BULL. AUSTRAL. MATH. SOC. 14H05, 12F20
VOL. 62 (2000) [193–204]

FIELD THEORY FOR FUNCTION FIELDS OF SINGULAR PLANE QUARTIC CURVES

KEI MIURA

We study the structure of function fields of plane quartic curves by using projections. Taking a point $P \in \mathbb{P}^2$, we define the projection from a curve C to a line l with the centre P. This projection induces an extension field $k(C)/k(\mathbb{P}^1)$. By using this fact, we study the field extension $k(C)/k(\mathbb{P}^1)$ from a geometrical point of view. In this note, we take up quartic curves with singular points.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic zero. We fix k as the ground field of our discussion. Let K be an algebraic function field in one variable over k. We would like to look into K, in particular to see what intermediate fields exist. For this purpose we develop a field theory for K.

Let C be an irreducible plane curve of degree d ($d \geqslant 1$) and $k(C)$ be a rational function field of C. Then K is expressed as $K = k(C)$. In this paper we study $K = k(C)$ from a geometrical viewpoint as we have done in [**3**].

Let $\varepsilon : X \to C$ be the birational morphism from the smooth model X onto C. Then $K = k(C) = k(X)$. Take a point $P \in \mathbb{P}^2$. The morphism $\pi_P : X \to \mathbb{P}^1$ is the rational function on X defined by

$$\pi_P : X \ni R \longmapsto \overline{P\varepsilon(R)} \in \mathbb{P}^1,$$

where $\overline{P\varepsilon(R)}$ is the line passing through P and $\varepsilon(R)$, $\mathbb{P}^1$ is the one-dimensional projective space of all lines in $\mathbb{P}^2$ passing through P. The degree of π_P is clearly $d - m_P$, where m_P is the multiplicity of C at P. (We put $m_P = 0$, if $P \notin C$.) Then we have a field extension $\pi_P^* : k(\mathbb{P}^1) \hookrightarrow K$. This field extension depends only on the point P. So we denote the function field $k(\mathbb{P}^1)$ by K_P, that is,

$$\pi_P^* : K_P \hookrightarrow K.$$

Received 21st September, 1999
The author would like to express his gratitude to Professor Hisao Yoshihara for suggesting this research and giving valuable advice.

REMARK 1. The above function field K_P is not necessarily a maximal rational subfield of K. However if C is a smooth plane curve of degree d ($d \geqslant 2$) and $P \in \mathbb{P}^2$ is a point, then K_P is a maximal rational subfield (see [**3**]).

Note that every subfield K' satisfying $k \neq K' \subset K_P$ is rational by Lüroth's theorem. We are interested in the structure of the field extension K/K_P. For example,

(1) When is the extension Galois ?

(2) Let L_P be the Galois closure of K/K_P. What can we say about L_P ?

(3) What is the Galois group $\mathrm{Gal}(L_P/K_P)$?

(4) How many fields do there exist between K and K_P ?

DEFINITION 1. The point $P \in \mathbb{P}^2$ is called a Galois point if K/K_P is a Galois extension.

Let $\widetilde{C}_P$ be the smooth curve with the function field L_P and $\widetilde{\pi}_P : \widetilde{C}_P \to X$ be the covering map induced by $L_P \supset K$. We denote the composite map $\pi_P \circ \widetilde{\pi}_P$ by θ_P, that is, $\theta_P : \widetilde{C}_P \to \mathbb{P}^1$. It is clear that θ_P is Galois. We call $\widetilde{C}_P$ the minimal splitting curve of $\pi_P : X \to \mathbb{P}^1$, after Tokunaga [**8**].

DEFINITION 2. We denote by G_P the Galois group $\mathrm{Gal}(L_P/K_P)$ and by $g(P)$ the genus of $\widetilde{C}_P$.

When C is smooth, we have studied questions (1), (2), (3) and (4) for the case $d = 4$ and quintic Fermat curve (see [**3, 4**]). In what follows we assume that C has at least one singular point. In the case $d = 1$, 2 or 3, the above questions are trivial. In this paper we study the questions in the case $d = 4$. *So henceforth we denote by C a plane quartic curve which is not smooth.*

REMARK 2. For a singular point $P \in C$, we have that $\deg \pi_P = 1$ or 2. Hence we see that a singular point of C becomes a Galois point. We call this a *non-smooth* Galois point. In particular, we call the Galois point $P \in \mathbb{P}^2$ with $m_P = 0$ or 1 a *smooth* Galois point.

DEFINITION 3. We denote the number of smooth Galois points by $\delta(C)$.

We use the following notation:

$f(x,y) = 0$: the defining equation of (the affine part of) C.

$f_i(x,y)$: the homogeneous part of degree i of $f(x,y)$.

$g = g(X)$: the genus of X.

