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ON NONABELIAN H^2 FOR PROFINITE GROUPS

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Let G be a profinite group. We define an extension (E,j) of G by a group A to consist of an exact sequence of groups

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\kappa} G \longrightarrow 1$$

together with a section $j: G \longrightarrow E$ of κ satisfying:

$$(*) \qquad j(sg) = j(s)j(g), \quad j(gs) = j(g)j(s), \quad g \in G, s \in S,$$

for some open normal subgroup S of G, and the map

$$(**) \qquad G \times A \longrightarrow A, (g,a) \longmapsto j(g)aj(g)^{-1},$$

is continuous (A being discrete).

This notion of extension of a profinite group appears to be new. It can be viewed (as pointed out in sec. 7) as an algebraization of the corresponding topological notion in Springer [6].

Let T_G be the topos of continuous discrete G-sets. The aim of this paper is to interpret the cohomology set $H^2(T_G, L)$ for a band L of T_G (Giraud [2]) by extensions of G as defined above. We shall associate with an extension $E = (E,j)$ of G a gerbe F_E over T_G and show that any gerbe over T_G is equivalent to a gerbe of the form F_E.

In [1], Eilenberg and MacLane defined G-kernels (later called abstract kernels) for a group G to be pairs (A, α) consisting of a group A and a homomorphism $\alpha: G \longrightarrow \mathrm{Out}(A)$. In [6], Springer extended this definition to topological groups G by demanding that $\alpha: G \longrightarrow \mathrm{Out}(A)$ be continuous, $\mathrm{Out}(A)$ having the discrete topology. But if G is compact, it follows that $\alpha(G)$ is a compact, hence finite subset of $\mathrm{Out}(A)$, a restriction which makes little sense for infinite G. This shows that a different definition of abstract kernels for profinite groups is necessary. It is given in Sec. 4. We shall prove that the category of abstract kernels of G is equivalent to the category of bands of T_G.

As in the case of discrete groups, each extension (E,j) of a profinite group G yields naturally an abstract kernel $(A, \tilde{\alpha})$, and hence a band $L(A, \tilde{\alpha})$ of T_G. Let $L = L(A, \tilde{\alpha})^{\mathrm{op}}$. Our main result, Theorem 6.1, states that $E \longmapsto F_E$ induces a bijection

$$\mathrm{Ext}(G, A, \tilde{\alpha}) \cong H^2(T_G, L)$$

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where the lefthand side is the set of isomorphism classes of extensions of G defining the same $(A, \tilde{\alpha})$. If G happens to be finite, this is of course a special case of the result for discrete groups ([2], VIII, 7.4) originally due to Eilenberg and MacLane [1].

In an earlier version of this paper Theorem 6.1 was proved by using Giraud's interpretation of H^2 by topos extensions ([2], VIII, Theorem 6.2.5). I am grateful to P. Deligne for pointing out how to obtain a gerbe directly from a group extension, which led to the present simplified version of the paper.

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NOTATIONS. In the following G denotes a profinite group and $\mathcal{S}$ the set of open normal subgroups of G. We shall write $E = (E, \kappa, j)$ and $E' = (E', \kappa', j')$ for extensions of G as defined above; S_E will denote the set of $S \in \mathcal{S}$ satisfying $(*)$.

T_G denotes the topos of continuous discrete G-sets, i.e., (left) G-sets X such that $X = \bigcup_{S \in \mathcal{S}} X^S$. A family $(f_i \colon X_i \longrightarrow X, \quad i \in I)$ of morphisms in T_G is a covering of X if and only if $X = \bigcup_i f_i(X_i)$. An important fact used throughout the following is that $(G/S, S \in \mathcal{S})$ is cofinal in T_G (each $X \in T_G$ has a covering of the form $(G/S_x \longrightarrow X, x \in X)$ with $S_x \in \mathcal{S}$).

For $X \in T_G$, $T_G|_X$ denotes the category with objects the T_G-morphisms $Y \longrightarrow X$.

Given a category F and a functor $p \colon F \longrightarrow T_G$, the category $F(X)$ for $X \in T_G$ has objects $z \in F$ with $p(z) = X$, and sets of morphisms $\mathrm{Hom}_X(z, z')$ consisting of $\beta \colon z \longrightarrow z'$ with $p(\beta) = \mathrm{id}_X$.

1. **The localization $T_G|_{G/S} \longrightarrow T_G$.** We first show that the topos $T_G|_{G/S}$ for $S \in \mathcal{S}$ may be identified with T_S. For any morphism $f \colon Y \longrightarrow G/S$ in T_G let

$$Y_e = \{ y \in Y | f(y) = 1 \}.$$

Obviously, Y_e is an object of T_S.

