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INTEGRAL FUNCTIONALS IN THE DUALS OF L^λ-SPACES(1)

BY
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1. **Introduction.** Luxemburg and Zaanen [5] call an element φ of the topological dual of a normed or seminormed vector space V an integral if

$$(I\varphi) f_n \downarrow 0 \left(\text{i.e. } f_n \in V, f_n \geq f_{n+1}, \bigwedge_n f_n = 0\right) \quad \text{implies} \quad \lim_{n\to\infty} \varphi(f_n) = 0.$$

We denote the space of integrals by V^I. For the L^λ function spaces introduced by Ellis and Halperin [2] another Banach subspace of the dual emerges, namely the conjugate space L^{λ^*} which is the L^λ space determined by the conjugate length function λ^*. L^{λ^*} is contained in $(L^\lambda)^I$ but need not coincide with it.

There are measure spaces in which, for $L^\lambda = L^1$, $L^{\lambda^*} \neq (L^\lambda)^*$, $(L^\lambda)^I$. In [3] Ellis and Snow characterized the dual of L^1 in terms of integrals with respect to collections of functions defined in terms of an arbitrary ND-decomposition of the measure space. The space of analogous collections of elements for an arbitrary L^λ will be denoted by $(L^\lambda)^{\mathscr{A}}$ (§4 below). Always

$$L^{\lambda^*} \subset (L^\lambda)^{\mathscr{A}} \subset (L^\lambda)^I \subset (L^\lambda)^*.$$

In §§3–5 the spaces L^{λ^*}, $(L^\lambda)^{\mathscr{A}}$ and $(L^\lambda)^I$ are related and conditions whereby each coincides with $(L^\lambda)^*$ are given.

2. **Definitions and notation.** We consider an arbitrary complete measure space (X, S, μ) where S is a σ-algebra. We let $\mathbf{M}$ denote the measurable functions valued in the extended reals, $\bar{\mathbf{R}}$. In a partially ordered space $(B, \leq)$ with a zero we denote by B_+ the elements $f \in B$ with $f \geq 0$. A function $\lambda : \mathbf{M}_+ \to \bar{\mathbf{R}}_+$ is called a length function [2] if

$(L1)$ $\lambda(f) = 0$ if $f(x) = 0$ for almost all $x \in X$;
$(L2)$ $\lambda(f_1) \leq \lambda(f_2)$ if $f_1 \leq f_2$;
$(L3)$ $\lambda(f_1 + f_2) \leq \lambda(f_1) + \lambda(f_2)$;
$(L4)$ $\lambda(kf) = k\lambda(f)$, $k \in \mathbf{R}_+$;
$(L5)$ $f_i \uparrow_{i=1}^{\infty} f$ implies that $\lambda(f_i) \uparrow_{i=1}^{\infty} \lambda(f)$.

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For $f \in \mathbf{M}$ we define $\lambda(f)=\lambda(|f|)$ and set $\mathscr{L}^\lambda=\{f \in \mathbf{M}^*:\lambda(f)<\infty\}$ (where $\mathbf{M}^*$ denotes the finite real valued elements of $\mathbf{M}$). On $\mathscr{L}^\lambda$, ($L1$), ($L3$) and ($L4$) imply that λ defines a seminorm. Making identifications modulo λ-null functions gives a space L^λ which is a Banach space. With the usual pointwise order $\mathscr{L}^\lambda$ is a vector lattice and ($L2$) implies that λ is monotone on the positive cone of $\mathscr{L}^\lambda$. When $f \in \mathscr{L}^\lambda$, ($L5$) is an analogue of the Lebesgue monotone convergence property in the space of integrable functions. Similar statements are true in L^λ ordered pointwise modulo λ-null functions. With conventional lack of precision we shall not always distinguish $\mathscr{L}^\lambda$ and L^λ.

To each length function λ corresponds a conjugate length function λ^* defined on $\mathbf{M}_+$ by

$$\lambda^*(g) = \sup_{f:\lambda(f)\leq 1} \int fg\, d\mu \leq +\infty.$$

The space $\mathscr{L}^{\lambda^*}$ is called the conjugate of $\mathscr{L}^\lambda$. If $g \in \mathscr{L}^{\lambda^*}$, defining $\varphi_g:\mathscr{L}^\lambda\to\mathbf{R}$ by

$$\varphi_g(f) = \int fg\, d\mu,$$

$$\|\varphi_g\| = \sup_{\lambda(f)\leq 1} |\varphi_g(f)| = \sup_{\lambda(f)\leq 1} \left|\int fg\, d\mu\right|$$

$$= \sup_{\lambda(f)\leq 1} \int f|g|\, d\mu = \lambda^*(|g|) = \lambda^*(g)$$

and $\varphi_g \in (\mathscr{L}^\lambda)^*$. Furthermore if $\{f_n\} \in \mathscr{L}^\lambda$ and $f_n \downarrow 0$ then $f_n g^+ \downarrow 0$, $f_n g^- \downarrow 0$ and the general Lebesgue convergence theorem implies that $(I\varphi_g)$ holds.

