proportion of the whole population; and it is not upon such lives that assurances and annuities are granted.

The mortality, too, among these classes is undoubtedly greater than that among those who rank higher in the social scale—as I have recently had an opportunity of showing, in a paper published in the Transactions of the Institute of Actuaries, in the course of which this question was entered into at considerable length. On the other hand, it might be urged that a table containing so large a number of lives exposed to an undue rate of mortality would be, for that very reason, more suited to the requirements of public Companies.

A discussion, however, even of this one question, would lead to an inquiry into the respective merits and demerits of the existing tables of mortality, and would cause me to engross far too much of the available space in the pages of the *Assurance Magazine*. The effect, too, of the exclusion from the census returns—notwithstanding all the care to prevent it—of a considerable proportion of the nomadic population of the country, as well as of numerous other influences, would have to be considered.

In conclusion, I may state that Mr. Finlaison—who, in connection with his recent Report, has evidently investigated and considered this subject most fully—is clearly of opinion, that "at the present day, the census, and registration of births, deaths, and marriages, cannot with prudence be adopted as the bases of the true measure of the value of life."

I am, Sir,
Your obedient servant,
H. W. PORTER.

*Alliance Assurance Company,*
*16th March,* 1861.

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## ON MR. GOMPERTZ'S LAW OF HUMAN MORTALITY, AND MR. EDMONDS'S CLAIMS TO ITS INDEPENDENT DISCOVERY AND EXTENSION.

*To the Editor of the Assurance Magazine.*

Sir,—The remarks which appeared from the pen of Professor De Morgan in the number of the *Assurance Magazine* for last July, must have attracted the notice, not only of those who are interested in the history of the theory of life contingencies, but of all your readers who wish to see improvements in science attributed to their actual originators. It is no unusual thing for the title to a discovery to be contested; and it not uncommonly appears that different persons have made the same discovery independently. This is sometimes an extremely difficult point to decide; but in the present case, the means of arriving at a satisfactory decision are unusually ample. It must be clear to all who have read Professor De Morgan's remarks and Mr. Edmonds's rejoinder, that the charge brought by the former against the latter has been completely substantiated: viz., that Mr. Edmonds, following in the footsteps of Mr. Gompertz, and familiar with his writings, "has adopted his ideas without anything approaching to a sufficient acknowledgment." On this point, nothing further remains to be said. But Mr. Edmonds, in the course of his defence, as it may be termed, has introduced many other matters, which seem to me to call for further notice, although dismissed by Professor De Morgan as having no bearing on the question raised by him.

Mr. Edmonds, then, has made a minute comparison of his processes and results with those of Mr. Gompertz; and has made many reflections to the disparagement of the latter (*vide* pp. 174–178), which, if allowed to remain unanswered, may have more weight than they deserve with some of your readers, and lead them to undervalue the labours of Mr. Gompertz. Having been led by Professor De Morgan's remarks to make, for my own satisfaction, a somewhat close comparison of the writings of Mr. Gompertz and Mr. Edmonds, I am induced by the above consideration to lay before your readers the conclusions at which I have arrived.

In the first place, I notice that, curiously enough, Professor De Morgan, who appears as the voluntary champion of Mr. Gompertz, does not do him full justice. Mr. Gompertz's paper in the *Transactions of the Royal Society* is full of misprints—many of them being of the most serious character. Two such are corrected by Professor De Morgan in the extract he has made (pp. 87, 88); but he has left several uncorrected, which give rise to the "obscurity" to which Mr. Edmonds refers (pp. 178, 181). In order to demonstrate this, and with the secondary object of making Mr. Gompertz's important theorem more generally known,* I subjoin its demonstration. It will be seen, that besides correcting these misprints, I have expanded the reasoning in some of the steps, and have in several instances substituted for the notation of Mr. Gompertz, that more generally used at the present time.