PROPOSITION 1.1. *The functor $T_G|_{G/S} \longrightarrow T_S, Y \longmapsto Y_e$ is an equivalence.*

PROOF. Let $i \colon G/S \longrightarrow G$ be a section of the natural projection $G \longrightarrow G/S$ and choose $i(1) = 1$. Let $X \in T_S$. The set $X \times G/S$ admits a G-action

$$g(x, h) = (sx, gh), \qquad s = i(gh)^{-1} g i(h),$$

for $g \in G, x \in X$, and $h \in G/S$. This defines an object $X \ltimes G/S$ of $T_G|_{G/S}$ and a functor

$$(1) \qquad T_S \longrightarrow T_G|_{G/S} \quad X \longmapsto X \ltimes G/S.$$

For if $m \colon X \longrightarrow X'$ is a morphism in T_S then clearly $m \ltimes 1 = m \times 1$ is a G-morphism over G/S. The map $(X \ltimes G/S)_e \longrightarrow X, \quad (x, 1) \longmapsto x$, is an isomorphism in T_S. Also, for each morphism $f \colon Y \longmapsto G/S$ in T_G the map

$$Y \longrightarrow Y_e \ltimes G/S, \quad y \longmapsto \big(i(f(y))^{-1} y, f(y) \big),$$

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is an isomorphism of G-sets over G/S. Thus (1) is a quasi-inverse for $Y \mapsto Y_e$. ■

Consider now the diagram of topos morphisms

$$\begin{array}{ccc} T_S & \xrightarrow{\sim} & T_G|_{G/S} \\ & t \searrow \quad \swarrow u & \\ & T_G & \end{array}$$

where $u^*(Z) = Z \times G/S$, and $t^*(Z) = Z$ with natural S-action for $Z \in T_G$; it is commutative up to the (right adjoint of the) isomorphisms $t^*(Z) \cong (Z \times G/S)_e$. We therefore obtain

$$(T_S, t) \simeq (T_G|_{G/S}, u),$$

i.e., (T_S, t) interprets as the localization of T_G over G/S.

COROLLARY 1.2. *Let $\mathcal{A}$ be a sheaf on $T_G|_{G/S}$. Then*

$$A = \varinjlim_{S' \subset S} \mathcal{A}(G/S')$$

is a representing object for the sheaf $\mathcal{A}_e$ on T_S obtained from $\mathcal{A}$ by composition with (1); $A \ltimes G/S$ is a representing object for $\mathcal{A}$.

PROOF. If F is any sheaf on T_S, then $\varinjlim_{S' \subset S} F(S/S')$ is a representing object for F. But for $S' \subset S$ we have a G-isomorphism

$$G/S' \xrightarrow{\sim} S/S' \ltimes G/S, \quad h \mapsto (i(\bar{h})^{-1}h, \bar{h}),$$

which gives the result by Proposition 1.1.

REMARK 1.3. Suppose that S is a normal subgroup of an arbitrary group G. Replacing then T_G by the topos B_G of all G-sets, one obtains $B_G|_{G/S} \simeq B_S$ in the same way as above. For $S = 1$ this reduces to the well-known equivalence $B_G|_G \simeq$ Ens, (cf. [2], p. 113, Prop. 1.2.8.8).

2. **The gerbe F_E for an extension E.** Let E be an extension of G by A, and let S_E be the set of $S \in \mathcal{S}$ satisfying $(*)$. We shall regard any $X \in T_G$ as an E-set via $\kappa: E \longrightarrow G$, and any E-set as an S-set via the homomorphism $j|_S: S \longrightarrow E$. We define a category $F_E = F$ as follows (after P. Deligne). The objects of F are the pairs (Z, β) with Z an E-set and $\beta: Z \longrightarrow X, X \in T_G$, an E-map subject to the following conditions:

(i) A operates freely on Z,

(ii) the G-map $A \backslash Z \longrightarrow X$ induced by β is bijective,

(iii) $Z = \bigcup_{S \in \mathcal{S}_E} Z^S$.

Here $A \backslash Z$ denotes the set of A-orbits of Z. The morphisms $\eta: (Z, \beta) \longrightarrow (Z', \beta')$ in F are the E-maps $Z \longrightarrow Z'$. Any such η induces by (ii) a G-map $\bar{\eta}: X \longrightarrow X'$ such that $\beta' \eta = \bar{\eta}\beta$. This gives a functor

$$p: F \longrightarrow T_G \quad (Z, \beta) \longmapsto X.$$

It makes F a fibred category over T_G. For if $f\colon Y \longrightarrow X$ is a morphism in T_G and (Z,β) an object in $F(X)$, then

$$(Z,\beta)\times_X Y = (Z\times_X Y, \beta\times 1)$$

is an object in $F(Y)$, and the natural projection $Z\times_X Y \longrightarrow Z$ makes it an inverse image of (Z,β) under f.

PROPOSITION 2.1. *F_E is a gerbe over T_G.*

PROOF. Let $\eta\colon (Z,\beta)\longrightarrow (Z',\beta')$ be a morphism in $F(X)$. Choose $z_x\in Z$ with $\beta(z_x) = x$ for $x\in X$, and similarly $z'_x \in Z'$. Since η projects to id_X we have $\eta(z_x) = b_x z'_x$ for $b_x \in A$. Hence any morphism in $F(X)$ is an isomorphism.

For $S\in \mathcal{S}_{\mathcal{E}}$ we have an object

$$E/jS = (E/jS, \bar{\kappa}) \in F(G/S).$$

Let (Z,β) be another object in $F(G/S)$ and let $z_1\in Z$ with $\beta(z_1) = 1$. Choose $S'\subset S$ in S_E which leaves z_1 fixed. Then

$$E/jS' \longrightarrow Z\times_{G/S} G/S', \quad 1\longmapsto (z_1,1),$$

is an isomorphism in $F(G/S')$. It follows that for $X\in T_G$ any two objects in $F(X)$ are locally isomorphic because $(G/S, S\in S_E)$ is cofinal in T_G.