$m_P(C) = m_P$: the multiplicity of C at P. (We put $m_P = 0$, if $P \notin C$.)

$s_P(C) = s_P$: the number of the analytic branches of C at P.

$I_P(C_1, C_2)$: the intersection number of C_1 and C_2 at P.

$W = W(C)$: the sum of order of flex of C, that is,

$$W(C) = \sum_{Q \in C} \{I_Q(C, T_Q) - 2\},$$

where T_Q denotes the tangent line to C at $Q \in C$.

Let $\theta_P : \widetilde{C}_P \to \mathbb{P}^1$ be the Galois covering. Then we define a branch type of θ_P as follows (see [6]). Let $B_{\theta_P} = m_1Q_1 + \cdots + m_dQ_d$ $(2 \leqslant m_1 \leqslant \cdots \leqslant m_d)$ be the branch locus of θ_P. Here m_i is called the ramification index of θ_P at Q_i. That is, if R is a point of $\theta_P^{-1}(Q_i)$, then there are local coordinate systems ζ and η around R and Q_i respectively with $\zeta(R) = 0$ and $\eta(Q_i) = 0$ such that θ_P is locally given as: $\zeta \mapsto \eta = \zeta^{m_i}$. We say θ_P has branch type $(m_1, \ldots, m_d)$ if $B_{\pi_P} = m_1Q_1 + \cdots + m_dQ_d$.

2. Statement of Results

Under the situation above the main results are as follows. We state our results separately according to the case $P \in C$ or $P \notin C$.

In the case $P \in C$, we have the following:

Theorem 1. *Suppose C has a triple point. Then we have $g(P) = 0$ or 1 for any $P \in C$. If C has a tacnode-cusp, then C has no smooth Galois point.*

Theorem 2. *Suppose the singularities of C are at most double points. Then for any C and for any $P \in C$, we have that $g(P) = g$ or $3g + 1 - a$, $(0 \leqslant a \leqslant g + 1)$. If P is a general point of C, then G_P is isomorphic to S_3; the symmetric group on three letters, and $g(P) = 3g + 1$.*

In the case $P \notin C$, we have the following:

Theorem 3. *For any plane quartic C and any point $P \in \mathbb{P}^2 \setminus C$, G_P is isomorphic to one of the following: (1) S_4; the symmetric group on four letters, (2) A_4; the alternating group on four letters, (3) D_4; the dihedral group of order eight, (4) C_4; the cyclic group of order four or (5) V_4; Klein's four group. Furthermore, if C has no simple cusp of multiplicity three and $P \in \mathbb{P}^2 \setminus C$ is a general point, then $G_P \cong S_4$ and $g(P) = 12g + 13$.*

Corollary 4. *For a general point $P \in \mathbb{P}^2$, there exists no field between K and K_P.*

On Galois points, we have the following.

Theorem 5. *If C is a plane quartic curve of genus two (that is, $g(X) = 2$), then there is no point $P \in \mathbb{P}^2 \setminus C$ satisfying $G_P \cong C_4$.*

3. Proof of Theorem 1 and 2

First, we consider the ramification points of $\pi_P : X \to \mathbb{P}^1$ $(P \in \mathbb{P}^2)$. From the definition of π_P, we see that the following assertions hold true.

(i) If Q is a smooth point of C:
Then there exist $\widetilde{Q} \in X$ such that $\varepsilon(\widetilde{Q}) = Q$. Hence we have that the ramification index of π_P at $\widetilde{Q}$ (which we denote by $e_{\widetilde{Q}}$) equals $I_Q(C, \overline{PQ})$. For example, if $\overline{PQ}$ is the tangent line at Q, then $e_{\widetilde{Q}} \geqslant 2$.

(ii) If Q is a singular point of C:
Let $C_1, C_2, \cdots, C_s$ be the analytic branches at Q, and $\varepsilon^{-1}(Q) = \widetilde{Q}_1, \cdots, \widetilde{Q}_s$, where $s = s_Q(C)$. Then we have a one-to-one correspondence between C_j and $\widetilde{Q}_j$.
If $\overline{PQ}$ is not a tangent line to C_j, then $e_{\widetilde{Q}_j} = m_Q(C_j) = I_Q(C_j, \overline{PQ})$.
If $\overline{PQ}$ is a tangent line to some C_k, then $e_{\widetilde{Q}_k} = I_Q(C_k, \overline{PQ})$.

REMARK 3. From the Riemann-Hurwitz formula for π_P ($P \in \mathbb{P}^2$), we have

$$\sum_{R \in X} (e_R - 1) = 2g(X) + 6 - 2m_P.$$

We prove Theorems 1 and 2. For a singular point $P \in C$, the above questions (1) $\sim$ (4) are trivial (see Remark 2). So henceforth we assume that $P \in C$ is a non-singular point of C, that is, $m_P = 1$.

