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Linear Equations with Small Prime and Almost Prime Solutions

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Abstract. Let b_1, b_2 be any integers such that $\gcd(b_1, b_2) = 1$ and $c_1|b_1| < |b_2| \le c_2|b_1|$, where c_1, c_2 are any given positive constants. Let n be any integer satisfying $\gcd(n, b_i) = 1$, $i = 1, 2$. Let P_k denote any integer with no more than k prime factors, counted according to multiplicity. In this paper, for almost all b_2, we prove (i) a sharp lower bound for n such that the equation $b_1 p + b_2 m = n$ is solvable in prime p and almost prime $m = P_k$, $k \ge 3$ whenever both b_i are positive, and (ii) a sharp upper bound for the least solutions p, m of the above equation whenever b_i are not of the same sign, where p is a prime and $m = P_k$, $k \ge 3$.

1 Introduction

Let b be an integer and b_1, b_2, b_3 be non-zero integers. Many mathematicians considered the solvability and small prime solutions p_1, p_2, p_3 of the linear equation

$$b_1 p_1 + b_2 p_2 + b_3 p_3 = b. \tag{1.1}$$

The problem on bounds for prime solutions of equation (1.1) was first raised by Baker in connection with his well-known work [1] on the solvability of certain Diophantine inequalities involving primes. Later, this problem was studied by many authors (see [3, 6, 8, 9]).

In 1973, Chen [2] proved that every sufficiently large even integer n can be represented as a sum of a prime and a P_2. As usual, here and later, P_k denotes any integer with no more than k prime factors, counted according to multiplicity. In this paper, we consider the solvability and small solutions of the linear equation

$$b_1 p_1 + b_2 m = n, \tag{1.2}$$

where p is a prime and m is an almost prime.

In order to avoid degenerate cases, we need to impose certain local conditions to equation (1.2). Let b_1, b_2 be any integers such that

$$\gcd(b_1, b_2) = 1 \quad \text{and} \quad c_1|b_1| < |b_2| \le c_2|b_1|, \tag{1.3}$$

where c_1, c_2 are any given positive constants. Let n be any integer satisfying

$$\gcd(n, b_i) = 1, i = 1, 2. \tag{1.4}$$

Let M be a sufficiently large number, which will be specified later. We obtain the following.

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Theorem 1 *If both b_1 and b_2 are positive and satisfy* (1.3), *and n satisfies* (1.4), *then for almost all b_2 with $M/4 < b_2 \le M$, except for $O(M \log^{-A} M)$ values, equation* (1.2) *is solvable for prime p and almost prime $m = P_3$, provided that $n \ge |b_1||b_2|^{7.5}$.*

If b_1, b_2 are not of the same sign and satisfy (1.3) *and n satisfies* (1.4), *then for almost all b_2 with $M/4 < b_2 \le M$, except for $O(M \log^{-A} M)$ values, equation* (1.2) *is solvable for prime p and almost prime $m = P_3$ satisfying* $\max\{m, p\} \le |b_2|^{7.5}$.

We can generalize Theorem 1 to the following.

Theorem 2 *If both b_1 and b_2 are positive and satisfy* (1.3), *and n satisfies* (1.4) *then for almost all b_2 with $M/4 < b_2 \le M$, except for $O(M \log^{-A} M)$ values, equation* (1.2) *is solvable for prime p and almost prime $m = P_k$, provided that*

$$n \ge |b_1||b_2|^K, \quad \textit{where} \quad K \ge \frac{2(k + 1 - \log 4/\log 3)}{k - 1 - \log 4/\log 3}, \quad k \ge 3.$$

If b_1, b_2 are not of the same sign and satisfy (1.3) *and n satisfies* (1.4), *then for almost all b_2 with $M/4 < b_2 \le M$, except for $O(M \log^{-A} M)$ values, equation* (1.2) *is solvable for prime p and almost prime $m = P_k$ satisfying* $\max\{m, p\} \le |b_2|^K$.

The first result on this problem was due to Liu [7, Theorem 1.1], who proved the following.

