Appendix A

Homework Exercises

1. **First-derivative matrix**

   Let

$$f'_1 = Af_1 + Bf_2 + Cf_3$$
$$f'_N = Af_N + Bf_{N-1} + Cf_{N-2}$$

   (where the constants A, B, and C have different values in each formula).

   (a) Find expressions for the constants in each formula so that the error is proportional to Δ^2.

   (b) Use your results from part (a) to define a matrix D such that

$$f'_i = D_{ij} f_j \; ; \quad i = 1, 2, \ldots, N.$$

   (c) Type in the Matlab function **ddz** printed below. Verify that it corresponds to the finite difference approximation to the first-derivative that you defined in part (b). Then type in the script **ddz_err** (in a separate m-file). This script tests the accuracy of **ddz** for a given function ($f = z^5$ in this case). Run the script to demonstrate that the error is second order in Δ.

   (d) Try it with a few other functions to see if the result is generally valid. (Two is enough.) Now try it for the case $f = z^2$. Can you make sense of the result?

2. **Second-derivative matrix**

   Repeat the analyses above for the second-derivative. To begin with, assume that:

$$f''_i = Af_{i-1} + Bf_i + Cf_{i+1} \quad ; \quad i = 1, 2, \ldots, N$$
$$f''_1 = Af_1 + Bf_2 + Cf_3 + Df_4$$
$$f''_N = Af_N + Bf_{N-1} + Cf_{N-2} + Df_{N-3}$$

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(a) Find expressions for the constants $A, B \ldots$ in each formula so that the error is proportional to Δ^2 .

(b) Write a Matlab function called **ddz2**, similar to **ddz**, that computes a second-derivative matrix using your results from part (a). Modify the script **ddz_test** so that it computes the second-derivative using your function **ddz2** and tests its accuracy. Demonstrate that your approximation is accurate to second order.

3. **Differential Eigenvalue Problem**

(a) Analytically determine the values of the constant λ for which the following boundary value problem has solutions:

$$f'' = \lambda f \; ; \quad f(0) = f(\pi) = 0. \tag{A.1}$$

(b) Now do the same thing numerically. Start by defining a vector of equally spaced z values z_i; $\quad i = 1, 2, \ldots, N$, such that z_0 and z_{N+1}, if they were included, would be equal to 0 and π. Use your subroutine **ddz2** to compute the second-derivative matrix for z , then replace the top and bottom rows so as to be consistent with the boundary conditions $f_0 = f_{N+1} = 0$. Set $N = 10$. The eigenvalues of your matrix should now correspond to the values of λ that you found in part (a) (at least inasmuch as the finite difference derivative you derived is accurate). Check this by using the Matlab routine **eig** to find the eigenvalues, then the routine **sort** to sort them from smallest to largest. Plot the eigenvectors corresponding to the smallest and largest[1] eigenvalues. You should find that the former is a smooth, well-resolved function, whereas the latter has a lot of poorly resolved small-scale structure. Correspondingly, the smallest eigenvalues should match the analytical solution closely, whereas the largest will not.

[At the end of this assignment is a sample script that you can use as you wish.]

(c) Repeat part (b) using one-sided derivatives for the top and bottom rows instead of boundary conditions. What difference does this make to the result?

Matlab Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function d=ddz(z)
% First derivative matrix for independent variable z.
```

[1] In absolute value.

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```
% 2nd order centered differences.
% Use one-sided derivatives at boundaries.

% check for equal spacing
if abs(std(diff(z))/mean(diff(z)))>.000001
    disp(['ddz: values not evenly spaced!'])
    d=NaN;
    return
end

del=z(2)-z(1);N=length(z);

d=zeros(N,N);
for n=2:N-1
    d(n,n-1)=-1.;
    d(n,n+1)=1.;
end
d(1,1)=-3;d(1,2)=4;d(1,3)=-1.;
d(N,N)=3;d(N,N-1)=-4;d(N,N-2)=1;
d=d/(2*del);
return
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Script ddz_err
% Example script for OC680 Hmwk #1.
% This script tests the first-derivative matrix computed in ddz.
% The result
% shows that the method is 2nd order in the time step grid
% increment.
% The assignment is to do the same for the second-derivative.
NN=[10:10:100];
% compute error at each N
for i=1:length(NN);
    N=NN(i);
    % 0<z<1
    del(i)=1/N;
    z=[0:1:N-1]'*del(i);
    % specify test function f(z) and its (exact)
    % derivative fp(z)
    f=z.^5;
    fp=5*z.^4;
```

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```
    d=ddz(z);       % compute derivative matrix
    df=d*f;         % compute finite-difference approximation
                    % to the derivative
    err(i)=sqrt(mean((fp-df).^2)); % compute root-mean-squared
                                   % error
end

% plot error vs. N
figure
loglog(del,err,'*')
xlabel('\Delta')
ylabel('ERROR')
hold on

% regress to find power law and plot
p=polyfit(log(del),log(err),1)
err_th=exp(p(2))*del.^p(1);
plot(del,err_th,'-')
ttle=sprintf('ERROR = %.2f\\Delta^{ %.2f}',exp(p(2)),p(1))
title(ttle)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% HMWK 1 Part 3
%
clear
close all