By taking a suitable set of coordinates, we can assume that

(i) $P = (0,0)$,
(ii) $y = 0$ is the tangent line to C at P,
(iii) the singular points of C do not lie on $x = 0$,
(iv) $x = 0$ and C meet transversally,
(v) if l is a line passing through P and a point of C at infinity, then l is not a tangent line to C and l does not pass through the singular points of C.

Let l_t be the line $y = tx$. Then we may assume that the projection is defined as $\pi_P(C \cap l_t) = t$, if l_t does not pass through the singular points of C. In the affine plane $(x,t) \in \mathbb{A}^2$, let $\widehat{C}$ be the curve defined by

$$\widehat{f}(x,t) = f(x,tx)/x = \varphi_4(t)x^3 + \varphi_3(t)x^2 + \varphi_2(t)x + \varphi_1(t),$$

where $\varphi_i(t) = f_i(1,t)$ ($1 \leqslant i \leqslant 4$). Then $K = k(x,t)$ and $K_P = k(t)$, the extension K/K_P is obtained by $\widehat{f}(x,t) = 0$. (When $t = \infty$, we consider $x = sy$ instead, where $st = 1$. Indeed, $\widehat{C}$ is an affine part of the blow-up of C.) Then we may study $\pi_P : X \to \mathbb{P}^1$ by considering a projection from $\widehat{C}$ to the t-axis.

We can find the branch points of π_P by using the discriminant of $\widehat{f}(x,t)$. Let $\psi(t)$ be the discriminant of $\widehat{f}(x,t) \in k[t][x]$, that is,

$$\psi(t) = \left(\varphi_4(t)\right)^4 \prod_{i<j} (x_i - x_j)^2$$

where x_i are the roots of $\widehat{f}(x,t) = 0$ in $\overline{k(t)}$.

DEFINITION 4. The point $Q \in C$ is called a 1-flex [respectively 2-flex], if $I_Q(C, T_Q) = 3$ [respectively 4], where T_Q is the tangent line to C at Q.

Noting that $I_P(C, T_P) = I_{\widehat{P}}(\widehat{C}, T_{\widehat{P}}) + 1$, where T_P is the tangent line to C at $P \in C$, we see that $\widehat{P}$ is the intersection point of $\widehat{C}$ and $t = 0$. We have the following lemmas by copying the proof of [**3**, Lemma 3.2].

LEMMA 1. *The discriminant $\psi(t)$ is expressed as $\psi(t) = \psi_0(t)\psi_1(t)$. If $(t-\alpha)^n$ is a factor of $\psi_0(t)$, then we have $n = 1, 2$. Suppose that $\alpha \neq 0$. Then $n = 2$ [respectively $n = 1$] if and only if the line l_α becomes a tangent line to C at a 1-flex [respectively not a flex]. On the contrary, suppose that $\alpha = 0$. Then $n = 2$ if and only if P is a 2-flex, $n = 1$ if and only if P is a 1-flex or l_0 is a bitangent line.*

LEMMA 2. *Let $P_i = (a_i, b_i)$ be the singular points of C $(1 \leqslant i \leqslant r)$. If $(t-\beta)^m$ is a factor of $\psi_1(t)$, then the line $y = \beta x$ passes through some P_i.*

REMARK 4. It is well-known that

$$-\varphi_4(t)\psi(t) = \mathrm{Res}(\widehat{f}, \partial\widehat{f}/\partial x),$$

where $\mathrm{Res}(\widehat{f}, \partial\widehat{f}/\partial x)$ is the resultant of $\widehat{f}$ and $\partial\widehat{f}/\partial x$ with respect to x. By our coordinates condition (v), if $\varphi_4(\alpha) = 0$, then $t = \alpha$ is not a branch point. Hence we have $m = \sum I_{\widehat{Q}_i}(\widehat{C}, \widehat{C}_x)$, where $\{\widehat{Q}_1, \cdots, \widehat{Q}_q\} = \widehat{C} \cap \{t = \beta\}$, and $\widehat{C}_x$ denotes the curve defined by $\partial\widehat{f}/\partial x$.

We call $\psi_0(t)$ the smooth part of $\psi(t)$, and $\psi_1(t)$ the singular part of $\psi(t)$. In particular, we can find flexes and singular points of C by computing the resultant of $\widehat{f}(x, t)$ and $\partial\widehat{f}(x, t)/\partial x$.

Next, suppose that $P \in C$ is not a Galois point. Then we consider the branch points of $\widetilde{\pi}_P : \widetilde{C}_P \to X$. Referring to [8], we have the following proposition.

