



# Moments of the central $L$ -values of the Asai lifts

Wenzhi Luo

*Abstract.* We study some analytic properties of the Asai lifts associated with cuspidal Hilbert modular forms, and prove sharp bounds for the second moment of their central  $L$ -values.

## 1 Introduction

Let  $\mathbf{F}$  be a fixed real quadratic field over  $\mathbf{Q}$ , with ring of integers  $O = O_{\mathbf{F}}$  and the real imbeddings  $\sigma_1 = 1, \sigma_2$ . For simplicity, we assume the narrow class number of  $\mathbf{F}$  is 1, so the totally positive units are squares of units and every ideal has a totally positive generator. Let  $SL(2, O)$  be the Hilbert modular group. For any ideal  $\mathcal{C} \subset O$ , the Hecke congruence subgroups  $\Gamma_0(\mathcal{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, O), \ c \equiv 0 \pmod{\mathcal{C}} \right\}$ , act discontinuously on the upper half-space  $\mathbf{H}^2$  in the usual way with finite co-volumes, i.e., for

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{C}) \text{ and } z = (z_1, z_2) \in \mathbf{H}^2,$$

we have

$$\gamma(z) = \left( \frac{\sigma_1(a)z_1 + \sigma_1(b)}{\sigma_1(c)z_1 + \sigma_1(d)}, \frac{\sigma_2(a)z_1 + \sigma_2(b)}{\sigma_2(c)z_1 + \sigma_2(d)} \right).$$

Denote by  $M_k(\Gamma_0(\mathcal{C}))(k \in 2\mathbf{Z} \text{ and } \geq 2)$ , the space of Hilbert modular forms of parallel even weight  $(k, k)$ , level  $\mathcal{C}$  with trivial character, i.e., the space of holomorphic functions  $f(z)$  on  $\mathbf{H}^2$  such that for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathcal{C})$ ,  $f(\gamma(z)) = N(cz + d)^k f(z)$ , where for  $z = (z_1, z_2) \in \mathbf{H}^2$ ,

$$N(cz + d)^k = (\sigma_1(c)z_1 + \sigma_1(d))^k \cdot (\sigma_2(c)z_2 + \sigma_2(d))^k.$$

Any  $f(z)$  in  $M_k(\Gamma_0(\mathcal{C}))$  has the following Fourier expansion (we assume that the different of  $\mathbf{F}$  is generated by  $\delta = \delta_{\mathbf{F}} > 0$ , where and henceforth  $\xi > 0$  for  $\xi \in \mathbf{F}$  means that  $\xi$  is a totally positive element in  $\mathbf{F}$ , and denote  $v^{(i)} = \sigma_i(v)$ , the  $i$ th conjugate of  $v$

Received by the editors October 17, 2023; accepted February 25, 2024.

Published online on Cambridge Core March 4, 2024.

This research is partially supported by a Simons Foundation Collaboration Grant.

AMS subject classification: 11F41, 11F30, 11F66.

Keywords: Hilbert modular form, Asai lift, central  $L$ -values, Petersson formula.



for  $i = 1, 2$ ):

$$(1) \quad f(z) = \sum_{v \in O, v \geq 0} a(v) \exp(2\pi i \operatorname{Tr}(vz)),$$

where

$$\operatorname{Tr}(vz) = \sum_{i=1}^2 v^{(i)} z_i \delta^{(i)-1}.$$

Since any  $f(z)$  in  $M_k(\Gamma_0(\mathcal{C}))$  is invariant under  $\begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}$ , where  $\varepsilon$  is a unit in  $O$ , we have  $a(\varepsilon^2 v) = a(v)$ .

$f(z) \in M_k(\Gamma_0(\mathcal{C}))$  is called a Hilbert modular cusp form if the Fourier expansion of  $f(g(z))N(cz + d)^{-k}$  (see [Lu, p. 130]) has no constant term for all  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{F})$ . Space of all such cusp forms is denoted by  $S_k(\Gamma_0(\mathcal{C}))$ .