Theorem *If b_1, b_2 are co-prime positive integers, and m is either* 1 *or* 2 *satisfying*

$$b_1 + b_2 \equiv m \pmod 2,$$

then for any $\delta > 0$, there exists a positive constant C depending only on δ such that

$$b_1 p - b_2 P_3 = m \tag{1.5}$$

has a solution in p, P_3, each less than $C^{(\max b_j)^\delta}$.

Later, Coleman [4] improved the above result and obtained that for three pairwise co-prime b_1, b_2, m and $2|b_1 b_2 m$, taking P_2 instead of P_3 in (1.5), the equation still has a solution with p and P_2 each less than $\max\{N_0, b_1^B, b_2^B, c|m|\}$, where N_0 and B are effectively computable constants.

To prove Theorem 1, we shall apply the sieve method, which has been used by many authors (see [5], for details). Since the proof of Theorem 2 is similar to that of Theorem 1, we shall omit it and only prove Theorem 1 in the next sections.

Notation Throughout this paper, N is a sufficiently large number, ε is a sufficiently small positive constant, and c, c_1 and c_2 are positive constants. The letter A with or without subscripts always denotes sufficiently large positive constants, and p with or without subscripts always denotes prime numbers. Let $\nu(n)$ be the number of distinct prime factors of n, and let P_k denote any integer with no more than k prime factors, counted according to multiplicity. Let $(a, b) = \gcd(a, b)$, $a/b = \frac{a}{b}$, and $p \equiv n\,(d)$ means $p \equiv n \pmod d$.

As usual, $\varphi(q)$ and $\mu(q)$ stand for the functions of Euler and Möbius respectively, and $\tau(d)$ stands for the divisor function.

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2 Some Preliminary Lemmas

Let $\mathcal{A}$ denote a finite set of integers, which will be specified later, and $\mathcal{P}$ an infinite set of prime numbers. Let $z \geq 2$, and put

$$P(z) = \prod_{\substack{p<z\\ p\in\mathcal{P}}} p, \quad S(\mathcal{A}, z) = \sum_{\substack{a\in\mathcal{A}\\ (a,P(z))=1}} 1,$$

$$\mathcal{A}_d = \{a : a \in \mathcal{A}, d|a\}.$$

Lemma 1 *Suppose*

$$|\mathcal{A}_d| = \frac{\omega(d)}{d}X + r_d,$$

and assume the following conditions hold:

$$1 \leq \frac{1}{1-\frac{\omega(p)}{p}} \leq A_1; \tag{2.1}$$

$$-A_2 \log\log 3X \leq \sum_{v\leq p\leq w} \frac{\omega(p)}{p}\log p - \log\frac{w}{v} \leq A_2 \quad \textit{for } 2 \leq v \leq w; \tag{2.2}$$

$$\sum_{z\leq p<y} |\mathcal{A}_{p^2}| \leq A_3\Big(\frac{X\log X}{z} + y\Big) \quad \textit{for } 2 \leq z \leq y; \tag{2.3}$$

$$\sum_{d<\frac{X^\alpha}{\log^{A_4} X}} \mu^2(d)3^{\nu(d)}|r_d| \leq A_5\frac{X}{\log^2 X}, \quad X \geq 2,\ 0 < \alpha < 1. \tag{2.4}$$

Let δ be a real number satisfying $0 < \delta \leq \frac{2}{3}$, and let $r \geq 2$ be so large that $|a| \leq X^{\alpha(\Lambda_r-\delta)}$ for all $a \in \mathcal{A}$, where

$$\Lambda_r = r + 1 - \frac{\log 4/(1+3^{-r})}{\log 3}.$$

Then we have

$$|\{P_r : P_r \in \mathcal{A}\}| \geq \frac{\delta}{\alpha}\prod_{p}\frac{1-\omega(p)/p}{1-1/p}\frac{X}{\log X}.$$

This is [5, Theorem 9.3].

Lemma 2 *Let*

$$\pi(x; d, l) = \sum_{\substack{p\leq x\\ p\equiv l \pmod d}} 1, \quad (l, d) = 1.$$

Then for any given constant $A > 0$, there exists a constant $B = B(A) > 0$ such that

$$\sum_{d\leq D} \tau(d)\Big|\pi(x; d, l) - \frac{Lix}{\varphi(d)}\Big| \ll \frac{x}{\log^A x},$$

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where

$$Lix = \int_2^x \frac{dt}{\log t}, \quad D = \frac{x^{1/2}}{\log^B x}.$$

This follows from [10, Theorem 8.2].