Finally, F is a stack, i.e., for each covering $X_i\longrightarrow X$, $i\in I$, in T_G the functor

$$F(X)\longrightarrow \mathrm{Desc}_F\big((X_i)_i, X\big), \quad Z\longmapsto (Z\times_X X_i)_i,$$

is an equivalence, where the righthand side is the category of descent data for the covering $(X_i)_{i\in I}$. For any descent datum $((Z_i)_i, \phi_{ij})$ one obtains a descent object Z by setting

$$Z = \coprod_i Z_i/\sim$$

where $z_i\sim z_j$ if and only if $\phi_{ij}(z_j,x_i,x_j) = (z_i,x_i,x_j)$. ∎

In the following we state a few properties of the objects E/jS which will be needed in the sequel. Fix $S\in\mathcal{S}_{\mathcal{E}}$. First observe that $(E/jS)^S \cong A^S j(G)/j(S)$ is a group since $j(S)$ is a normal subgroup in $A^S j(G)$. We then have natural group isomorphisms

$$(2)\qquad Aut_E(E/jS)^{\mathrm{op}} \cong (E/jS)^S, \quad (E/jS)^S\times_{G/S} G \cong A^S j(G),$$

the former given by $\eta\longmapsto \eta(1)$.

Next let $Y\longrightarrow G/S$ be a morphism in T_G. Then there is a group isomorphism

$$\rho\colon \mathrm{Hom}_S(Y_e, A) \xrightarrow{\sim} Aut_Y(E/jS\times_{G/S} Y)^{\mathrm{op}}$$

defined by $\rho(m)(1,y) = (m(y),y)$ for all $y\in Y_e$. This yields an isomorphism

$$(3)\qquad A\ltimes G/S \xrightarrow{\sim} Aut_{G/S}(E/jS)^{\mathrm{op}}$$

of group sheaves on $T_G|_{G/S}$ by Cor. 1.2.

3. $F \simeq F_E$. Let $p: F \longrightarrow T_G$ be a gerbe over T_G. We want to show that there is an extension E of G such that $F \simeq F_E$.

LEMMA 3.1. *There exists $S \in \mathcal{S}$ and $x \in F(G/S)$ such that $Aut_F(x) \longrightarrow G/S, \eta \mapsto p(\eta)(1)$, is surjective.*

PROOF. This is easy to see since G/S is finite and since any two objects in $F(G/S)$ are locally isomorphic. ∎

In the following, we fix $S \in \mathcal{S}$ and $x \in F(G/S)$ as above. For $S' \subset S$ in $\mathcal{S}$ we denote by $x^{S'}$ the inverse image of x under $G/S' \longrightarrow G/S$ with respect to a fixed cleavage of F. Then the family

$$E(S') = Aut_F(x^{S'})^{\mathrm{op}} \times_{G/S'} G, \qquad S' \subset S,$$

is naturally a directed system of groups, and we obtain an exact sequence

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\kappa} G \longrightarrow 1$$

by setting $E = \varinjlim_{S' \subset S} E(S')$ and $A = \varinjlim_{S' \subset S} Aut_{G/S'}(x^{S'})^{\mathrm{op}}$. By Cor. 1.2, A^{op} is a representing object for the group sheaf $Aut_{G/S}(x)_e$ on T_S. (Note, however, that E is in general not an object of T_S).

Let $\{ h_1 = 1, \ldots, h_r \} \subset G$ be a (minimal) set of representatives for G/S, and choose $\phi_i: x \longrightarrow x$ in F which projects to $\cdot h_i: G/S \longrightarrow G/S$. Let $\phi_1 = \mathrm{id}$, and define $j: G \longrightarrow E$ by

$$j(sh_i) = (\phi_i, sh_i), \qquad s \in \mathcal{S}, i = 1, \ldots, r.$$

Then j is a section of κ and clearly $(*)$ holds. Moreover, the action of S on A induced by conjugation in E coincides with the action of S on A as an object of T_S. Hence we have obtained an extension $E = (E, j)$ of G.

For $z \in F(X), X \in T_G$, we set

$$\Theta(z) = \varinjlim_{S' \subset S} \mathrm{Hom}_F(x^{S'}, z).$$

Then $\Theta(z)$ is naturally an E-set and it is easy to see that $\beta: \Theta(z) \longrightarrow X, \beta(\eta) = p(\eta)(1)$, satisfies (i) and (ii) of Sect. 2. Also, $S' \subset S$ leaves the elements of $\mathrm{Hom}_F(x^{S'}, z)$ in $\Theta(z)$ fixed, and hence $(\Theta(z), \beta)$ is an object of $F_E(X)$. Furthermore, for any morphism $f: Y \longrightarrow X$ in T_G, there is a natural isomorphism $\Theta(f^*(z)) \cong \Theta(z) \times_X Y$ in $F_E(Y)$.