It is well-known (see [Ga]) that  $\dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C}))$  is finite, and (see [Sh])  $J =: \dim_{\mathbf{C}} S_k(\Gamma_0(\mathcal{C})) \sim \frac{\operatorname{vol}(\Gamma_0(\mathcal{C}) \backslash \mathbf{H}^2)}{(4\pi)^2} (k-1)^2$  as  $k \rightarrow \infty$ . Moreover,

$$\begin{aligned} \operatorname{vol}(\Gamma_0(\mathcal{C}) \backslash \mathbf{H}^2) &= [SL(2, O) : \Gamma_0(\mathcal{C})] \operatorname{vol}(SL(2, O) \backslash \mathbf{H}^2) \\ &= 2N(\mathcal{C}) \prod_{\mathcal{P}|\mathcal{C}} (1 + N(\mathcal{P})^{-1}) \times \pi^{-2} \zeta_{\mathbf{F}}(2) D^{3/2}, \end{aligned}$$

where  $\zeta_{\mathbf{F}}(s)$  is the Dedekind zeta-function of  $\mathbf{F}$  and  $D = D_{\mathbf{F}}$  is the discriminant. The Petersson inner product on  $S_k(\Gamma)$  is defined by

$$\langle g_1, g_2 \rangle = \int_{\Gamma \backslash \mathbf{H}^2} g_1(z) \overline{g_2(z)} \prod_{i=1}^2 y_i^{k-2} dx_i dy_i,$$

where  $z = (z_1, z_2)$  with  $z_i = x_i + y_i \sqrt{-1}$ ,  $i = 1, 2$ .

Now, let  $f$  be a cuspidal Hilbert modular form of parallel weight  $(k, k)$  for even  $k \geq 2$  and with respect to  $GL^+(2, O) \supset SL(2, O)$ . We assume  $f$  is a normalized Hecke eigenform with Fourier coefficients  $a_f(v) = a_f(1) \lambda_f(v) N(v)^{(k-1)/2}$ ,  $v \in O$ , where  $\lambda_f(\mu)$  is the eigenvalue of  $f(z)$  for the Hecke operator  $T_{(\mu)}$  (see, e.g., [Ga]). We have

$$\lambda_f(\mu) \lambda_f(v) = \sum_{(d), d | (\mu, v), d > 0} \lambda_f\left(\frac{\mu v}{d^2}\right).$$

The standard  $L$ -function associated with  $f$  is defined, for  $\Re(s) > 1$ , by

$$L(s, f) = \sum_{(\mu), \mu > 0} \lambda_f(\mu) N(\mu)^{-s},$$

which has Euler product

$$\prod_{(\pi), \pi > 0} (1 - \lambda_f(\pi) N(\pi)^{-s} + N(\pi)^{-2s})^{-1},$$

where  $\pi$  stands for prime element of  $O$ . It is well-known that  $L(s, f)$  has analytic continuation to the whole complex plane as an entire function. Let

$$\Lambda(s, f) = (2\pi)^{-2s} \Gamma^2(s + (k-1)/2) L(s, f).$$

We then have the functional equation

$$\Lambda(s, f) = \varepsilon_f D^{1-2s} \Lambda(1-s, f),$$

where  $\varepsilon_f$  is the root number of absolute value 1.

Asai [As] defined a new Dirichlet series by restricting the coefficients on rational integers,

$$L(s, \text{As}(f)) = \zeta(2s) \sum_{m=1}^{\infty} \lambda_f(m) m^{-s}, \quad \Re(s) > 1.$$