A function $f:X\to\mathbf{R}$ will be called a *step function* if $f=\sum_{i=1}^n c_i\chi e_i$, $e_i \in S$ (where χA denotes the characteristic function of A). With the added requirement that each e_i has finite measure f will be called a *simple function*. We denote by $\mathscr{M}$, M respectively the spaces of step functions and of simple functions and define $\mathscr{M}^\lambda = \mathscr{M} \cap \mathscr{L}^\lambda$; $M^\lambda = M \cap \mathscr{L}^\lambda$. $\bar{\mathscr{M}}^\lambda$ and $\bar{M}^\lambda$ denote the closures of these spaces in $\mathscr{L}^\lambda$ for the seminorm topology.

REMARK. If $f \in \mathscr{M}_+$ there is a sequence of step functions $f_n \in \mathscr{M}$ with $f_n \uparrow f$. If X is σ-finite there is a sequence of simple functions increasing to f.

3. **The spaces $(\mathscr{L}^\lambda)^I$.** For an arbitrary measure space (X, S, ν) we use the following approach to the integral.

If $f=\sum_1^n c_i\chi e_i \in \mathscr{M}_+$, we define

$$\int f\, d\nu = \sum_1^n c_i\nu(e_i) \in \bar{\mathbf{R}}.$$

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If $f \in \mathbf{M}_+$ we define

$$\int f\,d\nu = \sup\left\{\int f_a\,d\nu, f_a \in \mathscr{M}_+, f_a \leq f\right\} \in \bar{\mathbf{R}}_+.$$

Finally if $f \in \mathbf{M}$ and $\int f^+\,d\nu$ and/or $\int f^-\,d\nu < \infty$, we define

$$\int f\,d\nu = \int f^+\,d\nu - \int f^-\,d\nu.$$

Then $\mathscr{L}^1(X, S, \nu) = \mathscr{L}^1(\nu) = \{f \in \mathbf{M} : \int f\,d\nu \in \mathbf{R}\}$ is the usual space of integrable functions without identifications. If $f_n \in \mathbf{M}_+$ and $f_n \uparrow f \in \mathscr{L}^1$ then

$$\lim_{n\to\infty} \int f_n\,d\nu = \int f\,d\nu.$$

Given a space $\mathscr{L}^\lambda(X, S, \mu)$ and positive measure ν on S, we define

$$\lambda^*(\nu) = \sup\left\{\int f\,d\nu, f \in \mathscr{M}_+^\lambda, \lambda(f) \leq 1\right\} \leq +\infty.$$

(For $g \in M_+$, $d\nu = g \circ d\mu$, we have $\lambda^*(\nu) = \lambda^*(g)$.)

Given $\mathscr{L}^\lambda(X, S, \mu)$, set $S^\lambda = \{e \in S : \chi e \in \mathscr{L}^\lambda\}$. For $\varphi \in (\mathscr{L}^\lambda)_+^I$ define, on S^λ,

$$\mu_\varphi(e) = \varphi(\chi e) \in \mathbf{R},$$

and extend the definition to all of S by

$$\mu_\varphi(e) = \sup\{\mu(e'), e' \in S^\lambda, e' \subset e\}.$$

Then $\mu_\varphi : S \to \bar{\mathbf{R}}_+$ and standard arguments show that μ_φ is a measure on S that is absolutely continuous with respect to μ.

If $f \in \mathscr{L}_+^\lambda$ and $f_n \uparrow f$, $f_n \in \mathscr{M}_+$, then $f_n \in \mathscr{M}_+^\lambda$ and $f - f_n \downarrow 0$. Hence, using $(I\varphi)$

$$\int f\,d\mu_\varphi = \lim_{n\to\infty} \int f_n\,d\mu_\varphi = \lim_{n\to\infty} \varphi(f_n) = \varphi(f), \tag{3.1}$$

and (3.1) remains valid in $\mathscr{L}^\lambda$ since (3.1) holds for f^+ and f^-. Thus every $\varphi \in (L^\lambda)_+^I$ can be expressed as an integral with respect to the corresponding measure μ_φ. Furthermore

$$\lambda^*(\mu_\varphi) \leq \sup\left\{\int f\,d\mu_\varphi, \lambda(f) \leq 1\right\} = \sup\{\varphi(f), \lambda(f) \leq 1\} = \|\varphi\|.$$

If $f \in \mathscr{L}_+^\lambda$, $\lambda(f) \neq 0$, $\epsilon > 0$, $f_n \uparrow f$, $f_n \in \mathscr{M}_+^\lambda$, then for n sufficiently large,

$$\int f\,d\mu_\varphi \leq \int f_n\,d\mu_\varphi + \varepsilon;$$

$$\int \frac{f_n}{\lambda(f_n)}\,d\mu_\varphi \geq \int \frac{f_n}{\lambda(f)}\,d\mu_\varphi \geq \int \frac{f}{\lambda(f)}\,d\mu_\varphi - \frac{\varepsilon}{\lambda(f)}.$$

Since for $\varphi \in (L^\lambda)_+^*$, $\|\varphi\| = \sup\{\varphi(f), f \geq 0, \lambda(f) \leq 1\}$ it follows that $\lambda^*(\mu_\varphi) \geq \|\varphi\|$ and equality holds.

Conversely for an arbitrary countably additive measure ν on S with $\lambda^*(\nu)<\infty$, define

$$\varphi_\nu(f) = \int f\,d\nu, \qquad f \in \mathscr{L}^\lambda.$$

If $f_n \uparrow f$ with $f_n \in \mathscr{M}_+^\lambda$ for all n and $f \in \mathscr{L}_+^\lambda$, then

$$\int f\,d\nu = \lim_{n\to\infty} \int f_n\,d\nu \le \lim_{n\to\infty} \lambda(f_n)\lambda^*(\nu) \le \lambda(f)\lambda^*(\nu) < \infty.$$

Considering f^+ and f^- it follows that if $f \in \mathscr{L}^\lambda$ then $f \in \mathscr{L}^1(\nu)$ and $(I\varphi_\nu)$ is implied by the general Lebesgue convergence theorem. Thus $\varphi_\nu \in (\mathscr{L}^\lambda)_+^I$.