Let the rate of mortality at the age $x$ be denoted by $aq^x$, where $a$ and $q$ are constant quantities, and let $L_x$ be the number living at the age $x$; then the number dying in the small time $dx$ will be $aL_x \times q^x dx$; so that—

(1) $$dL_x = -aL_x \times q^x dx;$$

(2) $$\therefore \frac{1}{L_x} \cdot \frac{dL_x}{dx} = -a \times q^x.$$

(3) Integrating both sides, $$\log_e \frac{L_x}{d} = -\frac{aq^x}{\log_e q} = \frac{aq^x}{\log_e \frac{1}{q}};$$

$d$ being the constant introduced by integration.

Now, $\log_e \frac{L_x}{d} = \log_{10} \frac{L_x}{d} \cdot \log_e 10$, $\log_e \frac{1}{q} = \log_{10} \frac{1}{q} \cdot \log_e 10$; so that the last equation gives

(4) $$\log_{10} \frac{L_x}{d} = \frac{aq^x}{\log_{10} \frac{1}{q} \cdot (\log_e 10)^2}:$$

* Since writing the above, I have found that I have been anticipated in this object by Mr. Peter Gray, who has fully explained the nature and application of Mr. Gompertz's method of interpolation in a paper (*Assur. Mag.*, vol. vii., p. 121), which is well worthy of every student's attention. In comparing Mr. Gray's form of the demonstration with that given in the text, it must be borne in mind that the constant added in the integration being arbitrary, may either be written $+C$, as Mr. Gray has it, or $-\log_e d$, which is Mr. Gompertz's form.

(5) or, putting $\log_{10}\frac{1}{q}.(\log_e 10)^2 = \frac{a}{c}$,

(6) $$\log_{10}\frac{L_x}{d} = cq^x.$$

(7) Hence $$\frac{L_x}{d} = 10^{cq^x}.$$

And if $10^c = g$, or $g = \log_{10}^{-1}c$, *i.e.* the number whose common logarithm is $c$,

(8) $$\frac{L_x}{d} = g^{q^x},$$

(9) and $$L_x = dg^{q^x}.$$

On comparing the above with Mr. Gompertz's demonstration, as quoted by Professor De Morgan, it will be seen that the mysterious constant ($b$), which has caused so much perplexity, is simply a misprint for the sign of multiplication ($\times$). This will be at once admitted, when it is noticed that Mr. Gompertz writes down the *equation*

$$aL_x \times q^x\dot{x} = -(L_x)^{\cdot};$$

and then concludes that $$abq^x = -\frac{\dot{L}_x}{L_x}.$$

Professor De Morgan seems to have overlooked this circumstance, which proves that a constant $b$ could not have been introduced in the manner he suggests.

It will also be noticed that the constant $d$, which does not appear in Mr. Gompertz's process till step (6), correctly enters on the integration in (3). Again, in (5), Mr. Gompertz's $c$ should be $\frac{a}{c}$.

These corrections having been made, all "obscurity and ambiguity" disappear from Mr. Gompertz's process. Nor does that process contain, as Mr. Edmonds asserts (p. 181), "two superfluous and useless indeterminate constants." I have already disposed of $b$, and I shall presently show that $d$ is neither superfluous nor indeterminate.

The preceding corrections occurred to me on reading carefully Mr. Gompertz's process, as quoted by Professor De Morgan (p. 88). In order, however, to make the comparison already mentioned of Mr. Edmonds's writings with Mr. Gompertz's, I referred to the copy of Mr. Gompertz's papers in the library of the Institute of Actuaries. That copy, which was presented to the Institute by Mr. Gompertz, exhibits the precise corrections I have already indicated, in addition to those pointed out by Professor De Morgan, made in the margin of the volume—as I presume, by Mr. Gompertz himself. (There is, however, in step (5), an *over-correction* made.) A single glance at this copy will convince any person of the accuracy of my statement, that Mr. Gompertz's paper, as it appears in the *Transactions of the Royal Society*, is full of misprints. It is to be presumed that that gentleman, like so many others of scientific eminence, does not write a very legible hand. Perhaps also the Royal Society, at the time the papers in question were printed, followed the course adopted at the present time by a few Societies—happily not by the Institute of Actuaries—and did not allow the contributors to its *Transactions* the opportunity of correcting the proofs of their papers.