He showed that the function

$$\Lambda(s, \text{As}(f)) = D^{s/2} (2\pi)^{-2s} \Gamma(s + k-1) \Gamma(s) L(s, \text{As}(f))$$

admits analytic continuation to the whole  $s$ -plane with possible simple poles at  $s = 0, 1$ , and satisfies the functional equation

$$\Lambda(s, \text{As}(f)) = \Lambda(1-s, \text{As}(f)).$$

Moreover, if

$$\begin{aligned} L(s, f) &= \prod_{(\pi), \pi > 0} (1 - \lambda_f(\pi) N(\pi)^{-s} + N(\pi)^{-2s})^{-1} \\ &= \prod_{(\pi), \pi > 0} [(1 - \alpha_f(\pi) N\pi^{-s})(1 - \beta_f(\pi) N\pi^{-s})]^{-1}, \end{aligned}$$

then we have

$$L(s, \text{As}(f)) = \prod_p L_p(s),$$

where

$$L_p^{-1}(s) = \begin{cases} (1 - \alpha_f(\pi_1) \alpha_f(\pi_2) p^{-s})(1 - \alpha_f(\pi_1) \beta_f(\pi_2) p^{-s}) \\ (1 - \beta_f(\pi_1) \alpha_f(\pi_2) p^{-s})(1 - \beta_f(\pi_1) \beta_f(\pi_2) p^{-s}), & \text{if } p = \pi_1 \pi_2, \pi_1 \neq \pi_2; \\ (1 - \alpha_f(\pi) p^{-s})(1 - \beta_f(\pi) p^{-s})(1 - p^{-2s}), & \text{if } p = \pi; \\ (1 - \alpha_f^2(\pi) p^{-s})(1 - \beta_f^2(\pi) p^{-s})(1 - p^{-s}), & \text{if } p = \pi^2. \end{cases}$$

Ramakrishnan [Ra] and Krishnamurthy [Kr] proved that  $\Lambda(s, \text{As}(f))$  is in fact the  $L$ -function associated with an automorphic form on  $GL(4, A_Q)$ , the Asai lift  $\text{As}(f)$  of  $f$ . Then, in view of the Splitting Formula in [As] and assuming  $D = D_F$  is odd, we have

$$L(s, f \otimes f^t) = L(s, \text{As}(f)) L(s, \text{As}(f) \otimes \chi_D),$$

where

$$\chi_D(\cdot) = \left( \frac{D}{\cdot} \right)$$

is the Kronecker symbol, and

$$f^t(z_1, z_2) = f(z_2, z_1).$$

If  $f$  is a base change from an Hecke eigenform  $h \in S_k(SL_2(\mathbf{Z}))$ , then  $f$  is symmetric, i.e.,  $f = f^t$ , and

$$L(s, \text{As}(f)) = L(s, \text{sym}^2(h)) L(s, \chi_D),$$

while if  $f$  is a base change from an Hecke eigenform  $h \in S_k(\Gamma_0(D), \chi_D)$ , then also  $f = f^t$ , and

$$L(s, \text{As}(f)) = L(s, \text{sym}^2(h)) \zeta(s)$$

(see [As, Section 5]).

Moreover, Prasad and Ramakrishnan [PR] established the following (special case of) cuspidal criterion for  $\text{As}(f)$ .

**Theorem 1.1** (Prasad and Ramakrishnan) *With the same notation as above. If  $f$  is non-dihedral, then  $\text{As}(f)$  is non-cuspidal iff  $f$  and  $f^t$  are twist-equivalent; if  $f$  is dihedral, then  $\text{As}(f)$  is non-cuspidal iff  $f$  is induced from a quadratic extension  $K$  of  $F$  which is biquadratic over  $\mathbf{Q}$ .*

Choosing an orthonormal basis  $\{f_j(z)\}_{j=1}^J$  of  $S_k(\Gamma_0(\mathcal{C}))$  and denote the Fourier coefficients of  $f_j(z)$  by  $a_j(\cdot)$ . We normalize the Fourier coefficients  $a_j(\mu)$  by

$$\psi_j(\mu) = \left( \frac{N(\mathcal{C})((k-1)!)^2 D^{k+1}}{((4\pi)^2 N(\mu))^{k-1}} \right)^{1/2} a_j(\mu).$$