Writing $\varphi'=\varphi_\nu$, φ' determines a positive measure $\mu_{\varphi'}$ with $\varphi_\nu(f)=\int f\,d\mu_{\varphi'}$ for all f in $\mathscr{L}^\lambda$. Since the integrals with respect to ν and $\mu_{\varphi'}$ coincide on $\mathscr{M}_+^\lambda$,

$$\lambda^*(\nu) = \lambda^*(\mu_{\varphi'}) = \|\varphi_\nu\|.$$

Dropping the assumption that φ is positive, if $\varphi \in (\mathscr{L}^\lambda)^I$, then $\varphi=\varphi^+-\varphi^-$ with φ^+, φ^- and $|\varphi|=\varphi^++\varphi^-$ in $(\mathscr{L}^\lambda)_+^I$. Since $\mathscr{L}^\lambda$ is a seminormed vector lattice, $\|\varphi\| = \|\,|\varphi|\,\|$ for every $\varphi \in (\mathscr{L}^\lambda)^*$. We set $\mu_\varphi=\mu_{\varphi^+}-\mu_{\varphi^-}$ where it is defined (which includes S^λ), define

$$\mathscr{L}^1(\mu_\varphi) = \mathscr{L}^1(\mu_{|\varphi|}),$$

and write

$$\int f\,d\mu_\varphi = \int f\,d\mu_{\varphi^+} - \int f\,d\mu_{\varphi^-}, \qquad f \in \mathscr{L}^1(\mu_\varphi).$$

Since $\mathscr{L}^\lambda \subset \mathscr{L}^1(\mu_{|\varphi|})$ this is finite for every $f \in \mathscr{L}^\lambda$ and

$$\int f\,d\mu_\varphi = \varphi^+(f) - \varphi^-(f) = \varphi(f).$$

We set $\lambda^*(\mu_\varphi)=\lambda^*(\mu_{|\varphi|})$. Using the above results for $(\mathscr{L}^\lambda)_+^I$,

$$\lambda^*(\mu_\varphi) = \lambda^*(\mu_{|\varphi|}) = \|\,|\varphi|\,\| = \|\varphi\|.$$

THEOREM 3.1. *To each $\varphi \in (\mathscr{L}^\lambda)^I$ corresponds measures $\nu_1=\mu_{\varphi^+}$, $\nu_2=\mu_{\varphi^-}$ on S, determined by φ^+ and φ^-, with*

$$\varphi(f) = \int f\,d\nu_1 - \int f\,d\nu_2 \tag{3.2}$$

and $\|\varphi\|=\lambda^*(\mu_\varphi)=\lambda^*(\nu_1+\nu_2)$. Conversely if ν_1 and ν_2 are measures on S with $\lambda^*(\nu_i)<\infty$, $i=1, 2$, then (3.2) *gives $\varphi \in (\mathscr{L}^\lambda)^I$ with $\|\varphi\|=\lambda^*(\nu_1+\nu_2)$.*

THEOREM 3.2. [5]. $(L^\lambda)^I=(L^\lambda)^*$ *if and only if*
(I^λ) $f_n \downarrow 0$, $f_n \in \mathscr{L}^\lambda$, *implies that* $\lim_{n\to\infty} \lambda(f_n)=0$.

Proof. Since $|\varphi(f_n)| \le \|\varphi\|\,\lambda(f_n)$, (I^λ) implies $(I\varphi)$ for every $\varphi \in (L^\lambda)^*$. Conversely if $(I\varphi)$ holds for every $\varphi \in (L^\lambda)^*$, (I^λ) holds [5, p. 671, Lemma 22.6].

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THEOREM 3.3. *If* $(L^\lambda)^I=(L^\lambda)^*$ *then* $\bar{\mathscr{M}}^\lambda=L^\lambda$.

Proof. $\mathscr{M}^\lambda$ is a vector subspace of L^λ. Assuming that $\bar{\mathscr{M}}^\lambda\neq L^\lambda$, let $f'\in L^\lambda-\bar{\mathscr{M}}^\lambda$. A corollary of the Hahn-Banach theorem then shows that there exists $\varphi\in(L^\lambda)^*$ with $\varphi(f)=0$, $f\in\mathscr{M}^\lambda$, $\varphi(f')\neq 0$. Since

$$\varphi(f')=\varphi^+(f'^+)-\varphi^+(f'^-)-\varphi^-(f'^+)+\varphi^-(f'^-)$$

we can assume that $\varphi, f'>0$. Then

$$\varphi(f)=\int f\,d\mu_\varphi,\qquad f\in L^\lambda.$$

There exists a sequence $f_n\in M^\lambda$ with $f_n\uparrow f'$. Using the countable additivity of φ and the Lebesgue monotone convergence theorem, $0<\varphi(f')=\lim_{n\to\infty}\int f_n\,d\mu_\varphi=0$, a contradiction.