PROPOSITION 3.1. *$\Theta: F \longrightarrow F_E$ is an equivalence of gerbes.*

PROOF. It suffices to show that the morphisms

$$Aut_X(z) \longrightarrow Aut_X(\Theta(z)), \qquad z \in F(X), X \in T_G,$$

induced by Θ are isomorphisms. For then Θ yields an isomorphism $L(F) \longrightarrow L(F_E)$ on the bands of F and F_E and the assertion follows from ([2], p. 216, Prop. 2.2.6). Further, since $(G/S', S' \subset S)$ is cofinal in T_G and since any two objects of $F(G/S')$ are locally isomorphic, it is enough to consider the case $X = G/S$ and $z = x$.

The element $\mathrm{id}_x \in \Theta(x)$ satisfies $j(s)\,\mathrm{id}_x = \mathrm{id}_x$ for all $s \in \mathcal{S}$ so that

$$\eta: E/jS \longrightarrow \Theta(x), \quad 1 \longmapsto \mathrm{id}_x,$$

is an isomorphism in $F_E(G/S)$. But the composite of $\mathrm{Int}(\eta)$ with the morphism $Aut_{G/S}(x) \longrightarrow Aut_{G/S}(\Theta(x))$ induced by Θ yields the isomorphism (3) since $A \ltimes G/S \cong Aut_{G/S}(x)^{\mathrm{op}}$ by definition of A.

4. **Bands of T_G.** The purpose of this section is to provide a description of the bands of T_G analogous to that of the bands of the classifying topos $B_{\mathcal{G}}$ for a group object $\mathcal{G}$ in a topos T, Giraud ([2], p. 430, Prop. 6.1.2). Our method of proof will be similar to that in [2]. However, while the proof in [2] relies on the equivalence $B_{\mathcal{G}|\mathcal{G} \simeq \mathcal{T}}$, we here can only employ the equivalences $T_G|_{G/S} \simeq T_S$ for $S \in \mathcal{S}$. This makes things more complicated because we still have to deal with S-actions and with further base change for $S' \subset S$.

In the following let A be a group and $\alpha: G \longrightarrow Aut(A)$ be a map of G into the set of group automorphisms of A. Let $\mathrm{Out}(A) = Aut(A)/\,\mathrm{In}(A)$ where $\mathrm{In}(A)$ is the normal subgroup of inner automorphisms of A. Suppose that α satisfies the following conditions:

(i) the map $\bar{\alpha}: G \longrightarrow \mathrm{Out}(A)$ induced by α is a group homomorphism,

(ii) there exists $S \in \mathcal{S}$ such that

$$\alpha(sg) = \alpha(s)\alpha(g), \quad \alpha(gs) = \alpha(g)\alpha(s), \quad s \in \mathcal{S}, g \in \mathcal{G},$$

and $\alpha|_S$ makes A a (group) object of T_S.

We call such a pair (A, α) a G-kernel, and write $ga = \alpha(g)(a)$, $g \in G$, $a \in A$. Condition (i) means there exists a map $c: G \times G \longrightarrow A$ satisfying

$$(gh)a = c(g,h)\big(g(ha)\big)c(g,h)^{-1}, \quad a \in A,\ g,h \in G. \tag{4}$$

By (ii) we can choose c in such a way that

$$c(g,hs) = c(g,h) = c(gs,h), \quad g,h \in G,\ s \in \mathcal{S}, \tag{5}$$

i.e., c factors through $G/S \times G/S$. Then $c(G \times G)$ is finite and we may also suppose without restriction that

$$sc(g,h) = c(g,h), \qquad g,h \in G,\ s \in \mathcal{S}. \tag{6}$$

In the following S_α denotes the set of $S \in \mathcal{S}$ satisfying (ii) and for which there exists $c: G \times G \longrightarrow A$ satisfying (4)–(6). Let $S \in \mathcal{S}_\alpha$ and let $i: G/S \longrightarrow G$ be a section of the canonical map $G \longrightarrow G/S$ with $i(1) = 1$. Further, let $p_1, p_2: G/S \times G/S \longrightarrow G/S$ denote the projections.

LEMMA 4.1. *The map* $\phi_\alpha: p_2^*(A \ltimes G/S) \longrightarrow p_1^*(A \ltimes G/S)$,

$$\phi_\alpha(a,h,g,h) = \big((i(g)^{-1}i(h))a, g, g, h\big), \qquad a \in A,\ g,h \in G/S,$$

is an isomorphism of group objects in $T_G|_{(G/S)^2}$. It is a descent datum up to the inner automorphism defined by

$$(G/S)^3 \longrightarrow A \ltimes G/S,\ (g,h,k) \longmapsto \big(c(g^{-1}h, h^{-1}k), g\big).$$

