptyset$). Thus $\lambda \geq 2$.

Lemma **8.1** $\lambda \leq p_1 + 1$.

Proof For $c \in C_L - F'$, by definition, $|L^{-1}(c)| \leq k + 1$. By Lemma 3.2, for each $c \in F'$, $|L^{-1}(c)| \leq k + p_1 + 2$. Therefore

$$k|V| \leq \sum_{v \in V} |L(v)| = \sum_{c \in C_L} |L^{-1}(c)| \leq |F'|(k + p_1 + 2) + |C_L - F'|(k + 1).$$

Hence

$$|F'| \geq \frac{k|V| - (k+1)|C_L|}{p_1 + 1} = \frac{k\lambda - |C_L|}{p_1 + 1}. \tag{8.1}$$

As $|F'| < k$, we have

$$|C_L| > k(\lambda - p_1 - 1). \tag{8.2}$$

Since $\lambda \geq 2$, we have $|C_L| \leq 2k$. Plug this into (8.2), we have $\lambda \leq p_1 + 2$.

If $\lambda = p_1 + 2$, then $|C_L| = |V| - \lambda = 2k + 2 - (p_1 + 2) = 2k - p_1 \leq 2k - 3$ (as $p_1 \geq 3$). This implies that G has no 2-part (if $\{u, v\}$ is a 2-part of G, then $L(u) \cap L(v) = \emptyset$ and hence $|C_L| \geq 2k$). By (4.7), $2k - 1 = |V| - 3 \geq 3(k - p_1) + p_1$. Hence

$$p_1 \geq \frac{k+1}{2}. \tag{8.3}$$

By (8.1),

$$|F'| \geq \frac{k\lambda - |C_L|}{p_1 + 1} = \frac{k(p_1 + 2) - (2k - p_1)}{p_1 + 1} = \frac{(k+1)p_1}{p_1 + 1} = k - \frac{k - p_1}{p_1 + 1} > k - 1.$$

Hence $|F'| \geq k$, a contradiction. Thus $\lambda \leq p_1 + 1$. ∎

Lemma **8.2** $F = \bigcap_{v \in T} L(v)$.

Proof If $p_1 = \lambda - 1$, then each colour in $\bigcap_{v \in T} L(v)$ is contained in at least $\lambda - 1$ singleton lists, and hence is a frequent colour of Type (3).

If $p_1 \geq \lambda$, then each colour in $\bigcap_{v \in T} L(v)$ is contained in at least λ singleton lists, and hence is a frequent colour of Type (2).

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

In any case,

$$\bigcap_{v\in T} L(v) \subseteq F.$$

On the other hand, assume there is a frequent colour $c \notin \bigcap_{v\in T} L(v)$, say $c \notin L(v)$ for some $v \in T$, then let L' be the list assignment of G defined as $L'(x) = L(x)$ for $x \neq v$ and $L'(v) = L(v) \cup \{c\}$. By our assumption that (G, L) is a minimum counterexample with $\sum_{v\in V(G)} |L(v)|$ maximum, G and L' is not a counterexample to Theorem 1.2. So G has an L'-colouring f. But then f is a near acceptable L-colouring of G, in contrary to Lemma 6.1. Therefore $F \subseteq \bigcap_{v\in T} L(v)$. ■

***Lemma* 8.3** *There are at most $k - p_1 - 1$ frequent colours.*

Proof Assume to the contrary that $\{c_{p_1+1}, c_{p_1+2}, \dots, c_k\}$ is a set of $k - p_1$ frequent colours.

Assume $T = \{v_1, v_2, \dots, v_{p_1}\}$. We choose p_1 colours $c_1, c_2, \dots, c_{p_1}$ so that for $i = 1, 2, \dots, p_1$,

$$c_i \in L(v_i) - \{c_{p_1+1}, \dots, c_k\} - \{c_1, \dots, c_{i-1}\}.$$

As $|L(v_i)| \geq k$, the colour c_i exists.