4. **The spaces $(\mathscr{L}^\lambda)^{\mathscr{A}}$.** In [3] Zorn's lemma was used to show that in an arbitrary measure space (X, S, μ) there exists a decomposition

$$X=X_1\cup X_2,$$

with $X_1\cap X_2=\phi$; $X_2=\cup_{a\in\mathscr{A}}e_a$, $0<\mu(e_a)<\infty$, $a\in\mathscr{A}$; $\mu(e_a\cap e_{a'})=0$, $a\neq a'$, and such that if $e'\in S$, $e'\subset X_1$ then either $\mu(e')=0$ or $\mu(e')=+\infty$. For each $e\in S$ with $0<\mu(e)<\infty$, there is then a countable collection of subscripts $\{a_i\}$ with

$$\mu(e)=\sum_1^\infty\mu(e\cap e_{a_i}).$$

Such a decomposition was called an *ND*-decomposition.

When X is σ-finite, X_1 and X_2 are measurable, $\mu(X_1)=0$ and X_2 is expressed as a union of sets of finite positive measure with null intersections.

Fixing an *ND*-decomposition, where (e_a, S_a, μ) is the restriction of (X, S, μ) to e_a, we set

$$\mathbf{M}^{\mathscr{A}}=\prod_{a\in\mathscr{A}}\mathbf{M}(e_a, S_a, \mu),$$

and denote points by $g_{\mathscr{A}}=\{g_a\in\mathbf{M}(e_a, S_a, \mu); a\in\mathscr{A}\}$.

If $E\in S$ is σ-finite, then $\mu(E\cap e_a)=0$ except for a countable collection $\{a_i\}$. Defining

$$g_E=(\sup_i g_{a_i})\chi_E\quad\text{if}\quad \mu(E\cap e_a)\neq 0 \text{ for at least one } a\in\mathscr{A},$$
$$g_E=0\quad\text{if}\quad\mu(E\cap e_a)=0\quad\text{for all}\quad a\in\mathscr{A},$$

we have $g_E\in\mathbf{M}(X, S, \mu)$. For an arbitrary length function λ on (X, S, μ) we define

$$\lambda^*(g_{\mathscr{A}})=\sup\{\lambda^*(g_E), E\in S, E\ \sigma\text{-finite}\},$$

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and denote by $(\mathscr{L}^\lambda)^{\mathscr{A}}$ the elements $g_{\mathscr{A}} \in \mathbf{M}^{\mathscr{A}}$ with $\lambda^*(g_{\mathscr{A}})<\infty$. Then $(\mathscr{L}^\lambda)^{\mathscr{A}}$ is a vector subspace of $\mathbf{M}^{\mathscr{A}}$ on which λ^* is a seminorm; a norm with identifications modulo λ^*-null functions. We note that

$$(\mathscr{L}^\lambda)^{\mathscr{A}} \subset \prod_{a\in\mathscr{A}} \mathscr{L}^{\lambda^*}(e_a, S_a, \mu).$$

LEMMA 4.1. *If $\lambda^*(g_{\mathscr{A}})<\infty, f_0 \in \mathscr{L}^\lambda$, then for at most countably many $a \in \mathscr{A}$*

$$\int f_0 g_a \, d\mu \neq 0. \tag{4.1}$$

Proof. There is no loss of generality in assuming that each $g_a \geq 0$ almost everywhere in e_a and that $f_0>0$, $\lambda(f_0)=1$. Suppose that (4.1) holds for uncountably many $a \in \mathscr{A}$. There then exists $d>0$ and $\{a_i\}$ with

$$\int f_0 g_{a_i} \, d\mu > d, \qquad i = 1, 2, \ldots.$$

If $E = \cup e_{a_i}$,

$$\lambda^*(g_{\mathscr{A}}) \geq \lambda^*(g_E) \geq \int f_0 g_E \, d\mu = \sum_1^\infty \int f_0 g_{a_0} \, d\mu = +\infty,$$

giving a contradiction.

DEFINITION. For each $g_{\mathscr{A}} \in (\mathscr{L}^\lambda)^{\mathscr{A}}, f \in \mathscr{L}^\lambda$ define

$$e(g_{\mathscr{A}}, f) = e(f) = \cup \left\{e_a, a \in \mathscr{A} : \int |fg_a| \, d\mu > 0\right\}.$$

Set $g_f = g_{e(f)}$ if $e(f) \neq \phi$; $=0$ if $e(f)=\phi$. Then $e(f) \in S$, is σ-finite and $g_f \in \mathbf{M}$. We denote the complement of $e(f)$ by $\tilde{e}(f)$.

LEMMA 4.2. *If $\varphi \in (\mathscr{L}^\lambda)^I_+$ and μ_φ is the corresponding measure on S, then each set e_a, $a \in \mathscr{A}$, is μ_φ σ-finite.*

Proof. Let $\{e_b, b \in \mathfrak{B}\}$ denote the measurable subsets of e_a satisfying

$$0 < \mu_\varphi(e_b) < \infty, \qquad 0 < \mu(e_b) \leq \mu(e_a).$$

Order collections of disjoint subsets from $\{e_b\}$ by inclusion. Each chain then has an upper bound so that Zorn's lemma implies the existence of a maximal collection which is countable since $\mu(e_a)<\infty$. Let e_a' denote the union of the sets in this maximal collection. Then $e_a - e_a' \in S$. If $\mu(e_a - e_a')=0$, $\mu_\varphi(e_a - e_a')=0$. If $\mu(e_a - e_a')>0$, $\mu_\varphi(e_a - e_a')=0$ or $+\infty$ as does e^* for any $e^* \in S$, $e^* \subset e_a - e_a'$. Thus $e_a - e_a'$ is either μ_φ null or μ_φ purely infinite. Since the definition of μ_φ does not permit purely infinite sets we conclude that $\mu_\varphi(e_a - e_a')=0$ and thus e_a is μ_φ σ-finite.