The proof of this lemma is by simple calculations which we omit. ∎

In the following let $\mathrm{lien}(A \ltimes G/S)$ denote the band of $T_G|_{G/S}$ defined by the group object $A \ltimes G/S$, ([2], p. 186). The lemma shows that we have a descent datum

$$\big(\mathrm{lien}(A \ltimes G/S), \mathrm{lien}(\phi_\alpha)\big) \tag{7}$$

in the fibre over G/S of the stack $\mathrm{LIEN}(T_G)$ of bands over T_G. We shall denote by

$$L(A, \alpha) \in \mathrm{Lien}(T_G)$$

a descent object of (7) in the category of bands (over the final object) of T_G. Suppose we replace S by $S' \subset S$ and $i: G/S \longrightarrow G$ by any $i': G/S' \longrightarrow G$. Then

$$A \ltimes G/S' \xrightarrow{\sim} (A \ltimes G/S) \times_{G/S} G/S', (a,h) \longmapsto \big((i(\bar h)^{-1} i'(h))a, \bar h, h\big),$$

is an isomorphism of group objects in $T_G|_{G/S'}$ which transforms $\phi_{\alpha,S'}$ into the isomorphism induced by $\phi_{\alpha,S}$. This shows that $L(A,\alpha)$ is also a descent object for (7) with S replaced by any $S' \in \mathcal{S}_\alpha$.

PROPOSITION 4.2. *Each $L \in \mathrm{Lien}(T_G)$ is isomorphic to an $L(A,\alpha)$ for a G-kernel* (A,α).

PROOF. Since any object and morphism of $\mathrm{Lien}(T_G)$ is locally representable ([2], p. 191, 1.2.1) there exists $S \in \mathcal{S}$ and a group A in T_S such that $L(G/S) \cong \mathrm{lien}(A \ltimes G/S)$, and we may choose S in such a way that also the canonical descent datum for $L(G/S)$ is representable. Hence there exists an isomorphism $\phi : p_2^*(A \ltimes G/S) \longrightarrow p_1^*(A \ltimes G/S)$ such that $\mathrm{lien}(\phi)$ is a descent datum for L; ϕ has the form

$$\phi(a,h,g,h) = (\phi_{g,h}(a), g, g, h), \qquad a \in A,\ g,h \in G/S,$$

each $\phi_{g,h}: A \longrightarrow A$ being a group automorphism of A. Since ϕ is a G-map it is uniquely determined by the maps $\phi_{1,h}$, $h \in G/S$. The fact that $\mathrm{lien}(\phi)$ is a descent datum implies

$$\phi_{g,h}\phi_{h,k} \equiv \phi_{g,k} \bmod \mathrm{In}(A).$$

In particular, $\phi_{g,g} \equiv \mathrm{id}_A$, and we may suppose without restriction that $\phi_{1,1} = \mathrm{id}_A$. We now define

$$\alpha : G \longrightarrow Aut(A), \quad \alpha(si(h)) = s\phi_{1,h} \quad s \in S, h \in G/S,$$

where $i: G/S \longrightarrow G$ is a fixed section with $i(1) = 1$. Then $\alpha|_S$ is the given S-action on A, and it is not difficult to show that (A,α) is indeed a G-kernel. It follows that $L(A,\alpha) \cong L$ because ϕ equals ϕ_α of Lemma 4.1, both having the same $(1,h)$-components. ∎

The G-kernels form a category $K(G)$ where a morphism $f:(A,\alpha)\longrightarrow(B,\beta)$ is defined to be a group homomorphism $f:A\longrightarrow B$ such that there exists $b:G\longrightarrow B$ and $S\in\mathcal{S}$ satisfying

$$f(ga)=b_g\big(gf(a)\big)b_g^{-1},\qquad \text{and } b_s=1$$

for all $g\in G, a\in A$ and $s\in\mathcal{S}$. Given f we can choose b and S in such a way that $S\in\mathcal{S}_\alpha\cap\mathcal{S}_\beta$ and

$$b_{gs}=b_g\quad sb_g=b_g\qquad g\in G, s\in\mathcal{S}.$$

Then $\mathcal{b}:\mathcal{G}/\mathcal{S}\times\mathcal{G}/\mathcal{S}\longrightarrow p_1^*(\mathcal{B}\ltimes\ \mathcal{G}/\mathcal{S})$, $\mathcal{b}(\mathcal{g},\mathcal{h})=(\mathcal{b}_{\mathcal{g}^{-1}\mathcal{h}},\mathcal{g},\mathcal{g},\mathcal{h})$, is a morphism in T_G and

$$\phi_\beta(f\ltimes\ 1)=\mathcal{b}\big((f\ltimes\ 1)\phi_\alpha\big)\mathcal{b}^{-1}.$$

Thus lien$(f\ltimes\ 1)$ is a morphism of descent data in LIEN(T_G) yielding a morphism $L(A,\alpha)\longrightarrow L(B,\beta)$. Hence we obtain a functor

$$\lambda:K(G)\longrightarrow \mathrm{Lien}(T_G),\ (A,\alpha)\longmapsto L(A,\alpha).$$

Given $f:(A,\alpha)\longmapsto(B,\beta)$ and $b\in B$, then

$$f^b:A\longrightarrow B,\ a\longmapsto bf(a)b^{-1},$$

is also a morphism $(A,\alpha)\longrightarrow(B,\beta)$ in $K(G)$. Moreover, if $S\in S_\alpha\cap S_\beta$ and $b\in B^S$, then $\mathcal{b}:\mathcal{G}/\mathcal{S}\longrightarrow\mathcal{B}\ltimes\ \mathcal{G}/\mathcal{S},\mathcal{g}\longmapsto(\mathcal{b},\mathcal{g})$, is a G-morphism and $\mathcal{b}(f\ltimes\ 1)\mathcal{b}^{-1}=f^{\mathcal{b}}\ltimes\ 1$. Thus lien$(f\ltimes\ 1)=$ lien$(f^b\ltimes\ 1)$, and $\lambda(f)=\lambda(f^b)$. Hence λ induces a functor

$$\bar\lambda:\bar K(G)\longrightarrow\mathrm{Lien}(T_G)$$

where $\bar K(G)$ has the same objects as $K(G)$, but has morphisms the equivalence classes of morphisms $f:(A,\alpha)\longrightarrow(B,\beta)$ under the action of B.