% define z values
N=10;
z=pi*[1:N]'/(N+1);

% compute derivative matrix
d=ddz2(z);
dz=z(2)-z(1);

% To use 1-sided derivatives, comment out the next two lines.
d(N,:)=0;d(N,N-1)=1/dz^2;d(N,N)=-2/dz^2;
d(1,:)=0;d(1,1)=-2/dz^2;d(1,2)=1/dz^2;

% compute eigvals & eigvecs
[v ee]=eig(d);e=diag(ee);
```

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```
% sort
[~,ind] =sort(abs(e),'ascend');
e=e(ind);v=v(:,ind);

% Plot eigenvalues /i^2
% If the eigfn is well-resolved, this will be close to -1.
figure
plot([1:N],e./[1:N].^2','*','markersize',10)
xlabel('i');
ylabel('\lambda_i / i^2')
title('Is \lambda_i / i^2 = -1 ?')

% Plot first and last eigvecs.
% The first is well-resolved, and its eigval is close to -i^2.
% The last is poorly-resolved, and the eigval is not close
% to -i^2.
figure
subplot(1,2,1)
plot(v(:,1),z,'b*'); hold on
plot(v(:,1),z,'b')
ylabel('z')
title('First eigvec (smallest abs(eigval))')
subplot(1,2,2)
plot(v(:,end),z,'r*'); hold on
plot(v(:,end),z,'r')
title('Last eigvec (largest abs(eigval))')
```

4. **Benard Convection**

(a) Given

$$\sigma^2 + (\nu + \kappa)K^2\sigma + \nu\kappa K^4 + B_z\cos^2\theta = 0 \tag{A.2}$$

as derived in class, show that

$$\frac{\partial\sigma}{\partial\cos^2\theta} > 0.$$

[Hint: You don't have to solve the quadratic equation to do this. Just differentiate each term.] What assumptions do you have to make about σ and B_z for this to be true? Write a brief (one-sentence) justification for each assumption. At what value of $\cos^2\theta$ will σ be greatest (if all other parameters are fixed)?

(b) Minimize the function $(\tilde{k}^{*2} + n^2\pi^2)^3/\tilde{k}^{*2}$ with respect to $\tilde{k}^{*2}$. Give both the minimum value, and the value of $\tilde{k}^{*2}$ at which the minimum occurs, as functions of n . Show that the critical Rayleigh number is 657.5.

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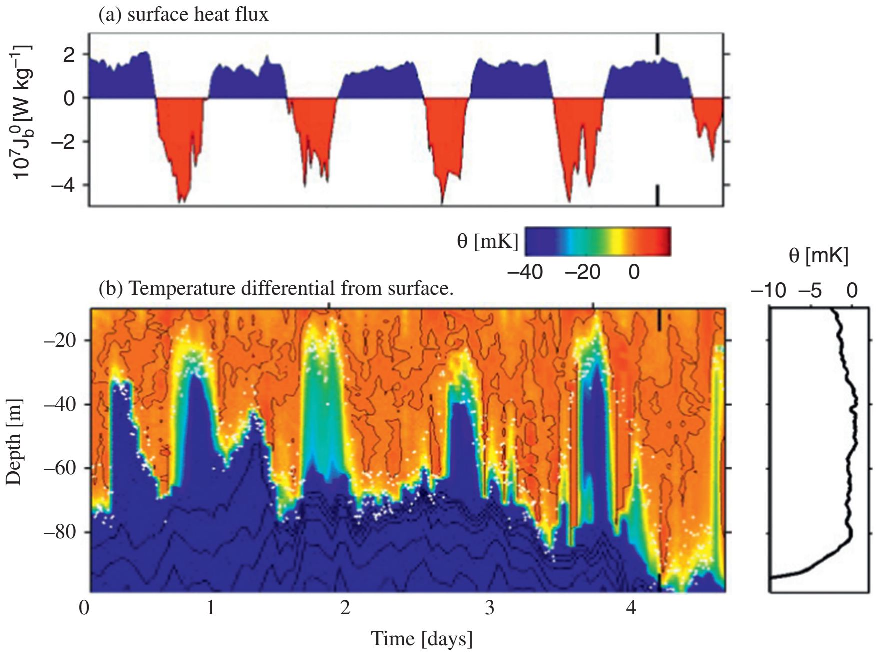

Figure A.1 The diurnal cycle of upper ocean convection (courtesy J. Moum). For reference, a temperature change $\Delta T = 2$mK is equivalent to a relative density change $\Delta\rho/\rho = 10^{-6}$.