Let $C' = \{c_1, c_2, \dots, c_k\}$ and define L' as follows:

$$L'(v) = \begin{cases} C' & \text{if } v \in T, \\ L(v) & \text{otherwise.} \end{cases}$$

By Lemma 8.1, $p_1 \geq \lambda - 1$. If $p_1 \geq \lambda$, then each colour in C' is Type-2 frequent with respect to L'. If $p_1 = \lambda - 1$, then each colour in C' is Type-3 frequent with respect to L'. By Lemma 7.1, (G, L') is not a mimnimum counterexample to Theorem 1.2. Since $C_{L'} \subseteq C_L$, we know that (G, L') is not a counterexample to Theorem 1.2. Hence G has an L'-colouring f.

Note that if $v \notin T$, then $f(v) \in L(v)$. We shall modify f to obtain a near acceptable L-colouring of G.

Let $T' = \{v_i : 1 \leq i \leq p_1, c_i \in f(T)\}$. As $|T - T'| = |f(T) - \{c_1, c_2, \dots, c_{p_1}\}|$, there is a bijection $g : T - T' \to f(T) - \{c_1, c_2, \dots, c_{p_1}\}$.

Let $f' : V \to C_L$ be defined as follows:

$$f'(v) = \begin{cases} f(v) & \text{if } v \notin T\ , \\ c_i & \text{if } v = v_i \in T'\ , \\ g(v) & \text{if } v \in T - T'\ . \end{cases}$$

Then f' is a near acceptable L-colouring of G, in contradiction to Lemma 6.1. ■

9 Final contradiction

We shall find a subset X of T and a set F'' of $k - p_1$ colours so that for each $c \in F''$,

$$|L^{-1}(c) \cap X| \geq \lambda.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

This would imply that all the $k - p_1$ colours in F'' are frequent (of Type (2)). This is in contrary to Lemma 7.1.

For any colour $c \in C_L - F$, $|L^{-1}(c)| \leq k + 1$. Let

$$b = \min\{k + 1 - |L^{-1}(c)| : c \in C_L - F\}.$$

***Lemma* 9.1** *There is a subset X of T such that*

(1) $|X| \geq p_1 - \lambda + 1$.
(2) $|L(X)| \leq k + b$.

Moreover, if $b = 0$ or $p_1 = \lambda - 1$, then $|X| \geq p_1 - \lambda + 2$.

Proof Let $c' \in C_L - F$ be a colour with $|L^{-1}(c')| = k + 1 - b$. By Lemma 8.2, there is a vertex $w \in T$ such that $c' \notin L(w)$. Define a list assignment L' as follows:

$$L'(v) = \begin{cases} L(v) \cup \{c'\} & v = w\,, \\ L(v) & \text{otherwise.} \end{cases}$$

By the maximality of $\sum_{v \in V(G)} |L(v)|$, G has an L'-colouring f. We must have $f(w) = c'$ and w is the only badly coloured vertex, for otherwise f is a proper L-colouring of G.

Now f is a pseudo L-colouring of G. By Lemma 5.1, in the bipartite graph B_f, V_f has a subset X_f such that $|X_f| > |Y_f| = |N_{B_f}(X_f)|$, and $V_f - X_f$ contains at most $\lambda - 1$ singletons of G.

It is easy to see that $w \in X_f$ and $c' \notin Y_f$. Let

$$X = \{v \in T : \{v\} \text{ is an } f\text{-class in } X_f\}.$$

Then $|X| = |T| - |(V_f - X_f) \cap T| \geq p_1 - \lambda + 1$ and by Lemma 5.2, if $p_1 = \lambda - 1$, then $|X| = |T| - |(V_f - X_f) \cap T| \geq p_1 - \lambda + 2$.

Since each f-class in X_f contains a vertex v for which $c' \notin L(v)$, we have

$$|L(X)| \leq |Y_f| < |X_f| \leq |V| - |L^{-1}(c')| = k + 1 + b.$$

So $|L(X)| \leq k + b$.

It remains to prove that if $b = 0$, i.e., $|L^{-1}(c')| = k + 1$, then $|X| \geq p_1 - \lambda + 2$.

