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SOME OBSERVATIONS AND SPECULATIONS ON PARTITIONS INTO d-TH POWERS

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Abstract

The aim of this article is to provoke discussion concerning arithmetic properties of the function $p_d(n)$ counting partitions of a positive integer n into dth powers, where $d \geq 2$. Apart from results concerning the asymptotic behaviour of $p_d(n)$, little is known. In the first part of the paper, we prove certain congruences involving functions counting various types of partitions into dth powers. The second part of the paper is experimental and contains questions and conjectures concerning the arithmetic behaviour of the sequence $(p_d(n))_{n\in\mathbb{N}}$, based on computations of $p_d(n)$ for $n \leq 10^5$ for $d = 2$ and $n \leq 10^6$ for $d = 3, 4, 5$.

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1. Introduction

Given $A \subset \mathbb{N}_+$ and $n \in \mathbb{N}$, a partition of a nonnegative integer n with parts in A is any representation of n in the form

$$n = a_1 + \cdots + a_k,$$

where $a_i \in A$. Two partitions that differ only in the order of their summands are considered to be the same and we can assume that $a_1 \geq a_2 \cdots \geq a_k$. In particular, if $A = \mathbb{N}_+$, then the number of partitions with parts in $\mathbb{N}_+$ is denoted by $p(n)$, the famous partition function introduced by Euler and extensively studied by Ramanujan.

The literature on arithmetic properties of functions counting various types of partitions is enormous. However, the theory concentrates mainly on the case when the set A is a sum of disjoint arithmetic progressions. In this case, the theory is especially rich because of the connections with modular forms and the general theory of q series (see, for example, [2]). In this case, the counting function $A(x) = \#\{a \in A : \ a \leq x\}$ is linear, that is, $A(x) = O(x)$. There is also a nice theory connected with the set of powers

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of a fixed integer m (so-called m-partitions); in this case, $A(x)$ has logarithmic growth, that is, $A(x) = O(\log x)$.

On the other hand, very little is known about the arithmetic behaviour of partition functions counting partitions into dth powers, where $d \in \mathbb{N}_{\geq 2}$ is fixed. In this case, $A = \{k^d : k \in \mathbb{N}\}$ and the growth of $A(x)$ is $O(x^{1/d})$, which is between the two cases mentioned earlier. Let $p_d(n)$ denote the number of partitions of n into dth powers. From general principles, the ordinary generating function of the sequence $(p_d(n))_{n\in\mathbb{N}}$ has the form

$$P_d(q) = \sum_{n=0}^{\infty} p_d(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^{n^d}}.$$

Up to now, the main line of research has been to investigate the asymptotic behaviour of $p_d(n)$. Hardy and Ramanujan claimed in [6], and Wright proved in [10], that

$$\log p_d(n) \sim (d+1)\Big(\frac{1}{d}\Gamma\Big(1+\frac{1}{d}\Big)\zeta\Big(1+\frac{1}{d}\Big)\Big)^{d/(d+1)} n^{1/d}.$$

Wright's very complicated proof was simplified by Vaughan in the case $d = 2$ (see [9]) and in the general case by Gafni (see [5]). The proofs of Vaughan and Gafni are based on the circle method. A new proof, using only the saddle point method, was presented by Tenenbaum *et al.* (see [8]). To the best of our knowledge, apart from identities between partitions into dth powers of various types, which can be deduced from simple manipulations of infinite products, and a recent result of Ciolan (see [3]), who proved that the number of partitions into squares with an even number of parts is asymptotically equal to that of partitions into squares with an odd number of parts, there are no theoretical or experimental results. The absence of such results was the main motivation for our research.