We then have the Petersson formula for Hilbert modular forms as proved in [Lu],

$$\sum_{j=1}^J \bar{\psi}_j(v) \psi_j(\mu) = \chi_v(\mu) D^{3/2} N(\mathcal{C}) (k-1)^2 + N(\mathcal{C}) (k-1)^2 D (2\pi)^2 \sum_{\varepsilon \in U} \sum_{c \in \mathcal{C}^\times / U} \frac{1}{|N(c)|} S(v, \mu \varepsilon^2; c) N J_{k-1}(4\pi \sqrt{\mu v} |\varepsilon|/|c|),$$

where  $\chi_v$  is the characteristic function of the set  $\{v \varepsilon^2, \varepsilon \in U\}$ ,  $U$  is the unit group of  $\mathbf{F}$ ,

$$S(v, \mu; c) = \sum_{h \pmod{c}}^* e\left(\frac{vh + \mu \bar{h}}{c}\right)$$

is the generalized Kloosterman sum, and  $e(x) = \exp(2\pi i \text{Tr}(x))$  for  $x \in \mathbf{F}$ . We will assume that in the above formula, the  $c$ 's are chosen among their associates the representatives satisfying  $|N(c)|^{1/2} \ll |c^{(i)}| \ll |N(c)|^{1/2}$ ,  $i = 1, 2$ .

If the  $L^2$ -normalized basis element  $f_j = \tilde{f}_j / |\tilde{f}_j|$  is a newform, where  $\tilde{f}_j$  is the corresponding arithmetically normalized newform with the first Fourier coefficient 1, then  $\psi_j(\mu) = \psi_j(1) \lambda_j(\mu)$ , where  $\lambda_j(\cdot)$  denotes the (normalized) Hecke eigenvalues of  $f_j$  as noted above. For  $\mathcal{C} = (1)$ , from the integral representation for  $L(s, \tilde{f}_j \otimes \overline{\tilde{f}_j})$ ,

and the factorization  $L(s, \tilde{f}_j \otimes \overline{\tilde{f}_j}) = \zeta_{\mathbf{F}}(s) L(s, \text{ad}(\tilde{f}_j))$ , we have

$$|a_j(1)|^{-2} = \|\tilde{f}_j\|^2 = 16D^{1+k} (4\pi)^{-2k-2} \Gamma^2(k) L(1, \text{ad}(\tilde{f}_j)) / L(1, \chi_D).$$

Thus for  $\mathcal{C} = (1)$ ,

$$\bar{\psi}_j(v) \psi_j(\mu) = \frac{(4\pi)^4 L(1, \text{ad}(\tilde{f}_j))}{16L(1, \chi_D)} \lambda_j(v) \lambda_j(\mu).$$

For each  $j$ ,  $1 \leq j \leq J$  and any  $\varepsilon > 0$ , we have (see [Ta])

$$\lambda_j(\mu) \ll N(\mu)^\varepsilon,$$

and by a straightforward extension of results of [Iw] and [HL] that

$$k^{-\varepsilon} \ll L(1, \text{ad}(\tilde{f}_j)) \ll k^\varepsilon.$$

In [Lu], we proved an asymptotic formula for the mean value of the linear form in  $\psi_j(\cdot)$  in the level aspect. In this paper, we establish an analogous result for the weight aspect as well in the context of the quadratic field  $\mathbf{F}$ , with an application to the second moment of  $L(1/2, \text{As}(f))$ . The generalization of Theorem 1.2 to the general totally real fields is straightforward.

**Theorem 1.2** *Let  $b(\cdot)$  be an arbitrary complex numbers such that  $b(\varepsilon^2\mu) = b(\mu)$  for  $\varepsilon \in U$ , and  $\eta > 0$ . Then for  $S_k(\Gamma_0(\mathcal{C}))$ , we have as  $k \rightarrow \infty$ ,*

$$\sum_{j=1}^J \left| \sum_{\mu} b(\mu) \psi_j(\mu) \right|^2 \ll (N(\mathcal{C})k^2 + X)(kXN(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2,$$

where the summation over  $\mu$ 's is restricted to  $\mu \in O^\times/U^2$ ,  $\mu > 0$ ,  $N(\mu) \leq X$ , and the implicit constant only depends on the quadratic field  $\mathbf{F}$  and  $\eta$ .