PROPOSITION 6. *Let $\Delta(X/\mathbb{P}^1)$ and $\Delta(\widetilde{C}_P/\mathbb{P}^1)$ be the branch loci of π_P and θ_P respectively. Then we have $\Delta(X/\mathbb{P}^1) = \Delta(\widetilde{C}_P/\mathbb{P}^1)$.*

Hence we have the following lemma.

LEMMA 3. *A point $Q \in X$ is a branch point of $\widetilde{\pi}_P$ if and only if the following conditions are satisfied:*

(a) *suppose $\pi_P(Q) = \alpha$, then $\pi_P^{-1}(\alpha) = \{Q,\ Q'\}$,*

(b) *π_P has ramification index one at Q, two at Q'.*

PROOF: Note that the Galois covering has the same ramification indices at each branch point. Suppose R is a branch point of $\widetilde{\pi}_P$ such that the ramification index of π_P at R is three. Put $\pi_P(R) = \beta$. Then θ_P has ramification index six at $t = \beta$. Since S_3 does not contain the cyclic group of order six as a subgroup, this is a contradiction. Next, R' is a branch point of $\widetilde{\pi}_P$ such that the ramification index of π_P at R' is two. Put $\pi_P(R') = \gamma$. Then θ_P has ramification index four at $t = \gamma$. Since the degree of θ_P is six, this is a contradiction. Hence by Proposition 6, this proves the lemma. ∎

Furthermore we have the following.

Lemma 4. *If P is not a Galois point, then $g(P) = 3g(X) + 1 - a$, $(0 \leqslant a \leqslant g(X) + 1)$.*

Proof: Let a and b be the numbers of ramification points of π_P whose ramification indices are three and two respectively. By Lemma 3, the number of branch points of $\widetilde{\pi}_P : \widetilde{C}_P \to X$ equals b. By the Riemann-Hurwitz formula for $\widetilde{\pi}_P$, we have that $2g(P) - 2 = 2(2g(X) - 2) + b$. By Remark 3, we have that $2a + b = 2g(X) + 4$. Hence we obtain that $g(P) = 3g(X) + 1 - a$. However, if $b = 0$, then π_P is a Galois covering by [8, Proposition 3.1]. Thus a can not be $g(X) + 2$. This proves the lemma. ∎

Remark 5. Referring to [8], we have the following assertion. A point P is a smooth Galois point if and only if $b = 0$. Hence $G_P \cong S_3$ if and only if $b \neq 0$.

Suppose C has a tacnode-cusp Q. Then we have that $m_Q = 3$ and $s_Q = 2$. Hence we infer that π_P always has a ramification point with ramification index two for any $P \in C$. Indeed since the line $\overline{PQ}$ is not tangent at Q, we have

$$\sum_{R \in \varepsilon^{-1}(Q)} (e_R - 1) = m_Q - s_Q = 1.$$

By the above remark, we see that C has no smooth Galois point.

If C has a simple cusp of multiplicity three, then for any $P \in C$, there exist a line passing through P which meets C at the cusp with intersection number three. Hence we infer that π_P always has a ramification point with ramification index three. Therefore we infer the following.

Lemma 5. *Suppose C has no simple cusp of multiplicity three. Then $a = 0$ for a general point $P \in C$.*

Combining the above results, we obtain the assertions in Theorem 1 and 2.

Remark 6. Suppose C has an ordinary triple point Q and the line l_α passes through P and Q. Then the line l_α passes through Q with the intersection number three, and $\varepsilon^{-1}(Q)$ consists of three points in X. So we see that π_P is unramified over $t = \alpha$.

Next, as an example, we consider the curve C defined by $y + g(x, y) = 0$, where $g(x, y)$ is a homogeneous polynomial of degree four and $g(x, 0) \neq 0$.

Claim 1. The curve C has a smooth Galois point $P = (0, 0)$.

Proof: Putting $f(x, y) = y + g(x, y)$, we have $\widehat{f}(x, t) = g(1, t)x^3 + t$. Since the field extension K/K_P is given by $x^3 = -t/g(1, t)$, the claim is clear. ∎

Then the homogeneous equation of C is $F(x, y, z) = yz^3 + g(x, y)$, where x, y, z are the homogeneous coordinates of $\mathbb{P}^2$.

Claim 2. The singular points of $C \subset \mathbb{P}^2$ exist only on the line $z = 0$, and satisfy $\partial g/\partial x = \partial g/\partial y = 0$.