5. **A Convective Mixed Layer**
Suppose that nocturnal convection in the upper ocean is driven by a density difference $\Delta\rho/\rho_0 = 10^{-6}$ over the upper 40 m (as in Figure A.1). Compute the Rayleigh number, using the following values:

$$\nu = 1.0 \times 10^{-6}\text{m}^2\text{s}^{-1}$$
$$\kappa = 1.4 \times 10^{-7}\text{m}^2\text{s}^{-1}$$
$$g = 9.81\text{ms}^{-2}.$$

Plot $\sigma(\tilde{k})$ for these parameter values. What is the horizontal wavelength $(2\pi/\tilde{k})$ of the fastest-growing instability? What are its growth rate and e-folding time? Give the e-folding time in hours, and compare it with the length of time over which convective conditions persist (say 12 hours). By what factor would the amplitude of this instability grow during that time?

6. **An Unstable Layer in an Inviscid Fluid**
In a fluid with $\nu = \kappa = 0$, suppose that the mean buoyancy gradient has the following profile:

$$B_z = B_{z0}(1 - 2\text{sech}^2\alpha z). \tag{A.3}$$

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Sketch this function and show that the fluid is stably stratified except for an unstable layer surrounding $z = 0$.

Solve (2.29) with boundary conditions $\hat{w} \to 0$ as $z \to \pm\infty$ for the special case $\tilde{k} = \alpha$. (Hint: try $\hat{w} = \text{sech}^2 \beta z$, where β is a constant to be determined.) In a later project you will solve this numerically for a full range of $\tilde{k}$.

Note: Hyperbolic functions provide a useful model for simple shear flows. Here are a couple of useful properties:

$$\frac{d}{dx}\tanh x = \text{sech}^2 x = 1 - \tanh^2 x.$$

7. **Numerical Analysis of Shear Instability**

(a) Write a Matlab function to find eigenvalues σ and eigenfunctions $\hat{w}$ of the Rayleigh equation in finite difference form:

$$\sigma A_{ij}\hat{w}_j = B_{ij}\hat{w}_j \tag{A.4}$$

for $i, j = 1, 2, \ldots, N$ with boundary conditions

$$\hat{w}_0 = \hat{w}_{N+1} = 0. \tag{A.5}$$

The matrices A and B are defined by

$$\begin{aligned} A_{ij} &= D_{ij} - \tilde{k}^2 I_{ij} \\ B_{ij} &= -ik(U_i A_{ij} - U_i'' I_{ij}) \end{aligned} \tag{A.6}$$

where D is the second-derivative matrix [including the boundary conditions (A.5)], $\tilde{k}$ is the wave vector magnitude $\sqrt{k^2 + \ell^2}$, $\vec{U}$ is the background velocity profile, $\vec{U}''$ is its second-derivative, I is the identity matrix and there is no sum on the repeated index i. After defining A and B, use

```
[w,e]=eig(B,A); sigma=diag(e);
```

to solve the generalized eigenvalue problem (A.4). Finish by sorting the eigenvalue/eigenvector pairs in order of descending growth rate. Output the pair with the largest growth rate. Your function should accept the vectors $\vec{z}$, $\vec{U}$, and the scalars k and ℓ as inputs and deliver σ and $\hat{w}$ for the mode with the maximum growth rate as outputs.

Hints:

- In Matlab, a simple way to left-multiply a vector onto a matrix, $v_i A_{ij}$ (with no sum on i), is like this: `diag(v)*A`.
- The identity matrix of size $N \times N$ is given by the built-in function `eye(N)`

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- It's a good idea to use the second-derivative matrix to compute U'' , but use one-sided derivatives rather than boundary conditions for the top and bottom rows, since it does not obey the same boundary conditions as $\hat{w}$.
- Make sure your z vector excludes the boundaries!
- Sort using `sort(...,'descend')`.

(b) Write a script to test the function you developed in part (a) for the following test case:

$$U^\star = \tanh(z^\star);\ \ k^\star = 0.45;\ \ \ell^\star = 0;\ \ z^\star \in (-4, 4);\ \ \Delta = 0.2.$$

The script should define the inputs for the function, call the function, then plot the outputs. The resulting plot should show $\hat{w}^\star$ versus $z^\star$, both as real and imaginary parts and in polar form (magnitude and phase versus z^*), and should include an annotation that gives the growth rate, e.g.,

```
title(sprintf('\\sigma*=\%.3f',your_value_of_sigma)).
```

You should get $\sigma^\star = 0.175$.
[Hint: Remember that your vector of $z^\star$ values should exclude the boundaries.]

8. **The Piecewise-Linear Shear Layer: Numerical Solution**
   Here you will solve the shear layer problem numerically for comparison with the analytical solution.
   (a) Repeat the derivation of $\sigma^\star = \sigma^\star(k^\star)$ for the piecewise-linear shear layer with all of the algebra included.
   (b) Test your result from (a) using the numerical function developed in project 7. Use the scaled variables, so that the velocity profile is:

$$U^\star = \begin{cases} 1, & z^\star > 1 \\ z^\star, & -1 \le z^\star \le 1 \\ -1, & z^\star < -1 \end{cases}$$

Compare plots of $\sigma^\star(k^\star)$ as well as eigenfunctions and growth rates of the fastest-growing mode for both the analytical and numerical solutions. Try a few different ranges for z, e.g. z=[−3 3]; z=[−6 6]; z=[−10 10], and plot $\sigma^\star(k^\star)$ for each. You should find that only when z=[−10 10] or larger is the analytical form of (3.34) reproduced. That is because the boundary conditions are different. In the analytical solution, we assumed that the vertical domain is infinite, so that $\hat{w} \to e^{-\tilde{k}|z|}$ as $|z| \to \infty$.
[Hint: When comparing eigenfunctions, remember that they are only defined up to a multiplicative constant, which may be complex. As a result,

eigenfunctions that should be the same can look totally different. The solution is to normalize. The easiest way is to divide the eigenfunction through by its value at some fixed height, e.g., $z = 0$.]