Assume  $\text{As}(f)$  is cuspidal. From [IK, p. 98], we have a series representation for the central  $L$ -value of  $L(s, \text{As}(f))$ ,

$$(3) \quad L(1/2, \text{As}(f)) = 2 \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^{1/2}} V_{1/2} \left( \frac{n}{\sqrt{D}} \right),$$

where

$$V_{1/2}(y) = \frac{1}{2\pi i} \int_{(2)} (4\pi^2 y)^{-u} \zeta(1+2u) \frac{\Gamma(1/2+u) \Gamma(k+u-1/2)}{\Gamma(1/2) \Gamma(k-1/2)} \frac{du}{u}.$$

Since

$$\frac{\Gamma(k+u-1/2)}{\Gamma(k-1/2)} \ll k^{\Re(u)}$$

by Stirling's formula, we see that  $V_{1/2}(y) \ll k^{-A}$  for any  $A \geq 1$ , if  $y > k^{1+\eta}$  for any  $\eta > 0$ . Thus, we have

$$L(1/2, \text{As}(f)) = 2 \sum_{n \leq k^{1+\eta}} \frac{\lambda_f(n)}{n^{1/2}} V_{1/2} \left( \frac{n}{\sqrt{D}} \right) + O(1).$$

From Theorem 1.2 and the above formula for  $L(1/2, \text{As}(f))$ , and by extending the orthonormal Hecke basis of  $S_k(\text{GL}_2^+(\mathcal{O}))$  to an orthonormal (Hecke) basis of  $S_k(\text{SL}(2, \mathcal{O}))$  and the positivity, we obtain the following theorem.

**Theorem 1.3** For the orthonormal Hecke basis  $\{f_j\}$  of  $S_k(\text{GL}_2^+(\mathcal{O}))$  and any  $\eta > 0$ , we have

$$\sum_{1 \leq j \leq J}^* |L(1/2, \text{As}(f_j))|^2 \ll k^{2+\eta},$$

where the  $*$  means that the summation is restricted to cuspidal Asai lifts  $\text{As}(f_j)$ , and the constant implicit only depends on the quadratic field  $\mathbf{F}$  and  $\eta$ .

It remains to prove Theorem 1.2, which is the goal of the next section.

## 2 Proof of the Theorem 1.2

From the Poisson integral representation [GR, p. 953, (8)], we have

$$\begin{aligned} J_{k-1}(x) &= \frac{\left(\frac{x}{2}\right)^{k-1}}{\sqrt{\pi} \Gamma(k-1/2)} \int_{-1}^1 (1-t^2)^{k-3/2} \cos(xt) dt \\ (4) \quad &\ll \left(\frac{ex}{2k}\right)^{k-1}, \end{aligned}$$

where the implicit constant is absolute.

To prove Theorem 1.2, we may assume that  $\mu$ 's are chosen among their associates mod  $U^2$  the representatives satisfying  $N(v)^{1/2} \ll v^{(i)} \ll N(v)^{1/2}$ ,  $i = 1, 2$ . We have by the Petersson formula (2),

$$\begin{aligned} &\sum_{j=1}^J \left| \sum_{\mu} b(\mu) \psi_j(\mu) \right|^2 \\ &= \sum_{\mu, v} b(\mu) \bar{b}(v) \sum_{j=1}^J \psi_j(\mu) \bar{\psi}_j(v) \\ &= \sum_{\mu} |b(\mu)|^2 D^{3/2} (k-1)^2 N(\mathcal{C}) \\ &\quad + (k-1)^2 DN(\mathcal{C}) (2\pi)^2 \sum_{\varepsilon \in U} \sum_{c \in \mathcal{C}^\times / U} \\ &\quad \times \frac{1}{|N(c)|} \sum_{\mu, v} b(\mu) \bar{b}(v) S(v, \mu \varepsilon^2; c) NJ_{k-1}(4\pi \sqrt{\mu v} |\varepsilon| / |c|) \\ &= \sum_1 + \sum_2, \text{ say.} \end{aligned}$$