In Section 2, we prove some congruences for functions counting various types of partitions into dth powers, where $d \in \mathbb{N}_{\geq 2}$. In particular, if $A_{2,p_2}(n)$ denotes the number of partitions into dth powers of integers not divisible by 2^d or p_2^d and $B_{2,p_2}(n)$ denotes the number of partitions of n into distinct dth powers not divisible by p_2^d, where each part has one among $2^d - 1$ colours, then $A_{2,p_2}(n) \equiv B_{2,p_2}(n) \pmod 2$.

In Section 3, we present many computational observations based on our computer experiments. In particular, we state several questions and conjectures concerning the arithmetic behaviour of the sequence $(p_d(n))_{n\in\mathbb{N}}$ for $d = 2, 3, 4, 5$.

2. A class of congruences

This short section is devoted to the proof of a class of congruences involving partitions into dth powers under certain restrictions. More precisely, let $p_1, p_2 \in \mathbb{N}_{\geq 2}$ and assume that $(p_1, p_2) = 1$. Let $A_{p_1,p_2}(n)$ denote the number of partitions into dth powers of integers not divisible by p_1^d or p_2^d. It is easy to see that the generating function

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for the sequence $(A_{p_1,p_2}(n))_{n\in\mathbb{N}}$ is

$$\mathcal{A}_{p_1,p_2}(q)=\sum_{n=0}^{\infty}A_{p_1,p_2}(n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{(p_1n)^d})(1-q^{(p_2n)^d})}{(1-q^{n^d})(1-q^{(p_1p_2n)^d})}.$$

We prove the following result. Here, $\mathbb{P}$ denotes the set of primes.

THEOREM 2.1. *Let $d\in\mathbb{N}_+, n\in\mathbb{N}$ and $p_1,p_2\in\mathbb{N}_{\geq 2}$. Let $B_{p_1,p_2}(n)$ denote the number of partitions of n into distinct dth powers not divisible by p_2^d, where each part has one among p_1^d-1 colours, and let $C_{p_1,p_2}(n)$ denote the number of partitions of n into dth powers not divisible by p_2^d, where each part has one among p_1^d-1 colours.*

(1) *If $p_1=2$ and p_2 is odd, then $A_{2,p_2}(n)\equiv B_{2,p_2}(n) \pmod 2$.*
(2) *If $p_1\in\mathbb{P}_{\geq 3}$ and $p_2\in\mathbb{N}_{\geq 2}$, $p_1\nmid p_2$, then, for $n\geq 1$,*

$$\sum_{i=0}^{n}A_{p_1,p_2}(i)C_{p_1,p_2}(n-i)\equiv 0 \pmod{p_1}.$$

PROOF. From the general theory, it is easy to see that the generating functions of the sequences $(B_{p_1,p_2}(n))_{n\in\mathbb{N}}, (C_{p_1,p_2}(n))_{n\in\mathbb{N}}$ are

$$\mathcal{B}_{p_1,p_2}(q)=\sum_{n=0}^{\infty}B_{p_1,p_2}(n)q^n=\prod_{n=1}^{\infty}\bigg(\frac{1+q^{n^d}}{1+q^{(p_2n)^d}}\bigg)^{p_1^d-1},$$
$$C_{p_1,p_2}(q)=\sum_{n=0}^{\infty}C_{p_1,p_2}(n)q^n=\prod_{n=1}^{\infty}\bigg(\frac{1-q^{(p_2n)^d}}{1-q^{n^d}}\bigg)^{p_1^d-1}.$$

We recall a well-known property of formal power series with integer coefficients: if $f\in\mathbb{Z}[[q]]$ and p is a prime number, then $f(q^{p^k})\equiv f(q)^{p^k} \pmod p$ for each $k\in\mathbb{N}_+$.