PROOF: Since $C \subset \mathbb{P}^2$ is defined by $F(x,y,z) = 0$, the claim is clear by considering $\partial F/\partial x = \partial F/\partial y = \partial F/\partial z = 0$. $\square$

Then we may assume that $g(x,y)$ is the one of the following:

(i) $g(x,y) = (y-\alpha x)^2(y-\beta x)(y-\gamma x)$,

(ii) $g(x,y) = (y-\alpha x)^2(y-\beta x)^2$,

(iii) $g(x,y) = (y-\alpha x)^3(y-\beta x)$,

(iv) $g(x,y) = (y-\alpha x)^4$,

where α, β, γ are mutually distinct elements of $k \setminus \{0\}$. Then the singular points of C for each case are the following:

(i) $(1:\alpha:0)$ is the only singular point and is a simple cusp of multiplicity two,

(ii) $(1:\alpha:0)$ and $(1:\beta:0)$ are the only singular points and are simple cusps of multiplicity two,

(iii) $(1:\alpha:0)$ is the only singular point and is an ordinary triple point,

(iv) $(1:\alpha:0)$ is the only singular point and is a simple cusp of multiplicity three.

In case (i), we see that $(1:\beta:0)$ and $(1:\gamma:0)$ are 1-flexes, and the line $y = \beta x$ [respectively $y = \gamma x$] is the tangent line at $(1:\beta:0)$ [respectively $(1:\gamma:0)$]. Of course P is a 2-flex. Furthermore the line $y = \alpha x$ is the tangent line at $(1:\alpha:0)$. Hence π_P has branch type $(3,3,3,3)$. Indeed π_P has ramification index three at $t = \alpha$, $t = \beta$, $t = \gamma$ and $t = 0$.

In case (ii), we see that π_P has branch type $(3,3,3)$ by an argument similar to (i). In particular, in cases (i) and (ii), there are no more smooth Galois points (by considering the branching data of the other points). Indeed for an other point $Q \in C$ $(Q \neq P)$ and a line l_λ passing through Q, we see that l_λ meets C at $(1:\alpha:0)$ with intersection number two and it intersects C transversally at the other point. Hence $\pi_Q^{-1}(\lambda)$ is a two point, so we see π_Q is not Galois.

In case (iii), we see that π_P has branch type $(3,3)$. Indeed π_P has ramification index three at $t = 0$ and $t = \beta$.

In case (iv), we see that π_P has branch type $(3,3)$. Referring to [**1**], we have $W(C) = 2$. Since P is a 2-flex, there is no more flex. Hence we have $\delta(C) = 1$ in cases (i), (ii) and (iv).

4. PROOF OF THEOREM 3 AND 5

Next we consider the case $P \in \mathbb{P}^2 \setminus C$. By taking a suitable set of coordinates, we may assume

(i) $P = (0,0) \notin C$,

(ii) the singular points of C do not lie on $x = 0$,

(iii) $x = 0$ and C meet transversally.

Let l_t be the line $y = tx$. Then we may assume that the projection is defined as $\pi_P(C \cap l_t) = t$, if l_t does not pass through the singular points of C. In the affine plane $(x, t) \in \mathbb{A}^2$, let $\check{C}$ be the curve defined by

$$\check{f}(x,t) = f(x,tx)/x = \varphi_4(t)x^4 + \varphi_3(t)x^3 + \varphi_2(t)x^2 + \varphi_1(t)x + c,$$

where $\varphi_i(t) = f_i(1,t)$ $(1 \leqslant i \leqslant 4)$ and c is a non-zero element of k. Then $K = k(x,t)$ and $K_P = k(t)$, the extension K/K_P is obtained by $\check{f}(x,t) = 0$. We can find flexes and singular points of C by an argument similar to that in Lemmas 1 and 2. So we know the ramification points of $\pi_P : X \to \mathbb{P}^1$. We use $\psi(t)$, $\psi_0(t)$ and $\psi_1(t)$ as in Lemmas 1 and 2.

Lemma 6. *Suppose C has no simple cusp of multiplicity three. If P is a general point, then π_P has just $2g(X) + 6$ ramification points, and its ramification indices are two.*

Proof: If P is a general point and l is a line passing through P, then one of the following assertions holds true:

(a) The line l intersects C transversally.

(b) The line l touches at one point $Q \in C$ with $I_Q(C,l) = 2$, and it intersects C transversally at the other points.

(c) The line l is not a tangent line at singular points.

Hence we prove the lemma from the Riemann-Hurwitz formula for π_P. ∎

Let P be a general point and $\theta_P : \widetilde{C}_P \to \mathbb{P}^1$ be the Galois covering. If $R \in \widetilde{C}_P$ is a ramification point, then we infer that the ramification number of θ_P at R is two by the above lemma. Now the theorem is proved by copying after the proof of [**7**, Theorem 4.4.5]. Since θ_P is unramified over $t = \infty$ and the inertia group at the ramification point R is generated by a transposition, the latter part of the theorem is a consequence of [**7**, Proposition 4.4.6 and Lemma 4.4.4]. From the Riemann-Hurwitz formula, we obtain that $2g(P) - 2 = 24(0-2) + 12(2g(X) + 6)$. Hence we have $g(P) = 12g(X) + 13$. Corollary 4 is a general fact of Galois theory and the structure of S_4.