PROPOSITION 4.3. *The functor $\bar\lambda$ is an equivalence.*

PROOF. It remains to show that $\bar\lambda$ is fully faithful. Let $f,f':(A,\alpha)\longrightarrow(B,\beta)$ be morphisms in $K(G)$ and assume $\lambda(f)=\lambda(f')$. Then there exists $S\in\mathcal{S}$ and a morphism $\mathcal{b}:\mathcal{G}/\mathcal{S}\longrightarrow\mathcal{B}\ltimes\ \mathcal{G}/\mathcal{S}$ in $T_G|_G/S$ such that $\mathcal{b}(f\ltimes\ 1)\mathcal{b}^{-1}=f'\ltimes\ 1$. Let $\mathcal{b}(1)=(\mathcal{b},1)$. Then obviously $f'=f^b$. Thus $\bar\lambda$ is faithful.

Next let $\eta:L(A,\alpha)\longrightarrow L(B,\beta)$ be any morphism in Lien(T_G). It is locally defined by a morphism of group objects

$$f:\mathcal{A}\ltimes\ \mathcal{G}/\mathcal{S}\longrightarrow\mathcal{B}\ltimes\ \mathcal{G}/\mathcal{S},\qquad \mathcal{S}\in\mathcal{S}_\alpha\cap\mathcal{S}_\beta,$$

which satisfies

$$\mathcal{b}\big(\phi_\beta(f\ltimes\ 1)\big)\mathcal{b}^{-1}=(f\ltimes\ 1)\phi_\alpha \tag{8}$$

for a morphism $\mathcal{b}:\mathcal{G}/\mathcal{S}\times\mathcal{G}/\mathcal{S}\longrightarrow p_1^*(\mathcal{B}\ltimes\ \mathcal{G}/\mathcal{S}),(\mathcal{g},\mathcal{h})\longrightarrow(\mathcal{b}(\mathcal{g},\mathcal{h}),\mathcal{g},\mathcal{g},\mathcal{h})$ in T_G. Then $f=f\ltimes\ 1$ where $f:A\longrightarrow B$ is a morphism of groups in T_S. Define $b:G\longrightarrow B$ by

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$b_s = 1, s \in \mathcal{S}$, and $b_{i(h)s} = b(1,h)$ for $h \neq 1$ in G/S, where $i: G/S \longrightarrow G$ is the given section defining the G-action on $A \ltimes G/S$ and $B \ltimes G/S$. It follows then from (8) that $f(ga) = b_g(gf(a))b_g^{-1}$ for $g \in G, a \in A$. Hence $f: (A,\alpha) \longrightarrow (B,\beta)$ is a morphism in $K(G)$, and clearly $\lambda(f) = \eta$. ∎

If $E \xrightarrow{\sim} E'$ are isomorphic extensions of G by A (Section 6) then the induced maps $\alpha, \alpha': G \longrightarrow Aut(A)$ are equivalent in the sense that

(9) $$\alpha|_S = \alpha'|_S \text{ for some } S \in \mathcal{S}, \ and\ \bar{\alpha} = \bar{\alpha}': \mathcal{G} \longrightarrow \mathrm{Out}(\mathcal{A}).$$

We therefore define an abstract G-kernel to be a pair $(A, \tilde{\alpha})$ where (A,α) is a G-kernel and $\tilde{\alpha}$ the class of α under the above equivalence relation. Given $\alpha \sim \alpha'$ there exists $S \in \mathcal{S}_\alpha \cap \mathcal{S}_{\alpha'}$, such that $\mathrm{lien}(\phi_\alpha) = \mathrm{lien}(\phi_{\alpha'})$. Hence both admit the same descent object and we may set

$$L(A,\alpha) = L(A,\tilde{\alpha}) = L(A,\alpha').$$

Furthermore, we have $\alpha \sim \alpha'$ if and only if $\mathrm{id}_A: A \longrightarrow A$ defines a morphism $(A,\alpha) \longrightarrow (A,\alpha')$ in $K(G)$. Prop. 4.3 gives then an equivalence

$$\mathcal{K}(\mathcal{G}) \longrightarrow \mathrm{Lien}(\mathcal{T}_{\mathcal{G}}),\ (\mathcal{A},\tilde{\alpha}) \longmapsto \mathcal{L}(\mathcal{A},\tilde{\alpha}),$$

where $\mathcal{K}(\mathcal{G})$ is obtained from $\bar{K}(G)$ by factoring out the (atomic) subcategory of morphisms represented by id_A.