If $\varphi \in (\mathscr{L}^\lambda)^I_+$ then for each e_a, $a \in \mathscr{A}$, μ_φ is absolutely continuous with respect to μ on e_a, e_a is μ-finite and μ_φ σ-finite and the Radon–Nikodym theorem gives $g_a : e_a \to \mathbf{R}^+$ with $g_a \in \mathbf{M}$ and such that

$$\mu_\varphi(e) = \int_e g_a \, d\mu, \qquad e \in S, \qquad e \subset e_a.$$

It then follows that for each $f \in \mathscr{L}^\lambda$,

$$\varphi(f_e) = \int_e f \, d\mu_\varphi = \int_e f g_a \, d\mu, \qquad e \in S, \qquad e \subset e_a, \qquad a \in \mathscr{A}. \tag{4.2}$$

Set $g_{\mathscr{A}} = \{g_a, a \in \mathscr{A}\}$. Then

$$\lambda^*(g_{\mathscr{A}}) = \sup\left\{\int_E f \, d\mu_\varphi ; E \ \sigma\text{-finite}, \lambda(f) \leq 1\right\} \leq \lambda^*(\mu_\varphi) = \|\varphi\|.$$

Thus to each $\varphi \in (\mathscr{L}^\lambda)^I_+$ corresponds $g_{\mathscr{A}} \in (\mathscr{L}^\lambda)^{\mathscr{A}}_+$ related by (4.2). It follows from (4.2) that if $f \in \mathscr{L}^\lambda$,

$$\varphi(|f| \, \chi e_a) \neq 0,$$

iff $e_a \subset e(g_a, f) = e(f)$, and in particular for at most a countable collection of subscripts a. For an arbitrary $\varphi \in (\mathscr{L}^\lambda)^I_+$,

$$e(\varphi, f) = \cup\{e_a, a \in \mathscr{A} : \varphi(|f| \, \chi e_a) \neq 0\}$$

will coincide with $e(g_{\mathscr{A}}, f)$ for the $g_{\mathscr{A}}$ determined by φ and will also be denoted by $e(f)$.

THEOREM 4.1. *There is an isometric isomorphism between* $(\mathscr{L}^\lambda)^{\mathscr{A}}$ (*with identifications*) *and the subspace of* $(\mathscr{L}^\lambda)^I$ *of different elements* φ *with* $\varphi(f\chi\tilde{e}(f)) = 0$ *for each* $f \in \mathscr{L}^\lambda$. *To each* $g_{\mathscr{A}}$ *corresponds* φ *by*

$$\varphi(f) = \int f g_f \, d\mu, \qquad f \in \mathscr{L}^\lambda. \tag{4.3}$$

Proof. Let $g_{\mathscr{A}} \in (\mathscr{L}^\lambda)^{\mathscr{A}}_+$. Defining φ by (4.3),

$$|\varphi(f)| \leq \lambda(f)\lambda^*(g_f) \leq \lambda(f)\lambda^*(g_{\mathscr{A}}) < \infty,$$

and $\varphi : \mathscr{L}^\lambda \to \mathbf{R}$.

Since $e(af) = e(f)$, $a \neq 0$; $e(f+f') \subset e(f) \cup e(f')$, (a countable union of sets e_a, $a \in \mathscr{A}$). It is easy to verify that $(f+f')g_{f+f'} = fg_f + f'g_{f'}$ almost everywhere and φ is linear and so in $(\mathscr{L}^\lambda)^*_+$. To see that $\varphi \in (\mathscr{L}^\lambda)^I_+$ let $f_n \in \mathscr{L}^\lambda$, $f_n \downarrow 0$. Since $e(f_n) \subset e(f_1)$,

$$\varphi(f_n) = \int f_n g_{f_n} \, d\mu = \int f_n g_{f_1} \, d\mu, \qquad n = 1, 2, \ldots.$$

The Lebesgue general convergence theorem then implies that $\lim_n \varphi(f_n) = 0$.

Finally we observe that $\int f\chi\tilde{e}(f) g_a \, d\mu = 0$ for every $a \in \mathscr{A}$ which implies that $\varphi(f\chi\tilde{e}(f)) = 0$ for each f in $\mathscr{L}^\lambda$.

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Let $g_{\mathscr{A}} \in (\mathscr{L}^\lambda)^{\mathscr{A}}_+$ and define φ by (4.3). Then $\|\varphi\| \le \lambda^*(g_{\mathscr{A}})$. As shown after Lemma 4.2, φ determines $g'_{\mathscr{A}} \in (\mathscr{L}^\lambda)^{\mathscr{A}}_+$ with $\lambda^*(g'_{\mathscr{A}}) \le \|\varphi\|$ and with (4.2) holding. Thus if $f \in \mathscr{L}^\lambda$, $a \in \mathscr{A}$,

$$\int f(g_a - g'_a)\, d\mu = \varphi(f) - \varphi(f) = 0.$$

This implies that $\lambda^*(g_{\mathscr{A}} - g'_{\mathscr{A}}) = 0$ and thus

$$\lambda^*(g_{\mathscr{A}}) = \lambda^*(g'_{\mathscr{A}}) \le \|\varphi\| \le \lambda^*(g_{\mathscr{A}})$$

so that $\|\varphi\| = \lambda^*(g_{\mathscr{A}})$.