We have thus seen that there is no error in Mr. Gompertz's demonstration of the theorem that "if the rate of mortality follows the law $aq^x$, then the number living at any age $x$ will be given by the formula $dg^{q^x}$;" and that Mr. Edmonds's doubts as to its correctness have arisen from the fact of its being obscured by numerous misprints. Let us now examine some of Mr. Edmonds's remarks upon that demonstration, and it will be seen that they contain errors equally grave with those falsely imputed to Mr. Gompertz, and which cannot be similarly explained. In the first place, then, Mr. Edmonds speaks (p. 180) of "$b$ being introduced as arising in the process of integration." It will be noticed that $b$ makes its appearance in step (2), while the integration is not performed till (3); also that $d$ is the constant introduced by integration: so that the only conclusion appears to be that the remark just quoted was written under a misconception as to the real nature of the process of integration. This conclusion is strengthened by the perusal of the remarks (p. 178) on the limits of integration, and by the suggestion made (p. 184) of "an error committed in the process of integration, of which $b$ represents the correction." Here a confusion of ideas is betrayed, of which it is difficult, if not impossible, to trace the origin. A similar confusion of ideas pervades the remarks (p. 181) on the constants $a$, $b$, $\alpha$. The truth seems to be that Mr. Edmonds, having concentrated his attention for a long time upon a particular problem, with which it must be confessed he has shown a considerable degree of familiarity; and having arrived at a correct solution in a particular form, in which he has been aided by a knowledge of Mr. Gompertz's paper (*vide* p. 177, bottom), is yet unable, in consequence of want of familiarity with the principles of the integral calculus, to follow Mr. Gompertz in his somewhat more general way of treating the same question.

We next come to Mr. Edmonds's remarks upon the constant $d$. He terms this (p. 178) a "superfluous quantity," and speaks of it as constituting a "defect" in the formula. Instead of being such, it is really an essential part of the formula, *and as such is employed by Mr. Edmonds himself.* He uses the letter $g$ to denote the same quantity as Mr. Gompertz's $d$; and in arriving at his formula, determines $g$ so that $y=1$ when $x=0$. On the other hand, Mr. Gompertz leaves $d$ undetermined; since for his purpose, there was nothing to be gained by determining its value. (This, I conjecture, is what Mr. Edmonds means by calling $d$, in p. 181, an indeterminate constant; but mathematicians mean by an "indeterminate" quantity, one whose value cannot be ascertained.) If it were required to determine the value of $d$ in the formula $L_x=dg^{q^x}$, the process would be as follows:—Suppose the number living at the age $n$* to be $L_n$; then $L_n=dg^{q^n}$, whence $L_x=L_n(g^{q^x-q^n})$, or $L_x=L_n(g^{q^n})^{q^{x-n}-1}$. It must be noticed that in Mr. Gompertz's formula $x$ denotes the age measured from birth, while in Mr. Edmonds's $x$ is only measured from birth during the period of infancy, as will be seen by his example (p. 179). So far from $d$ being a defect in the formula $dg^{q^x}$, the formula $10^{\frac{k^2\alpha}{\lambda p}(1-p^x)}$ is, on the contrary, defective, because it assumes $y=1$ when $x=0$; and the defect in it is tacitly supplied by Mr. Edmonds himself in practice. Thus, in his example, he arrives at the result $y_{10}=1\div 1{\cdot}076823$ $(={\cdot}928658)$; and $x$ being measured from

* I use $n$ here in preference to Mr. Gompertz's $a$, because the letter $a$ has already been used to denote another quantity in the expression $aq^x$.