(c) Resolve the discrepancy in boundary conditions between (a) and (b) by deriving and implementing an asymptotic boundary condition in your code. For consistency with $\hat{w} \to e^{\mp\tilde{k}z}$, require that $\hat{w}' = +\tilde{k}\hat{w}$ and $\hat{w}' = -\tilde{k}\hat{w}$ at $z = z_1$ and $z = z_N$, respectively. At $z = z_1$:

$$\hat{w}_1' = \frac{\hat{w}_2 - \hat{w}_0}{2\Delta} = \tilde{k}\hat{w}_1 \quad \Rightarrow \quad \hat{w}_0 = \hat{w}_2 - 2\tilde{k}\Delta\hat{w}_1.$$

Now substitute into the finite difference expression for the second-derivative:

$$\hat{w}_1'' = \frac{\hat{w}_0 - 2\hat{w}_1 + \hat{w}_2}{\Delta^2} = \frac{2\hat{w}_2 - 2(1 + \tilde{k}\Delta)\hat{w}_1}{\Delta^2}$$

After a similar process at the upper boundary, you should have:

$$D_{1,1} = -2(1 + \tilde{k}\Delta)/\Delta^2\ ;\ D_{1,2} = 2/\Delta^2\ ;\ D_{1,j} = 0 \text{ otherwise}$$
$$D_{N,N} = -2(1 + \tilde{k}\Delta)/\Delta^2\ ;\ D_{N,N-1} = 2/\Delta^2\ ;\ D_{N,j} = 0 \text{ otherwise.}$$

After making this replacement in the derivative matrix, show that you can match the analytical result with a much smaller domain.

9. **Transforming the Rayleigh Equation**
The vertical displacement η can be defined in terms of vertical velocity: $w = D\eta/Dt$.
(a) Linearize this equation by assuming small perturbations about a parallel shear flow $U(z)\hat{e}^{(x)}$. Assuming a normal mode solution, show that

$$\hat{w} = \imath k(U - c)\hat{\eta}.$$

(b) Now show how the Rayleigh equation:

$$\hat{w}_{zz} = \left(\frac{U_{zz}}{U - c} + \tilde{k}^2\right)\hat{w}$$

can be transformed into an equation for the vertical displacement eigenfunction:

$$[(U - c)^2\hat{\eta}_z]_z = \tilde{k}^2(U - c)^2\hat{\eta}.$$

10. **Energy Analysis for a Shear Layer**
Using the function developed in problem 7, compute $\sigma^\star(k^\star)$ for the hyperbolic tangent shear layer $U^\star = \tanh(z^\star)$. Your plot should cover the range $0 < k^\star < 1$. You should find that the growth rate and wavenumber of the fastest growing mode are close to the "test case" suggested in homework problem 7b.

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Using the value of $k^\star$ you identified as the fastest-growing mode, do the following.

(From here on, stars indicating scaled quantities are dropped.)

(a) Plot profiles of
   (1) $\hat{w}$ (amplitude and phase)
   (2) $\overline{u'w'}$
   (3) $\overline{\pi'w'}$

(b) Plot profiles of
   (1) the kinetic energy $K'(z) = \frac{1}{2}(\overline{u'u'} + \overline{w'w'})$,
   (2) the shear production rate $SP(z) = -\overline{u'w'}dU/dz$, and
   (3) the flux convergence $FC(z) = -d\overline{\pi'w'}/dz$.

(c) Describe the pattern of energy transfer in words, i.e., where it's created, where it's fluxed from and to.

(d) Plot profiles of $2\sigma_r K'(z)$ and $SP(z) + FC(z)$ on the same axes. Check that they are equal to within, say, a few percent. If not, debug and recheck your results for (b) and (c).

(e) Add the x dependence to the eigenfunction: $w'(x,z) = \left\{\hat{w}(z)e^{ikx}\right\}_r$ for $x \in [0, \lambda]$ and $\lambda = 2\pi/k$. The result will be a matrix, $w'_{ij} = w'(x_i, z_j)$. Make a contour or image plot of w'. Is the tilt consistent with positive shear production? [Hint: In Matlab, good choices for plotting functions of two variables are `contourf` and `pcolor`.]

11. **The Bickley Jet**

(a) Using the techniques developed above, investigate the stability of the Bickley jet:

$$U^\star = \text{sech}^2(z^\star).$$

To encompass the domain of instability, you'll need to scan wavenumbers in the range 0–2.