On the other hand if we start with $\varphi \in (\mathscr{L}^\lambda)^I_+$, φ determines $g'_{\mathscr{A}} \in (\mathscr{L}^\lambda)^{\mathscr{A}}_+$ with $\lambda^*(g'_{\mathscr{A}}) \le \|\varphi\|$. As in the preceding paragraph, $g'_{\mathscr{A}}$ determines φ' with $\|\varphi'\| = \lambda^*(g'_{\mathscr{A}})$. Furthermore for every $f \in \mathscr{L}^\lambda$,

$$\varphi'(f) = \varphi'(f\chi e(f)) = \varphi(f\chi e(f)),$$

It follows that $\varphi = \varphi'$ *if and only if* $\varphi(f\chi\tilde{e}(f)) = 0$ *for every* $f \in \mathscr{L}^\lambda$.

We next consider the behavior of the functionals in $(\mathscr{L}^\lambda)^I_+$ on the sets $\tilde{e}(f)$, $f \in \mathscr{L}^\lambda_+$. Since $e(f)$ is σ-finite, $\tilde{e}(f) \in S$. There are three possibilities

(i) $\mu(\tilde{e}(f)) = 0$. Then, setting $f^0 = f\chi\tilde{e}(f)$, $\lambda(f^0) = 0$ and so $\varphi(f^0) = 0$;

(ii) $0 < \mu(\tilde{e}(f)) < \infty$. Then $\tilde{e}(f)$ is the union of a null set and an at most countable collection of sets e_a, $a \in \mathscr{A}$. If $\mu(\tilde{e}(f) \cap e_a) \ne 0$, e_a is not contained in $e(f)$ and $\varphi(f^0\chi e_a) = \int f g_a\, d\mu = 0$. It follows again that $\varphi(f^0) = 0$.

(iii) $\mu(\tilde{e}(f)) = +\infty$. If X_1 contains a purely infinite set $s \in S$ with $\varphi(f_s) \ne 0$, $\varphi(f_s\chi\tilde{e}(f_s)) \ne 0$ and φ has no isometric correspondent in $(\mathscr{L}^\lambda)^{\mathscr{A}}$. A simple example is given by taking $X = \{a, b\}$, $S = \mathscr{P}(X)$, $\mu\{a\} = 1$, $\mu\{b\} = +\infty$, $\lambda(f) = \max\{f(a), f(b)\}$, $X_1 = \{b\}$, $X_2 = \{a\}$. Then $(\mathscr{L}^\lambda)^{\mathscr{A}}$ coincides with the functionals vanishing at b and is different from $(\mathscr{L}^\lambda)^I = (\mathscr{L}^\lambda)^*$.

On the other hand if $\tilde{e}(f)$ is σ-finite it follows as in (ii) that $\varphi(f^0) = 0$.

THEOREM 4.2. *If* $\varphi \in (\mathscr{L}^\lambda)^I_+$ *then* $\varphi \in (\mathscr{L}^\lambda)^{\mathscr{A}}_+$ *if and only if for every* $s \in S^\lambda$,

$$\varphi(\chi s) = \mu_\varphi(s) = \sup\{\varphi(\chi s') = \mu_\varphi(s'),\ s' \subset s,\ \mu(s') < \infty\}. \tag{4.4}$$

Proof. Consider $f \in \mathscr{L}^\lambda_+$, $f^0 = f\chi\tilde{e}(f)$. There then exists $\{f_n\} \in \mathscr{M}^\lambda_+$, $f_n \uparrow f^0$ with

$$\varphi(f^0) = \int f^0\, d\mu_\varphi = \lim_{n\to\infty} \int f_n\, d\mu_\varphi.$$

Now $f_n = \sum_i^n c_i\chi s_i$ with each $s_i \in S^\lambda$, $s_i \subset \tilde{e}(f)$. If $s \subset s_i$, $\mu_\varphi(s) < \infty$ then $s \in S^\lambda$,

$$\mu_\varphi(s) = \varphi(\chi s) \le \frac{1}{c_i}\varphi(f\chi s) = 0,$$

as in (i) and (ii) above. It follows that $\int f_n\, d\mu_\varphi = 0$ for each n and therefore $\varphi(f^0) = 0$.

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Conversely if $\varphi \in (\mathscr{L}^\lambda)_+^{\mathscr{A}}$ and $s \in S^\lambda$, then

$$\varphi(\chi s) = \varphi(\chi s \cap e(\chi s)) = \varphi\left(\chi \bigcup_1^\infty s \cap e_{a_i}\right)$$

for certain $a_i \in \mathscr{A}$. Then (4.4) follows from $(I\varphi)$.

THEOREM 4.3. *$(\mathscr{L}^\lambda)^{\mathscr{A}} = (\mathscr{L}^\lambda)^I$ if and only if* (4.4) *holds for every $\varphi \in (\mathscr{L}^\lambda)_+^I$. If the simple functions are dense in $\mathscr{L}^\lambda$,* (4.4) *holds for every $\varphi \in (\mathscr{L}^\lambda)_+^I$. $(\mathscr{L}^\lambda)^{\mathscr{A}} = (\mathscr{L}^\lambda)^*$ if and only if (I^λ) holds and $\bar{M}^\lambda = \mathscr{L}^\lambda$.*

Proof. The first statement follows from Theorem 4.2.