Now we define the cubic resolvent $\widetilde{g}(z)$ of $\check{f}(x,t) = 0$ as follows:

$$\widetilde{g}(z) = z^3 - \frac{\varphi_2}{\varphi_4}z^2 + \frac{\varphi_1\varphi_3 - 4c\varphi_4}{\varphi_4^2}z + \frac{4c\varphi_2\varphi_4 - c\varphi_3^2 - \varphi_1^2\varphi_4}{\varphi_4^3},$$

where $\varphi_i = \varphi_i(t)$ $(1 \leqslant i \leqslant 4)$. Clearly $\widetilde{g}(z) \in k(t)[z]$. Then we have the following facts from field theory (see [**2**]).

Fact 7. *Let M be the splitting field of $\widetilde{g}(z) = 0$ over $k(t)$. Then we have the following.*

(i) *$G_P \cong S_4 \Leftrightarrow \widetilde{g}(z)$ is irreducible over $k(t)$ and $\sqrt{\psi(t)} \notin k(t)$.*

(ii) *$G_P \cong A_4 \Leftrightarrow \widetilde{g}(z)$ is irreducible over $k(t)$ and $\sqrt{\psi(t)} \in k(t)$.*

(iii) *$G_P \cong V_4 \Leftrightarrow \widetilde{g}(z)$ splits into linear factors over $k(t)$.*

(iv) *$G_P \cong C_4 \Leftrightarrow \widetilde{g}(z)$ has exactly one root α in $k(t)$ and $h(x):=(x^2 - \alpha x + c/\varphi_4)(x^2 + (\varphi_3/\varphi_4)x + \varphi_2/\varphi_4 - \alpha)$ splits over M.*

(v) *$G_P \cong D_4 \Leftrightarrow \widetilde{g}(z)$ has exactly one root α in $k(t)$ and $h(x)$ does not split over M.*

Next, we would like to characterise the defining equations by the structure of G_P. First, we recall general facts in field theory.

Fact 8. *If G_P is isomorphic to S_4 or A_4, then there exists no field between K and K_P.*

In the cases when $G_P \cong V_4$, C_4 or D_4, we can find subfields between K and K_P by considering the subgroups of G_P. The following lemma is clear.

Lemma 7. *The following assertions are equivalent.*

(a) *K contains an intermediate subfield K' with $[K' : K_P] = 2$.*

(b) *K is expressed as $K = K_P(\xi)$, where ξ satisfies an irreducible polynomial $x^4 + ax^2 + b \in K_P[x]$.*

From the above lemma, we obtain that $K = k(x,t)$, where $x^4 + ax^2 + b = 0$. The coefficients a and b are in $k(t)$, so we denote these by $a(t)$, $b(t)$ respectively. By cancelling the denominator, we obtain $c(t)x^4 + d(t)x^2 + e(t) = 0$, where $c(t)$, $d(t)$ and $e(t) \in k[t]$, and they are assumed to be relatively prime. Putting $y = tx$, we obtain that

$$c(y/x)x^4 + d(y/x)x^2 + e(y/x) = 0.$$

In order to obtain a quartic equation, $\deg e(t)$ must be zero, $\deg d(t) \leqslant 2$ and $\deg c(t) \leqslant 4$. Whence we infer the following assertion. The group G_P is isomorphic to V_4, C_4 or D_4 if and only if the quartic curve C is birationally equivalent to the curve defined by $f_4(x,y)+f_2(x,y)+c = 0$, where $f_i(x,y)$ is a homogeneous polynomial of degree i $(i = 2,4)$, and c is a non-zero element of k. Then we have that $\check{f}(x,t) = \varphi_4(t)x^4 + \varphi_2(t)x^2 + c$ and $\widetilde{g}(z) = (z - \varphi_2/\varphi_4)(z^2 - 4c/\varphi_4)$. Applying Fact 1 to this case, we have the following:

(i) $G_P \cong V_4 \Leftrightarrow \sqrt{c/\varphi_4} \in k(t)$.

(ii) $G_P \cong C_4 \Leftrightarrow \sqrt{c/\varphi_4 \cdot (\varphi_2^2/\varphi_4^2 - 4c/\varphi_4)} \in k(t)$.

(iii) $G_P \cong D_4 \Leftrightarrow$ neither (i) nor (ii).

By using this fact, we present some examples.