the age of 12, this should be the number living at the age of 22. Instead of this, Mr. Edmonds is compelled to introduce the quantity we have denoted above by $L_n$: he takes the number living at 12 to be 100,000 ($=L_{12}$), and deduces the number living at 22 to be 92865·8; as will be seen on reference to his Table of Mean Mortality. The above is not a point of great importance; but it shows that whatever defect does exist, is in Mr. Edmonds's formula, and not in Mr. Gompertz's. Mr. Edmonds, however, falls into a far more serious error (p. 180) in calling $d$ a particular value of $y$. It is easily shown that $y$ cannot possibly be equal to $d$. For, in the formula $y=dg^{q^x}$, if $y=d$, we have $g^{q^x}=1$; hence $q^x=0$, and $x=+\infty$ or $-\infty$, according as $q$ is $<$ or $>1$; values which are of course inadmissible.

Mr. Edmonds's words in speaking of $d$ are as follows (p. 178):—"The defect in Mr. Gompertz's formula, caused by the addition of ($d$), is the same as that which would exist in a table of discount of money at compound interest if any other basis were adopted than the value of the sum of £1 receivable ($x$) years hence." It would be generally considered that £1 was the *basis* of such a table,* and not the value of £1 receivable $x$ years hence; but without dwelling on this minor inaccuracy, I notice that in the passage just quoted, two very different things are confounded with each other, viz., the *formula* for the value of a sum of money due at the end of $x$ years, and the *table* by which that value would be practically found. It is true that a table in which any other basis than £1 were assumed, would be *inconvenient* in practice (though it could not be correctly described as *defective*); but on the other hand, a formula in which the sum to be received is taken as £1 (*i.e.* $(1+i)^{-x}$), would be defective; the correct formula being $s(1+i)^{-x}$.

Again, we read on the same page that Mr. Gompertz has changed the sign of $c$; whereas the true state of the case is that Mr. Edmonds has changed its sign for his own purposes. The quantity $c$, which does not appear in Mr. Edmonds's own investigation, is taken by Mr. Gompertz to be equal to $\dfrac{a}{\log_{10}\frac{1}{q}\cdot(\log_e 10)^2}$, which in Mr. Edmonds's notation would be $\dfrac{k^2a}{\lambda\frac{1}{p}}$, or $-\dfrac{k^2a}{\lambda p}$. The latter, when using the letter $c$ for the first time on p. 178, takes it equal to $+\dfrac{k^2a}{\lambda p}$, and then says that Mr. Gompertz has changed the sign of $c$!

I now pass on to the remaining portion of my subject—a general comparison of the results obtained by Mr. Gompertz and Mr. Edmonds. Here at the outset I must direct attention to a grossly unfair misquotation by the latter of Mr. Gompertz's words. He wishes to give your readers the impression, that Mr. Gompertz's discovery is imperfect by reason of its not stating any limits of age, within which the formula is to be applied with the same constants; and for this purpose he professes to quote (p. 174, top) a passage from Mr. Gompertz's paper. *No such sentence is to be found in*

* Similar to this, is Mr. Edmonds's use of the word "formula" to denote an *equation*, as $L_x=dg^{q^x}$, instead of the *expression* ($dg^{q^x}$), which forms the second member of the equation.