Let p_1 be prime. Using the aforementioned property, we note the chain of modulo p_1 equivalences

$$\sum_{n=0}^{\infty}A_{p_1,p_2}(n)q^n=\prod_{n=1}^{\infty}\frac{(1-q^{(p_1n)^d})(1-q^{(p_2n)^d})}{(1-q^{n^d})(1-q^{(p_1p_2n)^d})}\equiv\prod_{n=1}^{\infty}\frac{(1-q^{n^d})^{p_1^d}(1-q^{(p_2n)^d})}{(1-q^{n^d})(1-q^{(p_2n)^d})^{p_1^d}}$$
$$\equiv\prod_{n=1}^{\infty}\bigg(\frac{1-q^{n^d}}{1-q^{(p_2n)^d}}\bigg)^{p_1^d-1} \pmod{p_1}.$$

If $p_1=2$, then

$$\prod_{n=1}^{\infty}\bigg(\frac{1-q^{n^d}}{1-q^{(p_2n)^d}}\bigg)^{2^d-1}\equiv\prod_{n=1}^{\infty}\bigg(\frac{1+q^{n^d}}{1+q^{(p_2n)^d}}\bigg)^{2^d-1}=\sum_{n=0}^{\infty}B_{2,p_2}(n)q^n$$

and the first result follows by comparison of coefficients in the extreme expressions.

If $p_1\geq 3$, then

$$\prod_{n=1}^{\infty}\bigg(\frac{1-q^{n^d}}{1-q^{(p_2n)^d}}\bigg)^{p_1^d-1}=\prod_{n=1}^{\infty}\bigg(\frac{1-q^{(p_2n)^d}}{1-q^{n^d}}\bigg)^{1-p_1^d}=\bigg(\sum_{n=0}^{\infty}C_{p_1,p_2}(n)\bigg)^{-1},$$

and so

$$\mathcal{A}_{p_1,p_2}(q)C_{p_1,p_2}(q) \equiv 1 \pmod{p_1}.$$

Comparing coefficients on both sides of this congruence gives the second part of the theorem. □

REMARK 2.2. If we assume that p_1, p_2 are both primes, then performing the same reasoning as in the proof of the theorem above with respect to the modulus p_2 instead of p_1 gives an additional congruence

$$\mathcal{A}_{p_1,p_2}(q)C_{p_2,p_1}(q) \equiv 1 \pmod{p_2}.$$

Thus

$$\mathcal{A}_{p_1,p_2}(q) \equiv \frac{a}{C_{p_1,p_2}(q)} + \frac{b}{C_{p_2,p_1}(q)} \pmod{p_1p_2},$$

where a, b are the unique solutions of the system of congruences

$$a \equiv 1 \pmod{p_1}, \quad a \equiv 0 \pmod{p_2}, \quad b \equiv 0 \pmod{p_1}, \quad b \equiv 1 \pmod{p_1}.$$

Consequently,

$$\sum_{i_1+i_2+i_3=n} A_{p_1,p_2}(i_1)C_{p_1,p_2}(i_2)C_{p_2,p_1}(i_3) \equiv aC_{p_2,p_1}(n) + bC_{p_1,p_2}(n) \pmod{p_1p_2}.$$

REMARK 2.3. The number $A_{2,p_2}(n)$ has another interpretation: if $D_{p_2}(n)$ denotes the number of partitions into dth powers of odd integers in which no part appears more then $p_2^d - 1$ times, then $A_{2,p_2}(n) = D_{p_2}(n)$. Indeed, this can be deduced from the general theorem concerning partition ideals (see [1, Theorem 8.4]) or can be directly proved by performing simple manipulations of infinite products. We owe this remark to George Andrews (personal communication, 27 April 2020).

3. Questions and conjectures concerning the sequence $(p_d(n))_{n\in\mathbb{N}}$

Let $d \in \mathbb{N}_{\geq 2}$ be fixed. In this section, we state some questions and conjectures concerning certain aspects of the arithmetic behaviour of functions counting dth power partitions.