EXAMPLE 1. If C is the curve defined by $x^4-2x^2y^2+y^4+x^2+y^2+1=0$ and $P=(0,0)$, then we have that $G_P \cong V_4$ and $g(X)=1$. Indeed we have

$$\begin{aligned}\check{f}(x,t) &= (t^4-2t^2+1)x^4+(t^2+1)x^2+1,\\ \widetilde{g}(z) &= \left(z+\frac{2}{t^2+1}\right)\left(z-\frac{2}{t^2+1}\right)\left(z-\frac{t^2+1}{(t^2-1)^2}\right),\\ \psi(t) &= 16(t^2-3)^2(3t^2-1)^2(t+1)^2(t-1)^2.\end{aligned}$$

The singular points of C are $(1:-1:0)$ and $(1:1:0)$ which are nodes. Since the lines $y=\pm x$ are not tangent lines at these nodes, π_P is unramified at $t=\pm 1$. Furthermore the line $y=\alpha x$, where α satisfies $(\alpha^2-3)^2(3\alpha^2-1)^2=0$, is the bitangent line of C. Hence π_P has branch type $(2,2,2,2)$. By calculating the equation $\check{f}(x,t)=0$, we have that

$$K = k(x,t) = k\big(t, \sqrt{t^2-3}, \sqrt{1-3t^2}\big).$$

Then we obtain three intermediate subfields between K and K_P by considering the subgroups of V_4. Indeed we have

$$K_1 = k\big(t,\sqrt{1-3t^2}\big),\ K_2 = k\big(t,\sqrt{t^2-3}\big),\ \text{and } K_3 = k\big(t,\sqrt{-3+10t^2-3t^4}\big).$$

Let C_i be the smooth curve defined by K_i $(1 \leqslant i \leqslant 3)$. Then we have that $g(C_1)=0$, $g(C_2)=0$ and $g(C_3)=1$, where $g(C_i)$ is the genus of C_i.

EXAMPLE 2. If C is the curve defined by $y^4-xy^3+x^2+1=0$ and $P=(0,0)$, then we have that $G_P \cong D_4$ and $g(X)=2$. Indeed we have

$$\begin{aligned}\check{f}(x,t) &= (t^4-t^3)x^4+x^2+1,\\ \psi(t) &= 16(t-1)(4t^4-4t^3-1)^2t^3.\end{aligned}$$

The point $(1:0:0)$ is the only singular point of C and it is a simple cusp of multiplicity two. Note that the line $y=\alpha x$ is the bitangent line of C, where α satisfies $4\alpha^4-4\alpha^3-1=0$. Furthermore let x_i $(1 \leqslant i \leqslant 4)$ be the roots of the equation $\check{f}(x,t)=0$. We have the following diagram of Galois correspondences:

$$\begin{array}{ccc} L_P & \longleftrightarrow & \{1\} \\ | & & | \\ K = k(t,x_1) & \longleftrightarrow & \langle (34) \rangle \\ | & & | \\ k(t,x_1^2) & \longleftrightarrow & \langle (12),(34) \rangle \\ | & & | \\ K_P = k(t) & \longleftrightarrow & D_4 = \langle (1324),(12) \rangle \end{array}$$

Let C' be the smooth curve defined by $k(t,x_1^2)$. Then we can check easily that the double covering $C' \to \mathbb{P}^1$ branches at four points satisfying $4t^4-4t^3-1=0$, so $g(C')=1$. Since the function field L_P is isomorphic to $k\big(t,x_1,\sqrt{\psi(t)}\big)$, we see that $\theta_P : \widetilde{C}_P \to \mathbb{P}^1$ has branch type $(2,2,2,2,2,2)$, so $g(P)=5$.

Next, we present an example of a curve C satisfying $G_P \cong A_4$ for some $P \notin C$. Then, from Fact 1, we infer the following.

REMARK 7. The group $G_P \cong A_4$ if and only if the field extension $M/k(t)$ is a cubic Galois extension.

Then we see that $L_P = k(t,x,z)$. Let R_P be the smooth curve defined by $M = k(t,z)$. Since $k[t,x,z]$ is a tensor product of $k[t,x]$ and $k[t,z]$ over $k[t]$, $\widetilde{C}_P$ is the desingularisation of the fibre product of X and R_P over $\mathbb{P}^1$.

EXAMPLE 3. If C is the curve defined by $x^4+x^3y+xy^3+y^4+2x^3+2y^3+2x+2y+1=0$ and $P=(0,0)$, then we have that $G_P \cong A_4$ and $g(X)=2$. Indeed we have

$$\begin{aligned} \check{f}(x,t) &= (t^4+t^3+t+1)x^4+(2t^3+2)x^3+(2+2t)x+1, \\ \psi(t) &= -432(t^2-t+1)^2(2t^2+t+2)^2(t+1)^4. \end{aligned}$$

The point $(1:-1:0)$ is the only singular point of C and it is a biflecnode. The line $y=\alpha x$, where α satisfies $(\alpha^2-\alpha+1)^2(2\alpha^2+\alpha+2)^2=0$, becomes a tangent line at the 1-flex of C, and the line $y=-x$ is a tangent line at the biflecnode. Therefore the number of ramification points of π_P is five, and its ramification indices are three. Furthermore the field extension $M/k(t)$ is a Galois extension given by $x^3=-4(2t^2+t+2)(t+1)^2(t^2-t-1)$. Hence $G_P \cong A_4$. We obtain a triple Galois covering $R_P \to \mathbb{P}^1$, its branch type is $(3,3,3,3,3)$. From the above consideration, we see that $\theta_P : \widetilde{C}_P \to \mathbb{P}^1$ has branch type $(3,3,3,3,3)$, so $g(P)=9$.