If $\bar{M}^\lambda = \mathscr{L}^\lambda$ and $s \in S^\lambda$ then, given $\varepsilon > 0$, there exists $f = \sum_1^m c_i \chi s_i \in M^\lambda$ with $\lambda(\chi s - f) < \varepsilon$. If $s' = \{x : f(x) > 0\} \cap s$, $\mu(s') < \infty$ and

$$\lambda(\chi s - \chi s') \le \lambda(\chi s - f) < \varepsilon.$$

Thus for any $\varphi \in (\mathscr{L}^\lambda)_+^I$,

$$\varphi(\chi s) - \varphi(\chi s') \le \|\varphi\| \, \lambda(\chi s - \chi s') < \|\varphi\| \, \varepsilon,$$

which implies (4.4).

If $\bar{M}^\lambda = \mathscr{L}^\lambda$, $(\mathscr{L}^\lambda)^{\mathscr{A}} = (\mathscr{L}^\lambda)^I$ and, if (I^λ) holds

$$(\mathscr{L}^\lambda)^{\mathscr{A}} = (\mathscr{L}^\lambda)^I = (\mathscr{L}^\lambda)^*$$

by Theorem 3.3.

Now assume that $(\mathscr{L}^\lambda)^{\mathscr{A}} = (\mathscr{L}^\lambda)^*$. Since $(\mathscr{L}^\lambda)^{\mathscr{A}} \subset (\mathscr{L}^\lambda)^I \subset (\mathscr{L}^\lambda)^*$ it follows from Theorem 3.3 that (I^λ) holds and $\bar{M}^\lambda = \mathscr{L}^\lambda$. Assuming that $\bar{M}^\lambda \neq \mathscr{L}^\lambda$, an argument similar to that in Theorem 3.3 gives the existence of $\varphi \in (\mathscr{L}^\lambda)_+^{\mathscr{A}}$, $f \in \mathscr{L}_+^\lambda$ with $\varphi(f) > 0$ and $\varphi(g) = 0$ for every g in M^λ. Then $\varphi(f) = \varphi(f\chi e(f))$ with $e(f)$ σ-finite. Since there then exists $\{g_n\} \subset M^\lambda$ with $g_n \uparrow f\chi e(f)$,

$$\varphi(f) = \lim_{n\to\infty} \varphi(g_n) = 0,$$

giving a contradiction.

We observe that the spaces $\mathscr{L}^\infty$ with X not finite gives examples where

$$(\mathscr{L}^\lambda)^{\mathscr{A}} \neq (\mathscr{L}^\lambda)^I = \mathscr{L}^1 \quad \text{with} \quad \bar{M}^\lambda \neq \mathscr{L}^\lambda.$$

Note that in this case $(\mathscr{L}^\lambda)^{\mathscr{A}} \neq (\mathscr{L}^\lambda)^*$.

5. **The spaces $\mathscr{L}^{\lambda *}$.** If $\varphi \in \mathscr{L}_+^{\lambda *}$ there exists $g \in \mathbf{M}_+$ with $\lambda^*(g) = \|\varphi\|$ and

$$\varphi(f) = \int fg \, d\mu \in \mathbf{R}, \qquad f \in \mathscr{L}^\lambda.$$

It is clear that $\varphi \in (\mathscr{L}^\lambda)_+^I$ with $\mu_\varphi(s) = \int_s g \, d\mu$, $s \in S^\lambda$.

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For an arbitrary ND-decomposition define $g_{\mathscr{A}}=\{g\chi e_a; a\in\mathscr{A}\}$. If $s\in S^\lambda$, $g\in\mathscr{L}^1(s)$, $\{x\in s:g(x)\neq 0\}$ is σ-finite and (4.4) follows easily. Thus $\mathscr{L}^{\lambda*}\subset(\mathscr{L}^\lambda)^{\mathscr{A}}$ and, in general,

$$\mathscr{L}^{\lambda*}\subset(\mathscr{L}^\lambda)^{\mathscr{A}}\subset(\mathscr{L}^\lambda)^I\subset(\mathscr{L}^\lambda)^*. \tag{5.1}$$

On the other hand *if* $\varphi\in(L^\lambda)^{\mathscr{A}}$, *then* $\varphi\in L^{\lambda*}$ *if and only if there exists* $g\in\bar{\mathbf{M}}$ *with* $\lambda^*[(g-g_a)\chi e_a]=0$ *for every* $a\in\mathscr{A}$.

We observe that if X is σ-finite or if every $f\in L^\lambda$ has σ-finite support then $L^{\lambda*}=(L^\lambda)^{\mathscr{A}}=(L^\lambda)^I$.

Speaking loosely, if $\varphi\in(L^\lambda)^{\mathscr{A}}$ then φ will be in $L^{\lambda*}$ if it is possible to piece the g_a together to form a measurable function g on X. Examples are given in [3] for L^1 where the $\{g_a\}$ determine a function g which is not measurable and where there can be no function g equal to each g_a almost everywhere in e_a, $a\in\mathscr{A}$.