Let us write

$$P_d(q) = \prod_{n=1}^{\infty} \frac{1}{1-q^{n^d}} = \sum_{n=0}^{\infty} p_d(n)q^n.$$

Using the standard method of logarithmic differentiation,

$$q\frac{P_d'(q)}{P_d(q)} = \sum_{n=1}^{\infty} \frac{n^d q^{n^d}}{1-q^{n^d}} = \sum_{n=1}^{\infty} \sigma^{(d)}(n)q^n,$$

where

$$\sigma^{(d)}(n) = \sum_{k^d \mid n} k^d,$$

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with the usual convention that $\sigma^{(d)}(0) = 0$. Simple manipulations give the following recurrence relation satisfied by the sequence $(p_d(n))_{n\in\mathbb{N}}$: that is,

$$np_d(n) = \sum_{i=0}^{n-1} \sigma^{(d)}(i)p_d(n-i), \quad p_d(0) = 1.$$

This formula can be used to compute $p_d(n)$ in terms of $p_d(i)$ for $i < n$. However, even for relatively small values of n, the computations are slow. It would be interesting to find a different recurrence formula for $p_d(n)$ that allows faster computation for large values of n.

For $d = 2, 3, 4, 5$, we compute the coefficients $p_d(n)$ for $n \le 10^5$, using the following approach. First, we note that

$$P_d(q) - \prod_{i=1}^{\lceil 10^{5/d}\rceil} \frac{1}{1-q^{i^d}} = O(q^{10^5+1}),$$

that is, instead of working with the infinite product $P_d(q)$, it is enough to work with a rational function. If we write

$$P_{d,k}(q) = \prod_{i=1}^{k} \frac{1}{1-q^{i^d}} = \sum_{n=0}^{\infty} p_{d,k}(n)q^n,$$

then $p_{d,k}(n) = p_d(n)$ for $n \le k^d$. Note that, for fixed k, d, the sequence $(p_{d,k}(n))_{n\in\mathbb{N}}$ satisfies a linear recurrence. More precisely, $p_{d,1}(n) = 1$ and, for $k \ge 2$,

$$p_{d,k}(n) = p_{d,k-1}(n) \text{ for } n \le k^d, \quad p_{d,k}(n) = p_{d,k-1}(n) + p_{d,k}(n-k^d) \text{ for } n \ge k^d.$$

We used this observation to compute $p_d(n)$ for $d = 2$ and $n \le 10^5$ and $p_d(n)$ for $d = 3, 4, 5$ and $n \le 10^6$. To compute $p_2(n)$ for $n \le 10^5$, we take $k = \lceil 10^{5/2}\rceil = 317$. Similarly, to compute $p_d(n)$ for $d = 3, 4, 5$ for $n \le 10^6$, we take $k = 10^2, 32, 16$, respectively. All computations were performed on an ordinary laptop with 16 GB of memory and an i7 type processor.

Based on our data, we formulate several question and conjectures.

Conjecture 3.1. Let $d \in \mathbb{N}_{\ge 2}$ and $m \in \mathbb{N}_{\ge 2}$ be given and take $r \in \{0, \dots, m-1\}$. Then there are infinitely many values of $n \in \mathbb{N}$ such that $p_d(n) \equiv r \pmod{m}$.

The next question concerns the asymptotic behaviour of the number of solutions of the congruence $p_d(n) \equiv r \pmod{m}$, where $m \in \mathbb{N}_{\ge 2}$ and $r \in \{0, \dots, m-1\}$.

Question 3.2. Let $d \in \mathbb{N}_{\ge 2}$ and $m \in \mathbb{N}_{\ge 2}$ be given and take $r \in \{0, \dots, m-1\}$. Are the values of $p_d(n) \bmod m$ equidistributed modulo m? More precisely, is it true that

$$\limsup_{N\to+\infty} \frac{\#\{n \le N : \ p_d(n) \equiv r \pmod{m}\}}{N} = \frac{1}{m}?$$

This is a very difficult question. We do not even know any equidistribution modulo m results for the classical partition function $p(n) = p_1(n)$ for any m. In fact, the expectation is that, for m co-prime to six, the values of $p_1(n) \bmod m$ are not

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TABLE 1. Values of $\#\{n \le 10^5 : p_2(n) \equiv r \pmod{m}\}$ for $0 \le r \le m-1 \le 9$.

