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ON A CLASS OF ONE-RELATOR GROUPS

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1. Introduction. In this paper, we consider the class of groups $G(l, m; k)$ which are defined by the presentation

$$\langle a, t;\ t^{-1}a^{-k}ta^{l}t^{-1}a^{k}t = a^{m}\rangle,$$

where k, l, m are integers, and $|l| > m > 0$, $k > 0$. Groups in this class possess many properties which seem unusual, especially for one-relator groups. The basis for the results obtained below is the determination of endomorphisms.

For certain of the groups, we are able to calculate their automorphism groups. One consequence of this is to produce examples of one-relator groups with infinitely generated automorphism groups. This answers a question raised by G. Baumslag (in a colloquium lecture at the University of Waterloo). Our examples are, perhaps, the simplest possible; J. Lewin [**10**] has found an example of a finitely presented group with an infinitely generated automorphism group. More specifically, it is shown, in Theorem 3.3, that when $m = 1$ or $m \nmid l$, every endomorphism of $G(l, m; k)$ is either onto a cyclic group, or else it is an automorphism (so that the group is Hopfian). Moreover, the outer automorphism group is isomorphic to a certain stem product of cyclic groups when $m > 1$, and to the additive group of rational numbers with denominator a non-negative power of l when $m = 1$ (the latter group is infinitely generated).

A further result, Theorem 3.4, shows that when m divides both l and k, and m is coprime l/m, the groups $G(l, m; k)$ are non-Hopfian. It has been a matter for conjecture whether there exist one-relator non-Hopfian groups other than the well known examples of [**4**].

Lastly, in Theorem 3.5, we provide an infinite sequence of non-isomorphic, Hopfian, one-relator groups, with the property that any successor in the sequence is a proper quotient of any predecessor by the normal closure of a single element.

The group $G(l, m; k)$ is an HNN-extension of the group $H(l, m)$ which is defined by the presentation

$$\langle a, b;\quad b^{-1}a^{l}b = a^{m}\rangle,$$

and studied by G. Baumslag and D. Solitar [**4**]. Endomorphisms of these groups have been determined by M. Anshel ([**1**], [**2**] or [**3**]). Recently, A. Karrass and D. Solitar have been able to calculate the outer automorphism

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group of $H(l, m)$ as a consequence of their work on treed HNN-groups with a unique decomposition (see also [**9**]). Unfortunately, their methods are not applicable to the groups $G(l, m; k)$. For the case that m is a proper divisor of l, their remains the problem of showing that certain endomorphisms are not epimorphisms.

We mention that the results of [**12**] can be used to show that the groups $G(l, m; k)$ $(|l| > m)$ are not residually finite. G. Baumslag has observed, in [**5**], that the only finite quotients of the groups $G(2, 1; k)$ are cyclic. One-relator presentations of some of the groups $G(l, m; k)$ have been studied by A. M. Brunner [**7**].

Our approach is similar to that of D. J. Collins [**8**], who has shown that if w is a word in the free group on a, b which is not a primitive or a proper power, then the group with presentation

$$\langle a, b; t: t^{-1}w^l t = w^m \rangle$$

is Hopfian.

The methods used below involve the techniques of HNN-extensions. We refer to [**13**] or [**14**]; reference may also be made to [**8**] or [**9**].

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2. Preliminaries. Let $G = G(l, m; k)$ denote the group with presentation

$$\langle a, t; t^{-1}a^{-k}ta^l t^{-1}a^k t = a^m \rangle,$$

where k, l, m are non-zero integers and $|l| \neq |m|$. Since the groups $G(l, m; k)$, $G(m, l; -k)$, $G(-l, -m; k)$, and $G(l, m; -k)$ are isomorphic, there is no loss in generality to assume below the normalization $|l| > m > 0$ and $k > 0$ without explicit mention.

Now G may be presented in the form

$$\langle a, b; t: b^{-1}a^l b = a^m, b = t^{-1}a^k t \rangle,$$

which exhibits G as an HNN-extension having base

$$H = \langle a, b; b^{-1}a^l b = a^m \rangle$$

and associated subgroups $H_{-1} = \text{sbgp}\{a^k\}$, $H_1 = \text{sbgp}\{b\}$. In addition, H is itself an HNN-extension with base $K = \text{sbgp}\{a\}$, and associated subgroups $K_{-1} = \text{sbgp}\{a\}$, $K_1 = \text{sbgp}\{a^m\}$.