5. $L(A,\alpha) \cong L(F_E)^{\mathrm{op}}$. Let E be an extension of G by A and define $\alpha: G \longrightarrow Aut(A)$ by $\alpha(g)(a) = j(g)aj(g)^{-1}$ for $a \in A, g \in G$. Then (A,α) is a G-kernel.

PROPOSITION 5.1. *The band $L(A,\alpha)$ is isomorphic to the opposite of the band $L(F_E)$ of the gerbe F_E.*

PROOF. Let $S \in \mathcal{S}_E$.There is an isomorphism

(10) $$p_2^*(E/jS) \xrightarrow{\sim} p_1^*(E/jS) \quad \text{in } F_E(G/S \times G/S)$$

which maps $(\bar{w},\bar{g},\bar{h})$ to $(\bar{w},\bar{g},\bar{h})$ with $w' = wj(h^{-1}g)$ for $w \in E$ and $\kappa(w) = h$. Note that $\bar{w}' \in E/jS$ does not depend on the choice of the representatives $w \in E$ and $g,h \in G$. Conjugation by (10) gives an isomorphism of group sheaves

$$\phi: p_2^*\big(Aut_{G/S}(E/jS)\big) \xrightarrow{\sim} p_1^*\big(Aut_{G/S}(E/jS)\big).$$

But the isomorphism

$$A \ltimes G/S \xrightarrow{\sim} Aut_{G/S}(E/jS)^{\mathrm{op}}$$

of (3) transforms ϕ into ϕ_α of Lemma 4.1, up to an inner automorphism. Hence we obtain an isomorphism $L(A,\alpha) \xrightarrow{\sim} L(F_E)^{\mathrm{op}}$ by descent.

6. $\mathrm{Ext}(G,A,\tilde{\alpha}) \cong H^2(T_G,L)$. Let E,E' be extensions of G by the same group A. We define an isomorphism $E \xrightarrow{\sim} E'$ to be an isomorphism $\theta : E \xrightarrow{\sim} E'$ of the underlying groups satisfying

$$(11) \qquad \kappa'\theta = \kappa, \quad \theta|_A = \mathrm{id}_A, \quad \text{and } \theta j|_S = j'|_S$$

for some $S \in \mathcal{S}$. Given such θ we obtain an equivalence

$$\Theta: F_E \longrightarrow F_{E'}$$

by setting $\Theta(Z) = Z$ viewed as an E'-set via θ; (11) implies that α is equivalent (in the sense of (9)) to $\alpha': G \longrightarrow Aut(A)$ defined by j'. Moreover, it follows from $\theta|_A = \mathrm{id}_A$ that Θ induces the identity on $L(A,\alpha) = L(A,\alpha')$.

In the following we fix a G-kernel (A,α) and set

$$L = L(A,\tilde{\alpha})^{\mathrm{op}}.$$

Let $\mathrm{Ext}(G,A,\tilde{\alpha})$ denote the set of isomorphism classes of extensions of G by A inducing the same abstract G-kernel $(A,\tilde{\alpha})$.

THEOREM 6.1. *The map*

$$(12) \qquad \mathrm{Ext}(G,A,\tilde{\alpha}) \longrightarrow H^2(T_G,L)$$

sending the class of an extension E to the class of the L-gerbe F_E is a bijection.

PROOF. Suppose there is an L-equivalence $\Theta: F_E \longrightarrow F_{E'}$, for extensions E,E'. Choose $S \in S_E \cap S_{E'}$, such that there exists

$$\psi : \Theta(E/jS) \xrightarrow{\sim} E'/j'S \quad \text{in } F'_E(G/S).$$

For $S' \subset S$, Θ yields

$$Aut_E(E/jS') \xrightarrow{\sim} Aut_{E'}(\Theta(E/jS) \times_{G/S} G/S')$$

since $E/jS' \cong E/jS \times_{G/S} G/S'$. The composite with $\mathrm{Int}(\psi \times 1)$ induces

$$A^{S'}j(G) \xrightarrow{\sim} A^{S'}j'(G)$$

via the isomorphisms (2). Passing then to the direct limit gives an isomorphism $\theta : E \xrightarrow{\sim} E'$. It is easy to see that θ satisfies $\kappa'\theta = \kappa$ and $\theta j(s) = j'(s)$ for $s \in \mathcal{S}$. Moreover, since θ induces the identity on L, it follows that $\theta|_A$ is an inner automorphism defined by an $a \in A$. Replacing then θ by $a^{-1}\theta\, a$ we obtain an isomorphism satisfying (11). This shows that (12) is injective.

Consider now an arbitrary L-gerbe F. By Prop. 3.1 there is an equivalence of gerbes

$$\Theta: F \longrightarrow F_{E'}$$

where E' is an extension of G by a group A'. Let (A', α') be the corresponding kernel. Then the isomorphism $L(A, \tilde{\alpha}) \xrightarrow{\sim} L(A', \tilde{\alpha}')$ induced by Θ comes from a group isomorphism $A \xrightarrow{\sim} A'$, and replacing the embedding $A' \longrightarrow E'$ by $A \xrightarrow{\sim} A' \longrightarrow E'$ gives an extension E of G by A having the same underlying group $E = E'$. But then $F_E = F_{E'}$ and $\Theta\colon F \longrightarrow F_E$ is now an L-equivalence. Hence we obtain that (12) is surjective, thereby completing the proof.