Assume to the contrary that $|L^{-1}(c')| = k + 1$ and $|X| = p_1 - \lambda + 1$. By Lemma 5.1, $|Y_f| \geq k + 1$ and hence $|X_f| \geq k + 2$, in contrary to $|X_f| \leq |V| - |L^{-1}(c')| = k + 1$.

This completes the proof of Lemma 9.1. ■

We order the colours in $L(X)$ as $c_1, c_2, \ldots, c_t$, so that

$$|L^{-1}(c_1) \cap X| \geq |L^{-1}(c_2) \cap X| \geq \ldots \geq |L^{-1}(c_t) \cap X|,$$

where $t = |L(X)|$. Let $F'' = \{c_1, c_2, \ldots, c_{k-p_1}\}$.

It suffices to show that

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \lambda,$$

and hence each colour $c_i \in F''$ is a frequent of Type (2).

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Let $Z = \{c_{k-p_1}, c_{k-p_1+1}, \ldots, c_t\}$. For each $v \in X$, $|L(v) \cap Z| \geq |L(v)| - (k - p_1 - 1) \geq p_1 + 1$. Hence

$$|Z||L^{-1}(c_{k-p_1}) \cap X| \geq \sum_{i=k-p_1}^{t} |L^{-1}(c_i) \cap X| = \sum_{v \in X} |L(v) \cap Z| \geq |X|(p_1 + 1). \quad (9.1)$$

By Lemma 9.1,

$$|Z| = |L(X)| - (k - p_1 - 1) \leq p_1 + 1 + b.$$

Plugging this into (9.1), we have

$$(p_1 + 1 + b)|L^{-1}(c_{k-p_1}) \cap X| \geq |X|(p_1 + 1).$$

This implies that

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \frac{|X|(p_1 + 1)}{p_1 + 1 + b}. \quad (9.2)$$

For each $c \in C_L - F$, $|L^{-1}(c)| \leq k + 1 - b$ (by definition of b). By Lemma 3.2, for $c \in F$, $|L^{-1}(c)| \leq k + p_1 + 2$. Hence

$$(2k+2)k \leq \sum_{v \in V} |L(v)| = \sum_{c \in C_L} |L^{-1}(c)| \leq |C_L - F|(k+1-b) + |F|(k+p_1+2). \quad (9.3)$$

Plugging $|C_L| = |V| - \lambda = 2k + 2 - \lambda$ and $|F| \leq k - p_1 - 1$ into (9.3), we have

$$(2k + 2)k \leq (2k + 2 - \lambda - (k - p_1 - 1))(k + 1 - b) + (k - p_1 - 1)(k + p_1 + 2). \quad (9.4)$$

(Note that the coefficient of $|F|$ in the right hand side of (9.3) is positive.)

This implies

$$b \leq \frac{(p_1 + 3 - \lambda - k)(k + 1) + (k - p_1 - 1)(k + p_1 + 2)}{k + p_1 + 3 - \lambda}. \quad (9.5)$$

If $\lambda = 2$, then since $p_1 \geq 3$, by plugging $|X| \geq p_1 - \lambda + 1$ (see Lemma 9.1) into (9.2), we have

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \frac{(p_1 - \lambda + 1)(p_1 + 1)}{p_1 + 1 + b} \geq \frac{(p_1 - 1)(p_1 + 1)}{p_1 + 1 + \frac{(p_1+1)(k-p_1-1)}{k+p_1+1}} = \frac{(p_1 - 1)(k + p_1 + 1)}{2k} \geq \frac{2(k + p_1 + 1)}{2k} > 1.$$

Since $|L^{-1}(c_{k-p_1}) \cap X|$ is an integer, $|L^{-1}(c_{k-p_1}) \cap X| \geq 2 = \lambda$ and we are done.