Let N denote the normal closure of a in H; then H/N is an infinite cyclic group with generator bN,

$$N = \text{sbgp}\{a_i: a_i = b^{-i}ab^i; i = 0, \pm 1, \dots\},$$

and H is the semidirect product $N\text{sbgp}\{b\}$. We shall use the analysis due to W. Magnus of N as a treed product (see § 4.4 of [**10**]).

The remainder of this section deals with several technical results on conjugacy in G and H. The reader is referred to [**2**] for an alternative approach to conjugacy in H.

It should be noted that, throughout, exponents are always integers.

LEMMA 2.1. *The centralizer of b^p in H is* $\text{sbgp}\{b\}$, *if $p \neq 0$.*

Proof. Let w commute with b^p. We may assume that w is b-reduced with smallest b-length amongst all words $b^r w b^s$. Then, since $b^{-p} w b^p = w$, Britton's Lemma implies that w is b-free. Consequently, $w = a^{l^p h} = a^{m^p h}$, which implies $h = 0$ since H is torsion free.

Let α and β be non-negative integers or ∞, and put

$$N_{\alpha,\beta} = \text{sbgp}\{a_{-\alpha}, a_{-(\alpha+1)}, \ldots, a_{\beta-1}, a_\beta\}.$$

If r and s are integers, *the exponent of r in s* is the largest non-negative integer e such that r^e divides s when $r \neq 1$, and is ∞ when $r = 1$.

LEMMA 2.2. *Let $C(a^n)$ denote the centralizer of a^n in N, and put $d =$* $\gcd(l, m)$. *Then $C(a^n) = N_{\alpha,\beta}$ where*

(1) *if $d \nmid n$, then $\alpha = \beta = 0$;*

(2) *if $d \mid n$ and $m \neq 1$, then α is the exponent of l/d in n/d, and β is the exponent of m/d in n/d;*

(3) *if $m = 1$, then $\alpha = \beta = \infty$.*

Proof. The inclusion $N_{\alpha,\beta} \subseteq C(a^n)$ is an immediate consequence of the relations ${a_{i+1}}^l = {a_i}^m$. To establish the reverse inclusion, suppose $w \in C(a^n)$ but $w \notin N_{\alpha,\beta}$. We may premultiply w by a suitable element of $N_{\alpha,\beta}$, so that we may now assume that w, in b-reduced form, has initial segment b^σ with $\sigma \leqq -(\alpha + 1)$ or $\sigma \geqq \beta + 1$ (when $m \neq 1$). Since $w^{-1}a^n w = a^n$, Britton's Lemma implies $b^{-\sigma}a^n b^\sigma \in K$. But an examination of the cases that occur shows that this last inclusion is invalid.

LEMMA 2.3. *If z^n $(n \neq 0)$ is conjugate in H to an element of* $\text{sbgp}\{a\}$ *(or* $\text{sbgp}\{b\}$*), then so is z.*

Proof. Suppose first that z^n is conjugate to an element of $\text{sbgp}\{a\}$. Replacing z by a conjugate, if necessary, we may assume z is both b-reduced and cyclically b-reduced; but then z^n is also so reduced, whence $z^n \in \text{sbgp}\{a\}$, and therefore $z \in \text{sbgp}\{a\}$.

Next, assume z^n is conjugate to an element of $\text{sbgp}\{b\}$; say, $u^{-1}z^n u = b^p$, where $u \in H$ and $p \neq 0$. Thus $u^{-1}zu$ centralizes b^p; therefore $u^{-1}zu \in \text{sbgp}\{b\}$ by 2.1.

LEMMA 2.4. *Suppose $u^{-1}a^r u = a^s$ and $u = b^q d$ with $d \in N$. Then $rm^q = sl^q$ and $d \in C(a^n)$.*

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Proof. This is established by using induction on the b-length of u.