REMARK 6.2. Suppose that A is abelian. Then there is a conanonical isomorphism

$$\mathrm{Ext}(G, A, \tilde{\alpha}) \xrightarrow{\sim} H^2(G, A)$$

where the righthand side denotes the second cohomology group of the continuous discrete G-module A, [4], [5]. This can be shown in the usual way (see e.g., [5], p.63, Thm. 14) and is left to the reader.

7. **Other notions of extensions of profinite groups.**

Let A be a group and let

$$1 \longrightarrow A \longrightarrow E \xrightarrow{\kappa} G \longrightarrow 1$$

be a topological extension of the profinite group G by A as defined in ([6], 1.13). In particular, A (discrete) embeds onto a closed normal subgroup of E and κ is open. It is known that κ has a continuous section. If E is profinite this follows from the cross-section theorem ([4], p. 2, Prop. 1; [5], p. 10, Thm. 3). Evidently, E is profinite if and only if A is finite.

PROPOSITION 7.1. *There exists a continuous and open section j of κ satisfying*

$$(*) \qquad j(sg) = j(s)j(g), \text{ and } j(gs) = j(g)j(s), \quad s \in S, g \in G \text{ for some } S \in \mathcal{S}.$$

PROOF. Since 1 is open in A there is an open subset V of E such that $V \cap A = \{1\}$. Then $\kappa|_V\colon V \longrightarrow \kappa(V)$ is a homeomorphism since κ is open. Let $S \in \mathcal{S}$ with $S \subset \kappa(V)$, and let $\{h_1 = 1, \ldots, h_r\} \subset G$ be a set of representatives of G/S. Define $j(s) = \kappa|_V^{-1}(s)$ and

$$j(sh_i) = j(s)h_i' \qquad \text{for } s \in \mathcal{S}, i = 1, \ldots, r,$$

where h_i' is a preimage of h_i under κ, and $h_1' = 1$. Clearly $j(sg) = j(s)j(g)$ for all $s \in \mathcal{S}, g \in \mathcal{G}$. Since each $j(S)j(g)$ is open in E, it follows that j is open. Also, j is continuous, for if $U \subset E$ is open, then $\kappa(U \cap j(G)) = j^{-1}(U)$ is open in G. Consider now the map

$$c\colon G \times G \longrightarrow A, \quad c(g,h) = j(g)j(h)j(gh)^{-1}$$

It is continuous since its composite with $A \longrightarrow E$ is so, and since A is discrete. Hence there exists an $S' \subset S$ in $\mathcal{S}$ such that $c(gS', hS') = c(g,h), g, h \in G$. But since $c(g,1) = 1$ we conclude $j(gs') = j(g)j(s')$ for all $s' \in \mathcal{S}', g \in \mathcal{G}$. ∎

For j as above and $a \in A$, the map $G \longrightarrow A, g \longmapsto j(g)aj(g)^{-1}$, is continuous, hence a is fixed under some $S \in \mathcal{S}$. Thus we have obtained an extension (E, j) in our sense.

Conversely, given any (E, j) we can define a topology on E such that $A \times G \longrightarrow E$, $(a, g) \longrightarrow aj(g)$, is a homeomorphism, with $A \times G$ having the product topology. Then it is easy to see that E is a topological extension of G by the discrete group A.

For topological extensions of G by an arbitrary locally compact group the reader is referred to ([2], VIII, Thm. 8.4).

In [3] certain extensions $1 \longrightarrow A \longrightarrow E \xrightarrow{\kappa} G \longrightarrow 1$ were considered for which there exists an $S \in \mathcal{S}$ and a group homomorphism

$$j_S \colon S \longrightarrow E \text{ such that } \kappa j_S = \mathrm{id}_S.$$

We therefore consider the problem of extending j_S to a section $j \colon G \longrightarrow E$ satisfying $(*)$. It is clear that j_S can be extended to a section j' satisfying $j'(gs) = j'(g)j'(s)$ for all $g \in G, s \in \mathcal{S}$. Then also

$$j'(sg) = j'(g)j'(g^{-1}sg), \qquad g \in G, s \in S. \tag{13}$$

Consider for $g \in G$ the map

$$c_g \colon S \longrightarrow A, \ c_g(s) = j'(sg)j'(g)^{-1}j'(s)^{-1}.$$

PROPOSITION 7.2. *Each $c_g, g \in G$, is a 1-cocycle of S in A; j_S can be extended to a section $j \colon G \longrightarrow E$ satisfying $(*)$ if and only if c_g splits.*

PROOF. That c_g satisfies $c_g(ss') = c_g(s)c_g(s')^s$ for $s, s' \in \mathcal{S}$, is easy to see using (13). Suppose that j exists. Set $a_g = j(g)j'(g)^{-1}$. Then $j(sg) = j'(s)a_g j'(g)$. On the other hand

$$j(sg) = a_g j'(g)j'(g^{-1}sg) = a_g j'(sg).$$

Multiplying both equations by $j'(g)^{-1}j'(s)^{-1}$ gives $a_g^s = a_g c_g(s)$. Thus c_g splits. The converse is proved in the same way.

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