Therefore $\lambda \geq 3$ and $|C_L| \leq 2k - 1$. By Lemma 3.2, G has no 2-parts. By the same reason as (8.3), we have

$$p_1 \geq \frac{k + 1}{2}.$$

Combining (8.1) with Lemma 8.3, together with $p_1 \geq \frac{k+1}{2}$, we have

$$\frac{k - 3}{2} \geq k - p_1 - 1 \geq |F'| \geq \frac{k\lambda - |C_L|}{p_1 + 1} = \frac{k\lambda - (2k + 2 - \lambda)}{p_1 + 1} = \frac{(k + 1)\lambda - 2k - 2}{p_1 + 1}.$$

Hence

$$\lambda \leq \frac{\frac{(k-3)(p_1+1)}{2} + 2k + 2}{k + 1} = \frac{p_1 + 1}{2} + 2 - \frac{2(p_1 + 1)}{k + 1} < \frac{p_1 + 1}{2} + 1.$$

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press

Since λ is an integer,

$$\lambda \leq \frac{p_1}{2} + 1. \tag{9.6}$$

Therefore

$$p_1 \geq 2\lambda - 2 \geq \lambda + 1.$$

Plugging this into (9.5), we have

$$\begin{aligned} b &\leq \frac{(p_1+3-\lambda-k)(k+1)+(k-p_1-1)(k+p_1+2)}{k+p_1+3-\lambda} \\ &\leq \frac{(p_1+3-\lambda-k)(k+1)+(k-p_1-1)(k+p_1+2)}{k+4} \quad (\text{ as } p_1 \geq \lambda+1 \text{ }) \\ &= \frac{(p_1+1)(k-p_1-1)+(k+1)(2-\lambda)}{k+4} \\ &\leq \frac{\frac{k-3}{2}(p_1+1)+(k+1)(2-\lambda)}{k+4} \quad (\text{by (8.3), i.e., } p_1 \geq \frac{k+1}{2} \text{ }) \\ &= \frac{1}{2}(p_1+1-2\lambda) + \frac{2k+2+3\lambda-\frac{7}{2}(p_1+1)}{k+4} \\ &\leq \frac{1}{2}(p_1+1-2\lambda) + \frac{k+1/2}{k+4} \\ &< \frac{1}{2}(p_1+1-2\lambda) + 1. \end{aligned}$$

It follows from (9.6) that $p_1 \geq 2\lambda - 2$.
If $p_1 \in \{2\lambda-2, 2\lambda-1\}$, then $b = 0$. This implies that $|X| \geq p_1 - \lambda + 2$.
It follows from (9.2) that

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \frac{|X|(p_1+1)}{p_1+1+b} \geq \frac{(p_1-\lambda+2)(p_1+1)}{p_1+1} \geq \lambda.$$

If $p_1 \geq 2\lambda$, then

$$b \leq \frac{1}{2}(p_1+1-2\lambda) + \frac{1}{2} \leq \frac{1}{2}(p_1+1-2\lambda) + \frac{1}{2}(p_1+1-2\lambda) = p_1+1-2\lambda.$$

Hence

$$|L^{-1}(c_{k-p_1}) \cap X| \geq \frac{(p_1-\lambda+1)(p_1+1)}{p_1+1+b} \geq \frac{(p_1-\lambda+1)(p_1+1)}{2(p_1+1-\lambda)} = \frac{p_1+1}{2} \geq \lambda.$$

This completes the whole proof of Theorem 1.2.

This paper characterizes all non-k-choosable complete k-partite graphs G with $2k+2$ vertices. If the number of vertices of G increases, and the chromatic number remains k, then the choice number of G may increase. It was proved in [16] that k-chromatic graphs with $n \geq 2k+1$ vertices have choice number at most $\lceil \frac{n+k-1}{3} \rceil$. It would be interesting to characterize graphs for which this upper bound on the choice number is sharp.

Acknowledgement

We thank the referee for a careful reading of the manuscript and for many valuable comments that improved the presentation of this paper.

References

[1] B. Bollobás and A. J. Harris. List-colourings of graphs. Graphs Combin., 1(2):115–127, 1985.

[2] F. Chang, H. Chen, J. Guo and Y. Huang. On-line choice number of complete multipartite graphs: an algorithmic approach. Electron. J. Combin. 22 (2015), no. 1, Paper 1.6, 16p.

[3] H. Enomoto, K. Ohba, K. Ota and J. Sakamoto. Choice number of some complete multi-partite graphs. Discrete Math., 244(2002), no.1–3, 55–66.