   (1) For each wavenumber, plot not only the fastest-growing but also the second-fastest. You'll find that there are two families of modes, the *sinuous* and *varicose* modes.
   (2) For the fastest growing mode of each family, repeat the analyses of project 10.
   (3) Write 1–2 paragraphs describing and comparing the properties of these modes. Which grows fastest? What are their spatial scales? Where are the critical levels? Where are the inflection points? Where is kinetic energy created? Do these locations coincide with critical levels, or with inflection points, or neither? You might get some ideas from Smyth & Moum (2002, section 3.1, figures 3 and 4).

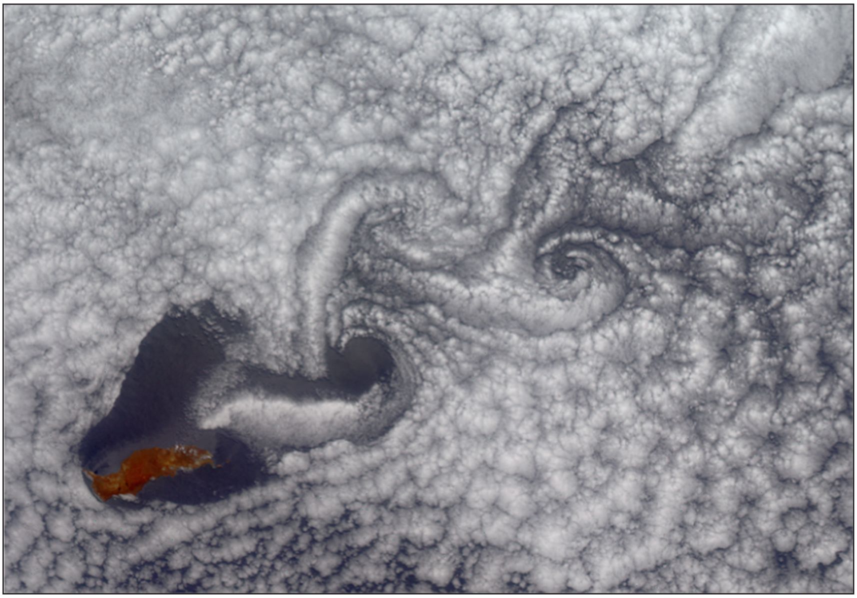

Figure A.2 Cloud patterns over Guadalupe Island, near the Baja peninsula (NASA). For use with project 11.

(b) Predict the ratio of wavelength of the fastest-growing mode to the jet width. Obtain the same ratio graphically from the NASA satellite photo below, and compare your results. (The comparison will of course be approximate. Be happy if you get agreement to within a factor of 2. Sketch on the satellite photo to indicate the lengths you used in your estimate.)

12. **Sinusoidal Flow**

Investigate the stability of the sinusoidal velocity profile

$$U^\star = \sin(z^\star)$$

with impermeable boundaries at $z^* = \pm H^*$.

(a) Choose the boundaries such that $H^* = \pi$. Solve the Rayleigh equation numerically for $0 < k^* < 1$ and plot $\sigma^*(k^*)$.

(b) Repeat the procedure for several smaller values of H^*. What is the effect of reducing H^*? At what H^* does the instability vanish?

(c) What aspect of the inflection point theorem does this illustrate?

13. **The Fourth-Derivative Matrix**

(a) Derive a second-order finite difference approximation to the fourth derivative having the following form:

$$f_i^{(4)} = C f_{i-2} + B f_{i-1} + A f_i + B f_{i+1} + C f_{i+2}. \qquad \text{(A.7)}$$

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Hint: To simplify the algebra, write (A.7) in this form:

$$f_i^{(4)} = Af_i + B(f_{i-1} + f_{i+1}) + C(f_{i-2} + f_{i+2}). \qquad \text{(A.8)}$$

In the Taylor series expansions for the pairs of terms in parentheses, every second term will cancel.

(b) In (A.7), the expressions for f_1, f_2, f_{N-1}, and f_N involve "ghost points" (at which f is not specified). Explain how these expressions can be evaluated using each of the following boundary conditions (in finite difference form):

- Rigid boundaries: $f_0 = f_{N+1} = 0$; $\quad f_0' = f_{N+1}' = 0$.
- Frictionless boundaries: $f_0 = f_{N+1} = 0$; $\quad f_0'' = f_{N+1}'' = 0$.

Note: It is sufficient to express the boundary conditions to second-order accuracy, e.g.,

$$f_0{}' = \frac{f_1 - f_{-1}}{2\Delta} = 0.$$

14. **Matrix Solution of the Orr-Sommerfeld Equation**

(a) Write a Matlab function to find eigenvalues and eigenvectors for the discretized Orr-Sommerfeld equation. Your function should accept as inputs a column vector of z values, the corresponding background velocity vector $U(z)$, the viscosity ν, the wavenumbers k, ℓ, and a choice of rigid or frictionless boundary conditions at each boundary. It should deliver as output the growth rate and vertical velocity eigenfunction for the fastest-growing mode. The function should compute U_{zz} internally.