We first prove Theorem 1.2 under the condition that  $k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$ . In view of (4) and bound  $|J_{k-1}(y)| \leq 1$ , we have  $J_{k-1}(y) \ll \left(\frac{ey}{2k}\right)^{k-1-\eta'} \ll \left(\frac{2y}{k}\right)^{k-1-\eta'}$ , for  $y > 0$  and  $0 \leq \eta' < 1/2$ , we have (choosing  $\eta'$  to be 0 or  $\eta$ ,  $0 < \eta < 1/2$  depending upon

whether  $|\varepsilon^{(i)}| \geq 1$  or not)

$$NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|) \ll (4(4\pi)^2\sqrt{(N\mu)(N\nu)}/k^2|N(c)|)^{k-1}(k^2|N(c)|)^\eta \prod_{1 \leq j \leq 2, |\varepsilon^{(j)}| \geq 1} |\varepsilon^{(j)}|^{-\eta} \\ \ll \left(\frac{1}{2|N(c_1)|}\right)^{k-1} (k^2|N(c)|)^\eta \prod_{1 \leq j \leq 2, |\varepsilon^{(j)}| \geq 1} |\varepsilon^{(j)}|^{-\eta},$$

where we write  $c = c_1\mathcal{C}$ .

Also we have trivially

$$|S(v, \mu\varepsilon^2; c)| \leq N(c).$$

Hence, the partial sum of  $\Sigma_2$  with the condition  $*$  on  $U$  that  $\varepsilon^{(0)} =: \max(|\varepsilon^{(1)}|, |\varepsilon^{(2)}|) \geq \exp(\log^2 N(\mathcal{C}))$ , is bounded by

$$k^{2+2\eta}(N(\mathcal{C}))^{1+\eta} \sum_{\varepsilon \in U}^* |\varepsilon^{(0)}|^{-\eta} \sum_{c_1 \in O^\times/U} \frac{2^{-k}X}{|N(c_1)|^{k-1-\eta}} \sum_{\mu} |b(\mu)|^2 \ll X \sum_{\mu} |b(\mu)|^2,$$

where we use the fact that the number of units  $\varepsilon$  satisfying  $x \leq \log \varepsilon^{(0)} < 2x$ , is  $O(x)$  since  $U$  is cyclic and generated by a fundamental unit of  $O$ .

It remains to deal with the remaining sum  $\Sigma_2'$  with the sum over the units  $\varepsilon$  in  $U$  satisfying the condition  $\#$ :  $\log \varepsilon^{(0)} < \log^2 N(\mathcal{C})$ . Note the above method clearly also works in this case if  $N(\mathcal{C}) \leq 2^{k/2}$ . Hence, we may assume  $N(\mathcal{C}) > 2^{k/2}$  and thus  $k \ll \log N(\mathcal{C})$ . We will apply the following lemma proved in [Lu].

**Lemma** Let  $c_1, c_2 > 0$  be constants,  $X \geq 1$ ,  $d(\cdot)$  arbitrary complex numbers, and  $c \in O$ . Then we have

$$\sum_{a \pmod{c}} \left| \sum_{N(v) \leq X, v \in O} ' d(v) e\left(\frac{va}{c}\right) \right|^2 = (|N(c)| + O(X)) \sum_{N(v) \leq X, v \in O} ' |d(v)|^2,$$

where “ $'$ ” means that the summation is restricted to those  $v$ 's such that  $v > 0$ ,  $c_1 N(v)^{1/2} \leq v^{(i)} \leq c_2 N(v)^{1/2}$ .