LEMMA 2.5. *Let w be a t-reduced word in G such that $w^{-1}a^n w \in H$ for some $n \neq 0$. Then, either $w \in H$, or else $w = utv$ where $u \in H$, $v \in N$, and $t^{-1}u^{-1}a^n ut \in \mathrm{sbgp}\{b\}$.*

Proof. If not, we may write w in t-reduced form

$$ut^{\epsilon}vt^{\delta}\ldots \quad (u \in H, v \in N,\ \epsilon = \pm 1 \text{ and } \delta = \pm 1).$$

Since $w^{-1}a^n w$ is not t-reduced, $u^{-1}a^n u \in H_{-\epsilon}$. However, $n \neq 0$ and H/N is infinite cyclic; thus $\epsilon = 1$ and $t^{-1}u^{-1}a^n ut = b^h$ for some $h \neq 0$. Now $v^{-1}b^h v \in H_{-\delta}$; by arguing as above, we obtain $v^{-1}b^h v = b^h$ and $\delta = -1$.

Therefore $v = 1$ by 2.1, and $ut^{\epsilon}vt^{\delta}\ldots$ is not t-reduced, which is a contradiction.

3. Endomorphisms. The main step in characterizing endomorphisms of the groups $G(l, m; k)$ is taken in the following lemma.

LEMMA 3.1. *Let $G = G(l, m; k)$, and let φ be an automorphism of G. Then, either the image of φ is cyclic, or else to within an inner automorphism*

$$a\varphi = a^r,\ b\varphi = c^{-1}bc,\ t\varphi = b^p dtc,$$

where $r \neq 0$, $p > 0$, $rm^p = l^p$, $d \in C(a^k)$, $c \in N$, and $b^{-1}c^{-1}bc \in C(a^{rm})$.

Proof. Since φ is an endomorphism

$$(1) \quad (b\varphi)^{-1}(a\varphi)^l(b\varphi) = (a\varphi)^m$$

and

$$(2) \quad b\varphi = (t\varphi)^{-1}(a\varphi)^k(t\varphi).$$

Assume the image of φ is not cyclic, or in other words, $a\varphi \neq 1$. The first thing we establish is that, up to an inner automorphism, $H\varphi \subseteq H$. Using the Collins Conjugacy Theorem (Theorem 2 of [**13**]), and (1), we may assume that $a\varphi \in H$. If, however, $b\varphi \notin H$, then (1) is not t-reduced and a conjugate of $(a\varphi)^l$ by an element of H lies in H_{ϵ} ($\epsilon = \pm 1$). Using 2.3, conjugating by an element of H and then t, if necessary, we obtain $a\varphi = a^r$ ($r \neq 0$). It now follows from 2.5 that $b\varphi \in H$.

In any case, we may assume that $H\varphi \subseteq H$. Applying the Collins Conjugacy Theorem to the HNN-extension H, we may, if necessary, conjugate by an element of H to achieve $a\varphi = a^r$ ($r \neq 0$) with $b\varphi \in H$. It follows from 2.4 that $b\varphi = bg$ with $g \in C(a^{rm})$. From (2) we see that $t\varphi \notin H$, and so 2.5 gives $t\varphi = utc$ with $u \in H$, $c \in N$, and $u^{-1}a^{rk}u = a^{hk}$ for some h. Using this information, and substituting in (2), from the fact that H/N is infinite cyclic we obtain $h = 1$ and $bg = c^{-1}bc$. Finally, apply 2.4 to obtain $u = b^p d$ and $rm^p = l^p$.

LEMMA 3.2. *For each* $w \in C(a^k)$, *the function*

$$\theta(w) : a \to a,\ b \to b,\ t \to wt$$

extends to an automorphism of G. The map $w \to \theta(w)$ *defines a one to one homomorphism of* $C(a^k)$ *into the automorphism group of G. If* $\theta(w)$ *is inner,* $w = 1$.

Proof. The first statements are established by routine calculations. The last follows because 2.1 and 2.5 imply that the centralizer of H in G is trivial.

The following theorem establishes that certain of the groups $G(l, m; k)$ are Hopfian.