You will need to write a subroutine **ddz4(z)** to compute the fourth derivative using your results from project 2. That routine need not include one-sided derivatives at the boundaries (because you will not actually use it to compute the fourth-derivative of anything).

Try your code for the following test case:

$$0 < z^\star < 1; \quad \Delta z^\star = 0.005; \quad U^\star = 4z^\star(1 - z^\star); \quad \nu^\star = 1/1e5;$$
$$k^\star = 1.55; \quad l^\star = 0.$$

Refer to your notes on scaling to make sure you understand what the starred variables mean, how they are input to your subroutine, and how to interpret the output. Use rigid boundary conditions. We get $\sigma^\star = 0.015 - 0.243\iota$.

(b) Suppose you needed to apply this result to a particular channel flow, with width 15 m and maximum flow speed 2 m/s. Give the wavelength in meters and the e-folding time in seconds (or minutes if that seems more sensible).

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Figure A.3 Schematic velocity profile for a triangular jet.

15. **Wave Resonance in a Jet**
The triangular jet profile shown in Figure A.3 has three kinks where vertical waves can propagate. Sketch the three waves such that each adjacent pair satisfies the criteria for resonance:
   (a) The vertical velocity perturbations of each wave amplify the crests and troughs of the other.
   (b) The propagation velocities allow for the waves to be stationary relative to each other.

   Make your own sketch if you prefer.
   Comparing with your analysis of the Bickley jet in homework 11 does your sketch represent the sinuous or the varicose mode?
16. **A Convectively Unstable Layer in an Inviscid Fluid, Revisited**
In an earlier project you developed a code to solve the Rayleigh equation. Adapt this code to solve (2.29), the equation for convection in a stationary, inviscid fluid with an arbitrary buoyancy profile. You will now use this code to address the unstable layer project 6 problem 2 more thoroughly.
   (a) Using the B_z profile (A.3), reproduced below for convenience, compute and plot the growth rate for a full range of $\tilde{k}$. For simplicity choose $\alpha = 1$.

$$B_z = B_{z0}(1 - 2\text{sech}^2\alpha z). \tag{A.9}$$

   (b) For the special case $\tilde{k} = \alpha$, do your numerical results for σ and $\hat{w}$ match the analytical solution? Make a plot to illustrate the comparison.
   (c) Does this case represent the fastest-growing mode?

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(d) If $\tilde{k} = \alpha$ is not the fastest-growing mode, compute $\tilde{k}$, σ, and $\hat{w}$ for the fastest-growing mode and discuss any differences you observe. Make a plot to illustrate the comparison.

(e) Is your result consistent with the upper bound on the growth rate given in (2.34)?

17. **Instability of a Separating Boundary Layer**

A bottom boundary layer flowing over an obstacle tends to separate on the downstream side. Figure A.4 is an aerial photo of Knight Inlet, a fjord on the coast of British Columbia. Tidal flow in the fjord must cross a shallow sill, which is the site of strong instability and turbulence. In this assignment, you will do stability analyses of this flow.

Here is a sketch of flow over the sill. It shows regions of instability: a stratified shear layer above, and a separating boundary layer lower down.

Below is an echosounder image of the flow.

We'll examine the stratified shear flow instabilities soon. Here, we'll look at the instabilities of the separating boundary layer, where stratification is not important. The velocity profile

$$U^\star = \begin{cases} z^{\star 2}\left(6 - 8z^\star + 3z^{\star 2}\right), & 0 \le z^\star < 1 \\ 1, & z^\star \ge 1 \end{cases}$$

is a model of a boundary layer about to separate.

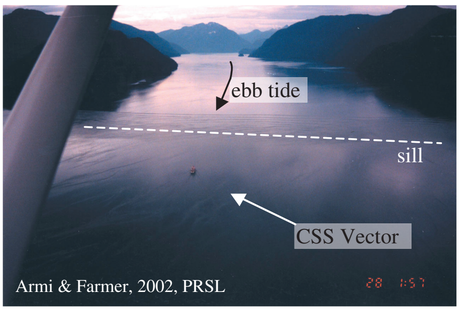

Figure A.4 Aerial photo of Knight Inlet, after Armi and Farmer (2002).

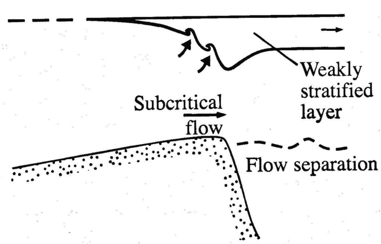

Figure A.5 Sketch of instabilities observed in sill flow (Armi and Farmer, 2002).