[4] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979), 125–157, Congress. Numer., XXVI, Utilitas Math., Winnipeg, Man., 1980.

[5] F. Galvin. The list chromatic index of a bipartite multigraph. J. Combin. Theory Ser. B, 63(1995), no.1, 153–158.

[6] S. Gravier, F. Maffray. Graphs whose choice number is equal to their chromatic number. J. Graph Theory, 27(1998), no.2, 87–97.

[7] R. Häggkvist and A. Chetwynd. Some upper bounds on the total and list chromatic numbers of multigraphs. J. Graph Theory, 16(1992), no.5, 503–516.

[8] P. Huang, T.Wong, and X. Zhu. Application of polynomial method to on-line list colouring of graphs. European J Combin 33(5) (2012), 872–883.

[9] W. He, L. Zhang, D. W. Cranston, Y. Shen, and G. Zheng. Choice number of complete multipartite graphs $K_{3\star 3,2\star(k-5),1\star 2}$. Discrete Math., 308(23):5871–5877, 2008.

[10] T. R. Jensen and B. Toft. Graph coloring problems. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[11] H. A. Kierstead. On the choosability of complete multipartite graphs with part size three. Discrete Math., 211(2000), no.1–3, 255–259.

[12] S. Kim, Y. Kwon, D. Liu, and X. Zhu. On-line list colouring of complete multipartite graphs. Electron. J. Combin. 19 (2012), no. 1, Paper 41, 13 pp.

[13] A. V. Kostochka, M. Stiebitz and D. R.Woodall. Ohba's conjecture for graphs with independence number five. Discrete Math., 311(2011), no.12, 996–1005.

[14] J. Kozik, P. Micek, and X. Zhu. Towards an on-line version of Ohba's conjecture. European J. Combin., 36(2014), 110–121.

[15] J. A. Noel. Choosability of graphs with bounded order: Ohba's conjecture and beyond. Master's thesis, McGill University, Montréal, Québec, 2013.

[16] J. A. Noel, D. B. West, H. Wu and X. Zhu. Beyond Ohba's Conjecture: A bound on the choice number of k-chromatic graphs with n vertices. European J. Combin., 43(2015), 295–305.

[17] J. A. Noel, B. A. Reed, and H. Wu. A proof of a conjecture of Ohba. J. Graph Theory, 79(2015), no.2, 86–102.

[18] K. Ohba. On chromatic-choosable graphs. J. Graph Theory, 40(2002), no.2, 130–135.

[19] K. Ohba. Choice number of complete multipartite graphs with part size at most three. Ars Combin., 72 (2004), 133–139.

[20] B. Reed and B. Sudakov. List colouring of graphs with at most $(2 - o(1))\chi$ vertices. Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002), 587–603, Higher Ed. Press, Beijing, 2002.

[21] B. Reed and B. Sudakov. List colouring when the chromatic number is close to the order of the graph. Combinatorica, 25(2005), no.1, 117–123.

[22] Y. Shen, W. He, G. Zheng, and Y. Li. Ohba's conjecture is true for graphs with independence number at most three. Appl. Math. Lett. 22(2009), no.6, 938–942.

[23] Y. Shen, W. He, G. Zheng, Y. Wang and L. Zhang. On choosability of some complete multipartite graphs and Ohba's conjecture. Discrete Math., 308(2008), no.1, 136–143.

[24] V. G. Vizing. Coloring the vertices of a graph in prescribed colors. Diskret. Analiz, 1976, no.29, Metody Diskret. Anal. v Teorii Kodov i Shem, 3–10, 101.

[25] D. Zeilberger, The method of undetermined generalization and specialization illustrated with Fred Galvin's amazing proof of the Dinitz conjecture. American Mathematical Monthly. 103 (3): 233–239.

Zhejiang Normal University, Jinhua, China
e-mail: jialuzhu@zjnu.edu.cn.

Zhejiang Normal University, Jinhua, China
e-mail: xdzhu@zjnu.edu.cn.

https://doi.org/10.4153/S0008414X24001184 Published online by Cambridge University Press