$m\backslash r$	0	1	2	3	4	5	6	7	8	9
2	50299	49702								
3	33373	33249	33379							
4	25252	24695	25047	25007						
5	19940	20125	19971	19955	20010					
6	16769	16454	16735	16604	16795	16644				
7	14121	14272	14320	14401	14257	14301	14329			
8	12679	12288	12496	12371	12573	12407	12551	12636		
9	11158	11081	11033	10941	11186	11239	11274	10982	11107	
10	10001	10025	10024	9866	10085	9939	10100	9947	10089	9925

TABLE 2. Values of $\#\{n \le 10^6 : p_3(n) \equiv r \pmod{m}\}$ for $0 \le r \le m-1 \le 9$.

$m\backslash r$	0	1	2	3	4	5	6	7	8	9
2	500013	499988								
3	333942	333563	332496							
4	250099	249905	249914	250083						
5	199907	200126	200490	199879	199599					
6	167109	166685	166026	166833	166878	166470				
7	142501	142721	142969	143340	142937	142913	142620			
8	125203	124636	125023	125198	124896	125269	124891	124885		
9	111451	111275	111186	111459	110992	110438	111032	111296	110872	
10	100033	100134	100021	99625	99713	99874	99992	100469	100254	99886

equidistributed. However, it is not clear what to expect in our situation because there are no connections to modular forms and Galois representations as in the case of the classical partition function. Our computations of the quantities

$$\#\{n \le m_d : \; p_d(n) \equiv r \pmod{m}\}$$

for $d = 2, 3, 4, 5$ seem to confirm Conjecture 3.1 and the equality stated in Question 3.2 (at least, for $m \le 10$); see Tables 1, 2, 3 and 4).

In the context of Euler's classical partition function, $p(n) = p_1(n)$, there are plenty of triples a, b, m, where $m \in \mathbb{N}_{\ge 5}$ and $a, b \in \mathbb{N}_+$, such that $p(an+b) \equiv 0 \pmod{m}$ for all $n \in \mathbb{N}$. This suggests the following question.

QUESTION 3.3. Let $d \in \mathbb{N}_{\ge 2}$ be fixed. Do there exist $m \in \mathbb{N}_{\ge 2}, r \in \{0, \ldots, m-1\}$ and positive integers a, b such that $p_d(an+b) \equiv r \pmod{m}$ for each $n \in \mathbb{N}$?

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TABLE 3. Values of $\#\{n \le 10^6 : p_4(n) \equiv r \pmod{m}\}$ for $0 \le r \le m-1 \le 9$.

$m\backslash r$	0	1	2	3	4	5	6	7	8	9
2	500517	499484								
3	333153	333474	333374							
4	250463	249010	250054	250474						
5	200555	199837	199524	200091	199994					
6	166388	166699	167354	166765	166775	166020				
7	143174	142713	143172	142658	142908	142621	142755			
8	125224	124544	125595	125373	125239	124465	124459	125101		
9	111012	111100	111214	111263	111238	111071	110878	111136	111089	
10	100310	99810	99660	99706	100135	100245	100027	99864	100385	99859

TABLE 4. Values of $\#\{n \le 10^6 : p_5(n) \equiv r \pmod{m}\}$ for $0 \le r \le m-1 \le 9$.