Using the Mellin–Barnes integral representation [MOS, Section 3.6.3, p. 82],

$$J_{k-1} \left( \frac{4\pi\sqrt{\mu^{(i)}v^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|} \right) \\ = \frac{1}{4\pi i} \int_{(2+\eta)} \left( \frac{2\pi\sqrt{\mu^{(i)}v^{(i)}}|\varepsilon^{(i)}|}{|c^{(i)}|} \right)^s \Gamma\left(\frac{k-1}{2} - \frac{s}{2}\right) \left[ \Gamma\left(1 + \frac{k-1}{2} + \frac{s}{2}\right) \right]^{-1} ds,$$

opening the Kloosterman sum, and by Cauchy's inequality, we infer that for  $c \in \mathcal{C}^\times/U$  and with  $s_i = 2 + \eta + \sqrt{-1}t_i$  ( $i = 1, 2$ ) and  $0 < \eta < 1/2$ ,

$$\sum_{\mu, v} b(\mu)\bar{b}(v)S(v, \mu\varepsilon^2; c) NJ_{k-1}(4\pi\sqrt{\mu\nu}|\varepsilon|/|c|) \\ \ll \int_{(2+\eta)} |ds_1| \int_{(2+\eta)} |ds_2| \left| \frac{\Gamma\left(\frac{k-1}{2} - \frac{s_1}{2}\right)}{\Gamma\left(1 + \frac{k-1}{2} + \frac{s_1}{2}\right)} \right| \cdot \left| \frac{\Gamma\left(\frac{k-1}{2} - \frac{s_2}{2}\right)}{\Gamma\left(1 + \frac{k-1}{2} + \frac{s_2}{2}\right)} \right|$$

$$\begin{aligned}
 & \times \max_{s_1, s_2} \sum_{h \pmod{c}} \left| \sum_{\mu, \nu} b(\mu) \bar{b}(\nu) \left( 4\pi^2 \sqrt{N(\mu)N(\nu)} / |N(c)| \right)^{2+\eta} \prod_{i=1}^2 (\sqrt{\mu^{(i)} \nu^{(i)}})^{\sqrt{-1}t_i} e\left(\frac{\mu h}{c}\right) \right| \\
 & \ll N(c)^{-(2+\eta)} \int_{(2+\eta)} \frac{|ds_1|}{k + |s_1|} \int_{(2+\eta)} \frac{|ds_2|}{k + |s_2|} \left| \frac{\Gamma\left(\frac{3}{2} - \frac{s_1}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{s_1}{2}\right)} \right| \left| \frac{\Gamma\left(\frac{3}{2} - \frac{s_2}{2}\right)}{\Gamma\left(\frac{3}{2} + \frac{s_2}{2}\right)} \right| \\
 & \times \max_{s_1, s_2} \sum_{h \pmod{c}} \left| \sum_{\mu} b(\mu) (N(\mu))^{1+\eta/2} \prod_{i=1}^2 (\mu^{(i)})^{\sqrt{-1}t_i/2} e\left(\frac{\mu h}{c}\right) \right|^2 \\
 & \ll N(c_1)^{-(2+\eta)} (|N(c)| + X) (N(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2,
 \end{aligned}$$

since  $k \ll \log N(\mathcal{C})$ , where as before, we write  $c = c_1 \mathcal{C}$ .

Thus the partial sum  $\Sigma'_2$  is bounded by

$$\begin{aligned}
 & k^2 (N(\mathcal{C}))^\eta \sum_{\varepsilon \in U}^\# \sum_{c_1 \in O^\times / U} \frac{1}{|N(c_1)|^{2+\eta}} (|N(c_1 \mathcal{C})| + X) \sum_{\mu} |b(\mu)|^2 \\
 & \ll (N(\mathcal{C}) + X) N(\mathcal{C})^\eta \sum_{\mu} |b(\mu)|^2,
 \end{aligned}$$

since

$$\sum_{\varepsilon \in U}^\# 1 \ll \log^2 N(\mathcal{C}).$$

Hence, Theorem 1.2 is true if  $k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$ .