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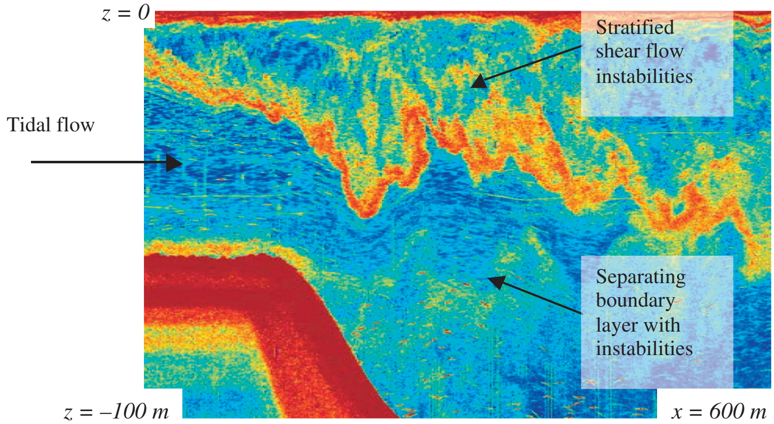

Figure A.6 Echosounder image of instabilities observed in sill flow. (Adapted from Armi and Farmer, 2002).

Before doing any stability analysis, what would you guess about the stability of this flow?

Use your Orr-Sommerfeld code (written as part of project 14) to investigate the stability of this profile. Place the upper boundary at $z^\star = 3$ and use a grid spacing $\Delta^\star = .02$. Use a rigid boundary condition at the bottom and a frictionless boundary at the top. Make a contour plot (using contourf or pcolor) of the growth rate versus wavenumber and Reynolds number for $0 \leq k^\star \leq 2.1$; $10 \leq \mathrm{Re} \leq 10^6$. Assume $\ell = 0$.

What is the minimum Reynolds number for which there is instability? (We get 280.) Is the instability stabilized or destabilized by viscosity? At this Reynolds number, is the frozen flow hypothesis valid?

In inviscid shear instability, the growth rate is approximately proportional to the maximum absolute value of the background shear,

$$S_{max} = \max_z |U_z|.$$

Find the maximum growth rate for $Re = 10^6$. Express this growth rate as a fraction of the maximum background shear.

Find the wavenumber of the fastest-growing mode for $Re = 10^6$. Express the corresponding wavelength as a multiple of the original boundary layer depth, 1. Now look at the instabilities on the separating boundary layer in the echosounder image (Figure A.6). Estimate the wavelength of those instabilities as a multiple of the original thickness of the boundary layer. (Don't forget to account for the aspect ratio of the image!) How does this result compare with the result from your stability analysis?

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How do you interpret the result?

- The stability theory for a separating boundary layer describes the observation perfectly.
- We're in the ballpark, but the data is imprecise and there may be other phenomena involved.
- This is not a shear instability of a separating boundary layer.
- ...?

18. **Instabilities in a Plunging Downslope Flow**
Starting from your Orr-Sommerfeld function you wrote for project 14, write a function to find eigenvalues and eigenvectors for a stratified shear flow in a viscous, diffusive fluid. Your function should accept as inputs a column vector of z values, the corresponding background velocity $U(z)$ and buoyancy gradient $B_z(z)$, the viscosity ν, the diffusivity κ, and the wavenumbers (k, ℓ) . Boundary conditions should be frictionless and fixed-buoyancy. (Later, you'll upgrade the function to include other choices.) The function should deliver as output the growth rate σ and the vertical velocity and buoyancy eigenfunctions for the fastest-growing mode.
The following test cases crudely model the stratified shear flow instabilities in the Knight Inlet observations. In each case, report the growth rate and wavenumber of the FGM, and assess the validity of the frozen flow approximation.

    (a) $U^\star = -\tanh(z^\star)$; $B^\star_{z^\star} = 0$; $z^\star \in [-4, 4]$; $\Delta^\star = 0.2$; $Re = 10^6$; $Pr = 1$. Boundaries are frictionless and fixed-buoyancy. (I get $\sigma^\star = 0.1753$ at $k^\star = 0.47$.)
    Compute the growth rate of the fastest-growing mode as a fraction of the maximum background shear. Compare with the corresponding result from project #1 of the this homework, the separating bottom boundary layer. How do these two results compare? Can you guess the reason for the difference?
    (b) $U^\star = -\tanh(z^\star)$ $B^\star_{z^\star} = 0.1\ \mathrm{sech}^2(z^\star)$ $z^\star \in [-4, 4]$; $\Delta^\star = 0.2$; $Re = 10^6$; $Pr = 1$. Boundaries are frictionless and fixed-buoyancy. (I get $\sigma^\star = 0.1146$ at $k^\star = 0.44$.)
    (c) Same as (b), but $Re = 100$. (I get $\sigma^\star = 0.0977$ at $k^\star = 0.43$.)
    (d) Same as (b), but $Re = 20$.
    In cases (a–c), the wavenumber of the fastest-growing mode remains more or less consistent. Use a typical value of this wavenumber, along with the scaling relations discussed in class, to estimate the wavelength of the instability as a multiple of the vertical scale over which the background velocity varies. Compare this result visually with the shear instabilities

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in the echosounder image below. How does this ratio of scales of the computed instabilities compare with what you see in the observations?

19. **Sheared Convection**

Here you will use your existing stratified shear flow function to investigate the effect of background shear on convective instability. Sheared convection is an important aspect of the dynamics of thunderstorms and of upper ocean response to intense surface forcing (e.g., hurricanes). You'll begin by testing the code by reproducing the results we obtained analytically in the first week of the course. You'll then repeat the analysis with a background shear flow added, and find something interesting.