*that paper!* Mr. Gompertz's words are as follows:—"*If* the average exhaustions of a man's power to avoid death were such that at the end of equal infinitely small intervals of time, he lost equal portions of his remaining power to oppose destruction which he had at the commencement of those intervals, *then* at the age $x$ his power to avoid death, or the intensity of his mortality, might be denoted by $aq^x$." And again: "This equation [$L_x = dg^{q^x}$] between the number of the living, and the age, becomes deserving of attention, not in consequence of its hypothetical deduction, which in fact is congruous with many natural effects, as for instance, the exhaustions of the receiver of an air-pump by strokes repeated at equal intervals of time, but it is deserving of attention because it appears corroborated *during a long portion of life* by experience; as I derive the same equation from various published tables of mortality *during a long period of man's life*, which experience therefore proves that the hypothesis approximates to the law of mortality *during the same portion of life;* and in fact the hypothesis itself was derived from an analysis of the experience here alluded to." I have been thus particular in quoting the above passages (in which the italics are mine), because the question of misquotation is one upon which all your readers will be able to form an opinion, whereas it requires for the full appreciation of many of the points here raised, a greater knowledge of the higher mathematics than they can all be expected to possess. By comparing together the above two passages from Mr. Gompertz's paper, any person may judge of the "fairness" or the truth of Mr. Edmonds's assertion (p. 174) that "it might fairly be inferred from the above statement that the vital force of man, measured by the ratio of the living to the dying, is in a constant state of decay *from birth to the end of life* at one and the same uniform rate." The reason Mr. Gompertz has stated no definite limits of age is obviously because he did not believe them to be fixed in the same nearly invariable manner as Mr. Edmonds does. This brings me to the next point I have to notice—the improvements the latter claims to have made on the former's discovery.

Mr. Edmonds has convinced himself that in the formula for the mortality at any age, $aq^x$, $q$ has three fixed values, invariable or nearly so, under all circumstances, and for all populations; and he endeavours, by reiterated assertions, to convince his readers of the truth of the proposition. I say *assertions*, because he has not advanced a single particle of *proof* in support of his proposition. In this single distinction is summed up the difference, or rather the contrast, between the writings of Mr. Gompertz and Mr. Edmonds. All that the former advances is *rigorously proved;* all that the latter claims to have discovered is *simply asserted.* Mr. Gompertz describes all his computations so as to enable a person with only slight knowledge of the subject to repeat them and test their acccuracy: he shows how he has found the values of his constants $g$ and $q$; and he arranges his results so as to exhibit at a glance the degree of accuracy with which they agree with previously computed tables of mortality. Mr. Edmonds has done nothing of this kind: he simply states the results he has arrived at, and apparently expects them to be received without demonstration.

If Mr. Edmonds's proposition of three invariable values for $q$ were established, it would no doubt be justly entitled to the name of a *discovery*, and it would be a great advance on what Mr. Gompertz has done. But in the absence of any proof, it is impossible to treat the proposition other-



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wise than as an unproved hypothesis. If Mr. Edmonds possesses the means of proving his proposition, and publishes sufficient details of the extensive comparison of tables described by him on p. 172, the scientific public will then be in a position to judge of the truth or error of his proposition; and I for one shall not be slow to avow my reception of anything put forward by Mr. Edmonds when supported by sufficient proofs. As the case stands at present, it is impossible to admit the truth of Mr. Edmonds's proposition, or to countenance his use of the term "discovery" in describing the conclusions at which he has arrived.

It forms no part of my purpose to comment on the bad taste displayed by Mr. Edmonds in many of his remarks, nor do I think it necessary to allude particularly to his unjustifiable use of the phrase "*true* law of human mortality" to describe his hypothesis; for these are points which must force themselves on the notice of every intelligent reader. I will therefore conclude this part of my subject by pointing out two other inaccuracies into which Mr. Edmonds has fallen. He states (p. 177) that he believes Mr. Galloway to be the only person who has made a practical use of Mr. Gompertz's formula. This is by no means the case. Mr. Jellicoe, in the first volume of the *Assurance Magazine*, p. 166, has applied the formula to the adjustment of Mr. Neison's Table of Mortality deduced from the deaths among the officers of the Indian army. He has also used it in a modified form (*Assur. Mag.*, vol iv., p. 206) to graduate his own Table of Mortality derived from the experience of the Eagle Insurance Company; and the same method has been recently adopted by Mr. Dove to adjust the results deduced from the experience of the Royal Insurance Company. Mr. Farren has also employed it in his Life Contingency Tables. Again: Mr. Edmonds (p. 174) says that "nearly all he [Mr. Gompertz] offers to show is, how *interpolations* may be made for intermediate ages when the number of survivors at the beginning, and the number of survivors at the end, of a large interval of age are given." (*Conf.* p. 176, bottom.) Here Mr. Edmonds quite overlooks the fact, that it requires a knowledge of the number of the survivors at *three* ages at least, to apply Mr. Gompertz's method; which is an obvious deduction from the fact that the formula $dg^{q^x}$ contains *three* disposable constants. When we remember that Mr. Edmonds erroneously rejects one of those constants as superfluous, it appears less surprising that he should have made this serious blunder.