In the case  $k^2 N(\mathcal{C}) < 8(4\pi)^2 X$ , we reduce it to the previous case by the famous embedding trick of Iwaniec. Choosing a prime ideal  $\mathcal{P} \subset O$  such that  $N(\mathcal{P}) k^2 N(\mathcal{C}) \asymp X$  and  $N(\mathcal{P}) k^2 N(\mathcal{C}) \geq 8(4\pi)^2 X$ . Note that  $[\Gamma_0(\mathcal{C}) : \Gamma_0(\mathcal{P}\mathcal{C})] \leq N(\mathcal{P}) + 1$ . Let  $H_k(\mathcal{C})$  denote an orthonormal basis of  $S_{2k}(\Gamma_0(\mathcal{C}))$ , and write

$$S_{\mathcal{C}}(b) = \sum_{f \in H_k(\mathcal{C})} \left| \sum_{\mu} b(\mu) \psi_f(\mu) \right|^2.$$

We deduce that

$$\begin{aligned}
 S_{\mathcal{C}}(b) & \leq (1 + N(\mathcal{P})^{-1}) S_{\mathcal{P}\mathcal{C}}(b) \\
 & \ll (N(\mathcal{P}\mathcal{C}) k^2 + X) (k X N(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2 \\
 & \ll X (k X N(\mathcal{C}))^\eta \sum_{\mu} |b(\mu)|^2,
 \end{aligned}$$

and this completes our proof.

**Acknowledgment** The author wishes to thank the referee for careful reading of the paper and for the valuable comments.

## References

- [As] T. Asai, *On certain Dirichlet series associated with Hilbert modular forms and Rankin's method*. Math. Ann. 226(1977), no. 1, 81–94.
- [Ga] P. B. Garrett, *Holomorphic Hilbert modular forms*, Wadsworth Inc., Monterey, California, 1990.



- [GR] I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products*, Academic Press, New York, 1965.
- [HL] J. Hoffstein and P. Lockhart, *Coefficients of Maass forms and the Siegel zero, appendix by D. Goldfeld, J. Hoffstein, and D. Lieman, an effective zero free region*. Ann. Math. **140**(1994), 161–181.
- [Iw] H. Iwaniec, *Small eigenvalues of Laplacian for  $\Gamma_0(N)$* . Acta Arith. **56**(1990), no. 1, 65–82.
- [IK] H. Iwaniec and E. Kowalski, *Analytic number theory*, American Mathematical Society Colloquium Publications, 53, American Mathematical Society, Providence, RI, 2004.
- [Kr] M. Krishnamurthy, *The Asai transfer to  $GL_4$  via the Langlands–Shahidi method*. Int. Math. Res. Not. IMRN **2003**(2003), no. 41, 2221–2254.
- [Lu] W. Luo, *Poincaré series and Hilbert modular forms. Rankin memorial issues*. Ramanujan J. **7**(2003), nos. 1–3, 129–140.
- [MOS] W. Magnus, F. Oberhettinger, and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics*, Springer, New York, 1966.
- [PR] D. Prasad and D. Ramakrishnan, *On the cuspidality criterion for the Asai transfer to  $GL(4)$ , Appendix A to the paper ‘Determination of cusp forms on  $GL(2)$  by coefficients restricted to quadratic subfields’ by M. Krishnamurthy*. J. Number Theory **132**(2012), no. 6, 1376–1383.
- [Ra] D. Ramakrishnan, *Modularity of solvable Artin representations of  $GO(4)$ -type*. Int. Math. Res. Not. IMRN **2002**(2002), no. 1, 1–54.
- [Sh] H. Shimizu, *On discontinuous groups acting on a product of upper half planes*. Ann. of Math. **77**(1963), 33–71.
- [Ta] R. Taylor, *On Galois representations associated to Hilbert modular forms*. Invent. Math. **98**(1989), 265–280.

Department of Mathematics, The Ohio State University, Columbus, OH 43210, United States

e-mail: [luo.43@osu.edu](mailto:luo.43@osu.edu)