THEOREM 3.3. *Let* $G = G(l, m; k)$ *and suppose that either* $m = 1$ *or* $m \nmid l$. *Then every endomorphism of G is either onto a cyclic group, or else is an automorphism of G. Furthermore, the outer automorphism group of G is isomorphic to* $C(a^k)$.

Proof. Suppose φ is an endomorphism of G whose image is not cyclic, and of the form given in Lemma 3.1. By Lemma 3.2 it suffices to show that (up to an inner automorphism) φ has the form $\theta(w)$ for some $w \in C(a^k)$. Now, by Lemma 2.2, $N_{\alpha+1,\beta} = C(a^r)$, and also $C(a^{rm}) = N_{\alpha,\beta}$. But $b^{-1}c^{-1}bc \in C(a^{rm})$, so we deduce that $c \in C(a^r)$. We could follow φ by the inner automorphism induced by c, so we may assume that

$$a\varphi = a^r,\ b\varphi = b,\ t\varphi = cb^p dt$$

where $c \in C(a^r)$, $r \neq 0$, $d \in C(a^k)$ and $rm^p = l^p$. When $m \nmid l$, it follows that $r = 1$, $p = 0$, and φ is of the required form. When $m = 1$, we have that N is abelian and $r = l^p$; in this case follow φ by the inner automorphism induced by b^p, and this completes the proof.

The outer automorphism groups of Theorem 3.3 admit of a more explicit description. Using Lemma 2.2 and Magnus' Freiheitssatz (see [**11**], Theorem 4.10), we can write down a presentation for $C(a^k)$. When $m = 1$, this is isomorphic to the additive group of rational numbers with denominators a non-negative power of l; when $m \nmid l$, it is isomorphic to a certain stem product of cyclic groups. It follows that nontrivial one relator groups can occur as outer automorphism groups of one-relator groups. As above, Lemmas 2.2 and 3.1 do give information about the endomorphisms of $G(l, m; k)$ when $m|l$ and $m \neq 1$; however, we have not succeeded in determining all ependomorphisms in this case. But we do have:

THEOREM 3.4. *Suppose that* $m|k$, $m|l$, *and* $\gcd(l/m, n) = 1$. *Then* $G(l, m; k)$ *is non-Hopfian.*

Proof. A routine computation shows that the function defined by $a\chi = a^{l/m}$,

$b\chi = b$, $t\chi = bt$ is an endomorphism of G. Further, it is not hard to see that the element

$$a^{-1}b^{-1}a^{-m}bab^{-1}a^{m}b,$$

which is contained in the kernel, is a non-trivial element of H, and hence of G.

It may be seen that the kernel of $\chi^q (q > 0)$ is the normal closure in G of the commutator $[a, b^{-q}a^m b^q]$. Further, it may be shown, using Lemma 3.1, that $\mathrm{Ker}\chi^q$ is fully invariant (this may be compared with [**1**], Theorem 2).

Our final result is the following:

THEOREM 3.5. *Let* $G_n = G(3, 2; 2^n)$ $(n > 0)$. *Then* $G_0, G_1, G_2, \ldots$ *is an infinite sequence of non-isomorphic, Hopfian, one-relator groups. If* $0 \leqq p < n$, *then* G_n *is a proper quotient of* G_p *by a fully invariant subgroup which is the normal closure of a single element.*

Proof. for each $0 \leqq p < n$, G_n has the presentation

$$\langle x, b; t: b^{-1}x^3b = x^2, [x, b^{n-p}xb^{-(n-p)}] = 1, b = t^{-1}x^{2^p}t\rangle$$

associated with its generating pair $(a^{2^{n-p}}, t)$ (see [**6**]). It follows that $G_n \cong G_p/M$, where M is the normal closure of $[a, b^{n-p}ab^{-(n-p)}]$; moreover, it is not hard to show that this is a non-trivial element of H, and consequently of G_p. Thus M is non-trivial. The remaining statements follow from 3.3.

We remark that the outer automorphism group A_n of G_n has the presentation

$$\langle a_0, a_1, \ldots, a_n; a_0{}^3 = a_1{}^2, \ldots, a_{n-1}{}^3 = a_n{}^2\rangle.$$

The groups A_n $(n > 0)$ are pairwise non-isomorphic (as may be seen directly, or by reference to [**15**]).

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