THEOREM 5.1. *If* $L^{\lambda*}=(L^\lambda)^*$ *then* (I^λ) *holds and* $\bar{M}^\lambda=L^\lambda$. *If* (I^λ) *holds, if* $\bar{M}^\lambda=L^\lambda$ *and* (5.2) *to each* $\varphi\in(L^\lambda)^*$ *corresponds a σ-finite set* E *with* $\varphi(f\chi E)=\varphi(f)$ *for every* $f\in L^\lambda$, *then* $L^{\lambda*}=(L^\lambda)^*$.

Proof. The first part is given by Theorem 4.2 since $L^{\lambda*}=(L^\lambda)^*$ implies that $L^{\lambda*}=(L^\lambda)^{\mathscr{A}}$.

Assuming (I^λ) and $\bar{M}^\lambda=L^\lambda$, then $(L^\lambda)^{\mathscr{A}}=(L^\lambda)^*$ by Theorem 4.2. To $\varphi\in(L^\lambda)^*$ corresponds $g_{\mathscr{A}}$ with $\lambda^*(g_{\mathscr{A}})=\|\varphi\|$. By (5.2) $\varphi(f)=\varphi(f\chi E)$ for every $f\in L^\lambda$ where E is σ-finite, $\mu(E\cap e_a)=0$ for all but a countable set of subscripts in $\mathscr{A}$, say a_i, $i=1,2,\ldots$. If $g=\sum_1^\infty g_{a_i}$ then

$$\varphi(f)=\int fg_f\,d\mu=\int fg\,d\mu,$$

for every $f\in L^\lambda$ and $\lambda^*(g)=\|\varphi\|$. We conclude that $L^{\lambda*}=(L^\lambda)^{\mathscr{A}}=(L^\lambda)^*$.

That (5.2) is not a necessary condition is shown by the following

EXAMPLE. Let $X=(0,1)$, $S=\mathscr{P}(X)$, $\mu(e)$=number of points in $e(=+\infty$ if e is infinite); $\lambda(f)=\int f\,d\mu$, $f\in\mathbf{M}_+$. Then $L^\lambda=L^1$. An ND decomposition of X is given by $X_1=\phi$, $X_2=\bigcup_{a\in(0,1)}\{a\}$ (where the sets are disjoint). Since (I^λ) holds in L^1 and $\bar{M}^\lambda=L^1$; $(L^1)^*=(L^1)^{\mathscr{A}}=L^\infty$. Clearly each g determines $g=\sum g_a\in M$ so that $(L^1)^*=L^{\lambda*}$. However $\chi X\in(L^\lambda)^*$ without (5.2) holding.

Halperin [4] has solved the problem of necessary and sufficient conditions for the reflexivity of L^λ. His conditions (1.3), (1.3)* correspond to (5.2) for $(L^\lambda)^*$ and $(L^{\lambda*})^*$ with $\mu(e_i)$ replaced by $\lambda(\chi e_i)$, $\lambda^*(\chi e_i)$. The condition (I^λ) together with $\bar{M}^\lambda=L^\lambda$ imply his (1.1) and $\bar{M}^{\lambda*}=L^{\lambda*}$ is his (1.2). We sketch a proof in the present context. It shows that when L^λ is reflexive (5.2) is a necessary condition.

THEOREM 5.2. *L^λ is reflexive if and only if*

(i) *(I^λ) and $(I^{\lambda*})$ hold in L^λ and $L^{\lambda*}$;*

(ii) *$\bar{M}^\lambda = L^\lambda$, $\bar{M}^{\lambda*} = L^{\lambda*}$;*

(iii) *(5.2) holds in $(L^\lambda)^*$ and in $(L^{\lambda*})^*$;*

(iv) *Every $f \in \mathbf{M}$ can be expressed as $f = f_1 + f_2$ with $\lambda^{**}(f) = \lambda(f_1)$, $\lambda^{**}(f_2) = 0$.*

Proof. *Necessity.* By [4, Lemma 3.3] if $L^\lambda = (L^\lambda)^{**}$ then $(L^\lambda)^* = L^{\lambda*}$ and $(L^{\lambda*})^* = L^{\lambda**} = L^\lambda$. By Theorem 5.1 (i) and (ii) are necessary. (5.2) is then a consequence of (ii) in $L^{\lambda*}$ and $L^{\lambda**} = L^\lambda$. As in [4] (iv) is also necessary.

Sufficiency. (i), (ii) and (iii) imply that $L^{\lambda*} = (L^\lambda)^*$ and $L^{\lambda**} = (L^{\lambda*})^*$ by Theorem 5.1. As in [4] (1.4) implies that $L^{\lambda**} = L^\lambda$ and completes the proof.

REFERENCES

1. S. Banach, *Théorie des opérations linéaires*, Monografje Matematyczne Warsaw, 1932.

2. H. W. Ellis and Israel Halperin, *Function spaces determined by a levelling length function*, Canad. J. Math. **5** (1963), 576–592.

3. H. W. Ellis and D. O. Snow, *On $(L^1)^*$ for general measure spaces*, Canad. Math. Bull. **6** (1963), 211–230.

4. Israel Halperin, *Reflexivity in the L^λ function spaces*, Duke Math. J. **21** (1954), 205–208.

5. W. A. J. Luxemburg and A. C. Zaanen, *Notes on Banach function spaces*, Proc. Acad. Science, Amsterdam, (Indag. Math). Note V, A. 66 (1963), 496–504; Note VI, A. 66 (1963), 655–668.

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