$m\backslash r$	0	1	2	3	4	5	6	7	8	9
2	500386	499615								
3	334253	332498	333250							
4	249768	249985	250618	249630						
5	199971	199526	200089	200380	200035					
6	167002	166054	166940	167251	166444	166310				
7	143141	142701	142907	143029	142768	143046	142409			
8	124187	125010	125168	125302	125581	124975	125450	124328		
9	111905	111078	110740	110779	111233	111095	111569	110187	111415	
10	100264	99955	100250	100380	100301	99707	99571	99839	100000	99734

In the range of our calculations, we were unable to find a single quadruple (m, r, a, b) and $d \in \{2, 3, 4, 5\}$ such that $p_d(an + b) \equiv r \pmod{m}$ for $n = 0, 1 \dots, 100$. In order to guarantee that $100a + b \le 10^5$, we considered the range $a \in \{2, \dots, 999\}$ and $b \in \{0, \dots, a-1\}$. This may suggest that even if there are quadruplets (m, r, a, b) such that $p_d(an + b) \equiv r \pmod{m}$ for all n, they are rare.

A sequence $(a_n)_{n\in\mathbb{N}}$ is convex if $2a_n \le a_{n-1} + a_{n+1}$ for $n \ge 1$. We formulate the following general conjecture.

CONJECTURE 3.4. Let $d \in \mathbb{N}_{\ge 2}$. Then there is an integer N_d such that, for all $n \ge N_d$,

$$2p_d(n) \le (p_d(n-1) + p_d(n+1))\Big(1 - \frac{1}{n^d}\Big).$$

In particular, the sequence $(p_d(n))_{n\ge N_d}$ is convex.

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This conjecture can be seen as a natural generalisation of log-concavity of the classical partition function $p(n) = p_1(n)$. We checked that

$$2p_2(n) \leq (p_2(n-1) + p_2(n+1))(1 - 1/n^2) \quad \text{for } n \in \{379, \ldots, 10^5 - 1\},$$
$$2p_3(n) \leq (p_3(n-1) + p_3(n+1))(1 - 1/n^3) \quad \text{for } n \in \{6769, \ldots, 10^6 - 1\},$$
$$2p_4(n) \leq (p_4(n-1) + p_4(n+1))(1 - 1/n^4) \quad \text{for } n \in \{239603, \ldots, 10^6 - 1\},$$

that is, we believe that $N_2 = 379, N_3 = 6769, N_4 = 239603$. It seems that the number N_5 (if it exists) is $\geq 10^6$.

A sequence $(a_n)_{n\in\mathbb{N}}$ of positive reals is log-concave if $a_n^2 \geq a_{n-1}a_{n+1}$ for $n \geq 1$, that is, the sequence $(-\log a_n)_{n\in\mathbb{N}}$ is convex. We formulate the following general conjecture.

CONJECTURE 3.5. Let $d \in \mathbb{N}_{\geq 2}$. Then there is an integer M_d such that, for all $n \geq M_d$,

$$p_d^2(n) \geq p_d(n-1)p_d(n+1)\Big(1 + \frac{1}{n^d}\Big).$$

In particular, the sequence $(p_d(n))_{n\geq M_d}$ is log-concave.

We checked that

$$p_2^2(n) \geq p_2(n-1)p_2(n+1)(1 + 1/n^2) \quad \text{for } n \in \{1086, \ldots, 10^5 - 1\},$$
$$p_3^2(n) \geq p_3(n-1)p_3(n+1)(1 + 1/n^3) \quad \text{for } n \in \{15656, \ldots, 10^6 - 1\},$$
$$p_4^2(n) \geq p_4(n-1)p_4(n+1)(1 + 1/n^4) \quad \text{for } n \in \{637855, \ldots, 10^6 - 1\},$$

that is, we believe that $M_2 = 1042$, $M_3 = 15656$, $M_4 = 637855$. It seems that the number M_5 (if it exists) is $\geq 10^6$.

It is very likely that this conjecture can be resolved using the classical asymptotic formula for $p_d(n)$ of Wright (see [10]) or its current improvements. An analogous result for Euler's partition function $p(n)$, that is, the case $d = 1$ of the conjecture, was proved by DeSalvo and Pak (see [4]) and recently generalised by Hou and Zhang (see [7]).

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