Lemma 3 *With the notations in Lemma 2, let*

$$R(D,q) = \sum_{d \le \frac{D}{q}} \mu^2(d) 3^{\nu(d)} \Big| \pi(x; dq, l) - \frac{Lix}{\varphi(dq)} \Big|.$$

Then for any $A > 0$ and $0 < \theta < 1/2$, there exists a constant $B = B(A) > 0$ such that for $q \le x^\theta$, except for $O(x^\theta \log^{-A} x)$ values, we have

$$R(D,q) \ll \frac{x}{q \log^A x}, \quad \text{where } D = \frac{x^{1/2}}{\log^B x}.$$

Proof Let

$$r_{d,q} = \pi(x; dq, l) - \frac{Lix}{\varphi(dq)}.$$

By Lemma 2, we have

$$\begin{aligned} \sum_{q \le x^\theta} \sum_{d \le \frac{D}{q}} r_{d,q} &= \sum_{q \le x^\theta} \sum_{d \le \frac{D}{q}} \Big| \pi(x; dq, l) - \frac{Lix}{\varphi(dq)} \Big| \\ &\ll \sum_{d \le D} \tau(d) \Big| \pi(x; d, l) - \frac{Lix}{\varphi(d)} \Big| \\ &\ll x \log^{-5A} x. \end{aligned}$$

Then we have

$$\begin{aligned} \sum_{q \le x^\theta} R(D,q) &= \sum_{q \le x^\theta} \sum_{\substack{d \le D/q \\ 3^{\nu(d)} \ge \log^{3A} x}} + \sum_{q \le x^\theta} \sum_{\substack{d \le D/q \\ 3^{\nu(d)} < \log^{3A} x}} \mu^2(d) 3^{\nu(d)} r_{d,q} \\ &\le \frac{1}{\log^{3A} x} \sum_{q \le x^\theta} \sum_{\substack{d \le D/q \\ 3^{2\nu(d)} \ge \log^{3A} x}} \mu^2(d) 3^{2\nu(d)} r_{d,q} + \log^{3A} x \sum_{q \le x^\theta} \sum_{d \le D/q} r_{d,q} \\ &\ll x \log^{-3A+1} x \sum_{q \le x^\theta} \frac{1}{q} \sum_{d \le D/q} \frac{\mu^2(d) 3^{2\nu(d)}}{d} + x \log^{-2A} x \\ &\ll x \log^{-3A+1} x \sum_{q \le x^\theta} \frac{1}{q} \sum_{n \le x/q} \frac{\tau^4(n)}{n} + x \log^{-2A} x \ll x \log^{-2A} x, \end{aligned}$$

where we have used the fact (see [10]) that $\mu^2(n)3^{2\nu(n)} \le \tau^4(n)$ and

$$\sum_{n\le x} \frac{\tau^r(n)}{n} \ll (\log x)^{2^r}.$$

Thus by the above, we have

$$\sum_{\substack{q\le x^\theta \\ R(D,q)>\frac{x}{q\log^A x}}} 1 \ll \frac{\log^A x}{x} \sum_{q\le x^\theta} qR(D,q) \ll \frac{x^\theta \log^A x}{x} \sum_{q\le x^\theta} R(D,q) \ll x^\theta \log^{-A} x.$$

So Lemma 3 is proved. ■

3 Proof of Theorem 1

Let N be a sufficiently large number with $N \ge \max\{|b_1|^{7.5}|b_2|, |b_1||b_2|^{7.5}\}$ that also satisfies the following hypotheses:

(i) If b_1, b_2 are positive, then $n \ge 4\max\{b_1, b_2\}$, and

$$N = \min\Big\{\frac{\varphi(b_1)n}{b_1}, \frac{\varphi(b_2)n}{b_2}\Big\}.$$

(ii) If b_1, b_2 are not of the same sign, then $N \ge 4\max\{|n|, |b_1|, |b_2|\}$.