Finally we prove Theorem 5. First we find the conditions when $G_P \cong C_4$. Since K/K_P is a Galois extension, its Galois group is isomorphic to C_4. Hence K can be expressed as $k(x,t)$ where $x^4 = a(t)/b(t) \in k(t) = K_P$. Putting $y=tx$, we have $b(y/x)x^4 = a(y/x)$. In order to obtain a quartic equation, $\deg a(t)$ must be zero. Whence we infer the following assertion. The covering $\pi_P : X \to \mathbb{P}^1$ is Galois with its Galois group $G_P \cong C_4$ if and only if the quartic curve C is birationally equivalent to the curve defined by $g(x,y)+c=0$, where $g(x,y)$ is a homogeneous polynomial of degree four and c is a non-zero element of k.

Then we study the curve defined by $g(x,y)+c=0$ with $g(x,y)$ and c as above. The homogeneous equation of the curve is $F(x,y,z)=cz^4+g(x,y)$, where x,y,z are the homogeneous coordinates of $\mathbb{P}^2$. Then we may assume that $g(x,y)$ is the one of the following:

(i) $g(x,y)=(y-\alpha x)^2(y-\beta x)(y-\gamma x)$,

(ii) $g(x,y)=(y-\alpha x)^2(y-\beta x)^2$,

(iii) $g(x,y)=(y-\alpha x)^3(y-\beta x)$,

(iv) $g(x,y)=(y-\alpha x)^4$,

where α, β, γ are mutually distinct elements of k. Then the singular points of C for each case are same as the case $P \in C$. But we see that $F(x,y,z)$ is not irreducible in cases

(ii) and (iv). So we consider cases (i) and (iii).

In case (i), we see that the singular point $(1:\alpha:0)$ is locally defined by $y^2 = z^4$, and the line $y = \alpha x$ is the tangent line at this point. Furthermore we see that $(1:\beta:0)$ and $(1:\gamma:0)$ are 2-flexes. Then we have $g(X) = 1$ and the branch type of $\pi_P : X \to \mathbb{P}^1$ is $(2,4,4)$. Indeed π_P has ramification index two at $t = \alpha$ and four at $t = \beta$ and $t = \gamma$.

In case (iii), we see that the singular point $(1:\alpha:0)$ is a simple cusp of multiplicity three, and the line $y = \alpha x$ is the tangent line at this point. Furthermore we see that $(1:\beta:0)$ is a 2-flex. Then we have $g(X) = 0$ and the branch type of $\pi_P : X \to \mathbb{P}^1$ is $(4,4)$.

In particular, we obtain Theorem 5. Thus we complete the proofs.

References

[1] S. Iitaka, K. Ueno and Y. Namikawa, *Descartes no seishin to Daisûkika*, (in Japanese) (Nippon Hyoron Sha, Tokyo, 1980).

[2] L. Kappe and B. Warren, 'An elementary test for the Galois group of a quartic polynomial', *Amer. Math. Monthly* **96** (1989), 133–137.

[3] K. Miura and H. Yoshihara, 'Field theory for function fields of plane quartic curves', *J. Algebra* **226** (2000), 283–294.

[4] K. Miura and H. Yoshihara, 'Field theory for the function field of the quintic Fermat curve', *Comm. Algebra* **28** (2000), 1979–1988.

[5] M. Namba, *Geometry of projective algebraic curves* (Marcel Dekker, New York, Basel, 1984).

[6] M. Namba, *Branched coverings and algebraic functions*, Pitman Research Notes in Mathematics **161** (Longman, Harlow, 1987).

[7] J.P. Serre, *Topics in Galois theory* (Notes written by H. Darmon), Research Notes in Mathematics **1** (Jones and Bartlett Publ., Boston, London, 1992).

[8] H. Tokunaga, 'Triple coverings of algebraic surfaces according to the Cardano formula', *J. Math. Kyoto Univ.* **31** (1991), 359–375.

Graduate School of Science and Technology
Niigata University
Niigata 950-2181
Japan
e-mail: kmiura@scux.sc.niigata-u.ac.jp