(a) **Test your software.**

Using the Matlab script given below along with your subroutine for stratified shear flow, plot growth rate versus wavenumber and Rayleigh number for a flow with

$$0 < z^\star < 1; \quad \Delta^\star = 0.05; \quad U^\star = 0; \quad \nu^\star = Pr = 7; \quad \kappa^\star = 1;$$

$$B^\star = -Ra\,Pr\beta; \quad \beta = z^\star; \quad B^\star_{z^\star} = -Ra\,Pr \cdot 1$$

Compute with both frictionless and rigid boundaries. In both cases, buoyancy can obey fixed-buoyancy conditions. Check to make sure that the critical values of Ra and k delivered by your code match the theoretical values.

The following script will guide you through most of this, but you must insert the call to your function for stability analysis of stratified shear flow with the appropriate inputs and outputs. If you prefer to write your own script, that is fine.

```
% Hmwk 7, project 1A
clear
close all
fs=18;
lw=1.6;

% set parameters
Pr=7;        % Prandtl number
irigid=0;    % boundary conditions (0=frictionless, 1=rigid)

% define z values
del=.05;
z_st=[del:del:1-del]';
```

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```
% define profiles
beta=z_st;
beta_zst=ones(size(z_st));
U_st=zeros(size(z_st));

% analytical results for critical Rayleigh number
% and wavenumber
if irigid==0
    Ra_c=(27/4)*pi^4; % frictionless boundaries (as derived
                      % in class)
    k_c=pi/sqrt(2);
    bcw='ff'
elseif irigid==1;
    Ra_c=1708;  % rigid boundaries (Kundu 11.3)
    k_c=3.12;
    bcw='rr';
end
bcb='cc';

% ranges for loops over k and Ra
ks=[0:.2:12];nks=length(ks)
Ras=10.^[2:.2:5];nRa=length(Ras);
l_st=0;     % 2D modes only

% loop over k and Ra
for i=1:nks
    k_st=ks(i);
    for j=1:nRa
        Ra=Ras(j);
        Bz_st=-Ra*Pr*beta_zst;

       %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
       % put call to your stratified shear flow routine here
       [s_st] = ...
       %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

        sig(i,j)=real(s_st);
    end
    disp([num2str(i/nks) '  done'])     % indicate progress
end

% plot growth rates along with analytical values for the
% critical Ra and k
```

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```
figure
contourf(ks,Ras,sig',max(sig(:))*[0:.05:.95]);shading flat
hold on
plot([k_c-1 k_c+1],Ra_c*[1 1],'k','linewidth',lw)
plot(k_c*[1 1],Ra_c*[1/1.5 1.5],'k','linewidth',lw)
set(gca,'yscale','log')
colorbar
ylabel('Ra','fontsize',fs,'fontangle','italic',
'fontweight','bold')
xlabel('k*','fontsize',fs,'fontangle','italic',
'fontweight','bold')
title('Benard convection scaled growth rate', 'fontsize',
fs,'fontweight','normal')
set(gca,'fontsize',fs-2)

print('-djpeg','hmwk7_1a')
```

(b) **The sheared case.**
First, investigate the possibility that oblique modes (modes whose wave vector points in a different direction than the sheared flow) are the most unstable by doing the following. Modify the script you used for part A so that the Rayleigh number is fixed at 1000, and loop instead over both $k^\star$ and $\ell^\star$. The range $0 \leq k^\star, \ell^\star \leq 4$ is sufficient. Assume frictionless boundaries. How do growth rates vary as the angle of obliquity is increased?
Now apply a uniformly sheared background flow in the x direction, $U^\star = RePr\, z^\star$, setting $RePr = 20$. Run the script again. Now how do growth rates of oblique modes compare with those of 2D modes?
Consider convective modes with wave vector parallel to the sheared flow, i.e., $\ell^\star = 0$. What effect does the presence of shear have on these modes? Does it increase the growth rates, decrease them, or leave them unchanged? Now consider the whole range of $k^\star$ and $\ell^\star$. What is the angle of obliquity for the fastest-growing mode? Knowing the effect of shear on convective modes, could you have *predicted* the angle of obliquity for the fastest-growing mode? [Hint: review section 4.3.2.]

20. **Instabilities of the Eady Model**
Based on section 8.8, implement a solution procedure

$$[\sigma, \hat{q}, \hat{w}, \hat{b}] = \mathcal{F}(z, U_z, B_z, f, k, l).$$

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(a) **Baroclinic modes**
Test your code by computing growth rates as a function of k with $\ell = 0$ and $Ri = 100$. Compare the growth rate and wavenumber of the fastest-growing mode with (8.60) and (8.61) or (8.62).

(b) **Symmetric modes**
Computing growth rates as a function of ℓ with $k = 0$ and $Ri = 0.75$. Is the growth rate independent of ℓ as in the analytical solution for symmetric instability? If not, why not?

(c) **Comparison**
What is the critical value of Ri above which the baroclinic mode grows faster than the symmetric mode?