Let $N_i = \frac{N}{\varphi(b_i)}$, $i = 1, 2$, and define

$$\mathcal{A} = \{a : b_1 p + b_2 a = n, N_1/4 < p \le N_1, N_2/4 < a \le N_2\},$$
$$\mathcal{A}_d = \{a : d|a, a \in \mathcal{A}\}.$$

We have

$$\begin{aligned} |\mathcal{A}_d| &= \big|\{p : b_1 p \equiv n\,(b_2 d),\ (d, nb_1) = 1,\ N_1/4 < p \le N_1\}\big| \\ &= \big|\{p : p \equiv \overline{b_1} n\,(b_2 d),\ (d, nb_1) = 1,\ N_1/4 < p \le N_1\}\big|, \end{aligned}$$

where $\overline{b_1}$ is an integer satisfying $b_1\overline{b_1} \equiv 1\,(b_2 d)$.

By Lemma 2, we have $|\mathcal{A}_d| = \frac{\omega(d)}{d}X - r_d$, where $X = \frac{1}{\varphi(b_2)}(LiN_1 - Li(N_1/4))$,

$$\omega(d) = \frac{\varphi(b_2)d}{\varphi(b_2 d)}, \mu(d) \ne 0, (d, nb_1) = 1, \tag{3.1}$$

and

$$r_d = \pi(N_1/4, N_1; b_2 d, \overline{b_1} n) - \frac{1}{\varphi(b_2 d)}(LiN_1 - Li(N_1/4)), \mu(d) \ne 0, (d, nb_1) = 1,$$

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where

$$\pi(y,x;d,l)=\sum_{\substack{y<p\le x\\ p\equiv l\,(d)}}1,\ (l,d)=1.$$

By Lemma 3, for almost all $b_2 \le N_1^{\frac{1}{7.5}}$, except for $O(N_1^{\frac{1}{7.5}}\log^{-A}N_1)$ values, we have

$$\sum_{d\le\frac{D}{b_2}}\mu^2(d)3^{\nu(d)}|r_d| \ll \frac{N_1}{b_2\log^A N},$$

where $D=\frac{N_1^{1/2}}{\log^B N}$.

Thus condition (2.4) in Lemma 1 holds.

By (3.1), we have

$$\omega(p)=\frac{\varphi(b_2)p}{\varphi(b_2p)}=\begin{cases}\frac{1}{\varphi(p)} & \text{if } (p,b_2)=1,\\ 1 & \text{if } (p,b_2)\ne 1.\end{cases}$$

Then it is easy to check that conditions (2.1) and (2.2) hold. We have

$$\begin{aligned}\sum_{\substack{z<p<y\\ p\in\mathcal{P}}}|\mathcal{A}_{p^2}| &\le \sum_{\substack{z<p<y\\ p\in\mathcal{P}}}\ \sum_{\substack{m\le N_1\\ b_1m\equiv n\,(b_2p^2)}}1\\ &\le \sum_{\substack{z<p<y\\ p\in\mathcal{P}}}\Big(\frac{N_1}{b_2p^2}+1\Big)\le\frac{N_1}{b_2z}+y\le\frac{X}{z\log X}+y.\end{aligned}$$

By the above, condition (2.3) also holds. So far, we can prove Theorem 1 by Lemma 1.

Let $\Lambda_3=3+1-\frac{\log 4/(1+3^{-3})}{\log 3}$, then $\Lambda_3>3+1-\frac{\log 4}{\log 3}$. For $D=N_1^{1/2}\log^{-B}N$ and $b_2\le N_1^{1/7.5}$, we have

$$d\le\frac{D}{b_2}\ll X^{11/26}\log^{-B}X.$$

For $a\in\mathcal{A}$, we have $a\le N_2\le X^{7.5/6.5}$. Since

$$\frac{11}{26}\Big(3+1-\frac{\log 4}{\log 3}\Big)>\frac{7.5}{6.5},$$

we can find a small $\delta>0$, such that $\frac{11}{26}(\Lambda_3-\delta)\ge\frac{7.5}{6.5}$. Thus by Lemma 1, we have

$$|\{P_3:P_3\in\mathcal{A}\}|\ge\frac{\delta}{\alpha}\prod_p\frac{1-\omega(p)/p}{1-1/p}\frac{X}{\log X},$$

where $\alpha=11/26$.

Theorem 1 follows. ■

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