We have seen that it is impossible to admit, on the evidence adduced hitherto, that Mr. Edmonds has *extended* the theory of Mr. Gompertz: it remains now to examine his claims to an independent discovery of what is common to them. Taking then Mr. Edmonds's own statement (pp. 172–178), it appears that he ascertained that the rate of mortality at different ages might be represented with sufficient accuracy by the formula $ap^x$, and he then learnt that Mr. Gompertz had shown that the number of survivors at any age would be given by the formula $dg^{p^x}$. In other words, Mr. Edmonds *very nearly* discovered what had been previously discovered by Mr. Gompertz; or, in his own phrase, if he had only had a little more time, he would certainly have discovered it. It is of no avail to make conjectures (*vide* pp. 175, 176) as to the course by which Mr. Gompertz arrived at his formula. The *fact* remains that he did arrive at it and publish it, and that Mr. Edmonds first became acquainted with the formula through Mr. Gompertz's writings. It was no doubt a great disappointment to Mr. Edmonds to find that he had been anticipated in this manner, but it may be a consolation to him to reflect that his position is by no means a singular one. There has probably been scarcely a single discovery of any kind made, but some unfortunate man has been on the point of making it too, and would have done so, if he had only had sufficient time allowed.

In conclusion, I beg to state that in all I have said, I have been influenced by no personal feeling towards Mr. Edmonds. I wish to allow him all the credit justly due to him. I believe that in accomplishing the object he had in view—viz., that of forming a new hypothetical table of mortality—he has shown considerable judgment and skill; and I shall willingly admit the universality of his constants whenever I see sufficient proofs of their accuracy produced. My object has been simply truth; and if, in the course of my remarks, I have been led to use language which may seem severe, I regret the necessity I have been under. I feel I have trespassed rather unreasonably on your space; but trust that what I have said will enable others who have but little time to devote to such subjects, to come to a correct conclusion as to the claims put forward by Mr. Edmonds. My task would have been much lighter if Mr. Edmonds had confined himself to an explanation of his own theory; for then I should not have thought it necessary to dwell upon the numerous errors into which he has fallen. This course has been rendered necessary by his uncalled for criticisms of Mr. Gompertz. Your readers will probably be better able to form an opinion as to the proper weight to be assigned to those criticisms, the whole of which I have not thought it necessary to review, when they learn that the writer of them has himself fallen into many serious errors.

I am, Sir,

Your obedient servant,

T. B. SPRAGUE.

25, *Pall Mall,*
*7th February,* 1861.

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## EXPRESSION FOR THE VALUE OF A TERM ASSURANCE, LIFE AGAINST LIFE.

*To the Editor of the Assurance Magazine.*

Sir,—Perhaps the following brief expression for the value of a short period assurance on ($x$) against ($y$) may be worth a place in your *Journal.*

By the combined use of the ordinary Commutation Tables in Jones, and the tables of Gray, Smith, and Orchard, for such an assurance to be current for $m$ years and paid for in $n$ annual premiums, the formula becomes

$$\frac{D_{xy}\,A_{\overline{\overset{1}{x}.y}}-D_{x+m,\,y+m}\,A_{\overline{\overset{1}{x+m},\,y+m}}}{N_{x-1,\,y-1}-N_{x+n-1,\,y+n-1}}.$$

I am, Sir,

Your most obedient Servant,

H. AMBROSE SMITH.

*Aberdeen, 7th May,* 1861.