ANZIAM J. 58(2016), 162–181
doi:10.1017/S144618111600016X

OPTIMAL MEAN–VARIANCE REINSURANCE WITH COMMON SHOCK DEPENDENCE

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(Received 26 January, 2015; accepted 17 February, 2016; first published online 30 August 2016)

Abstract

We consider the optimal proportional reinsurance problem for an insurer with two dependent classes of insurance business, where the two claim number processes are correlated through a common shock component. Using the technique of stochastic linear–quadratic control theory and the Hamilton–Jacobi–Bellman equation, we derive the explicit expressions for the optimal reinsurance strategies and value function, and present the verification theorem within the framework of the viscosity solution. Furthermore, we extend the results in the linear–quadratic setting to the mean–variance problem, and obtain an efficient strategy and frontier. Some numerical examples are given to show the impact of model parameters on the efficient frontier.

2010 *Mathematics subject classification*: primary 93E20; secondary 91B28, 91B30.

Keywords and phrases: common shock component, compound Poisson process, stochastic linear–quadratic problem, Hamilton–Jacobi–Bellman equation, proportional reinsurance.

1. Introduction

Using reinsurance, insurers are able to transfer some of their risks to another party, potentially at the expense of making less profit. Thus, finding the optimal reinsurance strategy to balance their risk and profit is of great interest to them. In fact, optimal reinsurance problems have attracted a lot of interest in the actuarial literature in the past few years, and the technique of stochastic control theory and the Hamilton–Jacobi–Bellman (HJB) equation are frequently used to cope with these problems (see, for example, [7, 10, 12, 15, 16]).

The mean–variance framework proposed by Markowitz [13] has become one of the milestones in mathematical finance. The author aimed to seek the best allocation among a number of (risky) assets in order to achieve the optimal trade-off between the

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expected return and its risk (say, variance) over a fixed time horizon. Since then, the mean–variance criterion has become popular in measuring the risk in finance theory. There are now numerous papers on the mean–variance problem and its extension to finance. For example, Li and Ng [9] developed an embedding technique to change the original mean–variance problem into a stochastic linear–quadratic (LQ) control problem in a discrete-time setting. This technique was extended by Zhou and Li [19], along with an indefinite stochastic LQ control approach, to the continuous-time case. Before 2005, all applications using the mean–variance criterion focused on classical financial portfolio allocation problems. In his study of the optimal reinsurance strategy problem for the classical compound Poisson insurance risk model, Bäuerle [2] first pointed out that the mean–variance criterion could also be of interest in insurance. Under the mean–variance framework, by using the stochastic LQ control theory, the explicit solutions of the efficient strategy and efficient frontier are derived. There have been further extensions and improvements in insurance applications (see, for example, [4] and the references therein).

The contribution of the present paper is to consider the optimal mean–variance reinsurance for a compound Poisson risk model with two dependent classes of insurance business, generalizing the results of Bäuerle [2] from an independent risk model to a dependent risk model, and increasing the number of control variables from one to two. The analysis becomes more complicated as a result.

Although research on optimal reinsurance is increasing rapidly, only a few papers deal with the problem in relation to dependent risks. Under the criteria of maximizing the expected utility of terminal wealth and maximizing the adjustment coefficient, Centeno [5] studied the optimal excess of loss retention limits for two dependent classes of insurance risks. Bai et al. [1] also investigated the optimal excess of loss reinsurance to minimize the ruin probability for the diffusion risk model. Liang and Yuen [11] considered the optimal proportional reinsurance with dependent risks under the variance premium principle. Using a nonstandard approach, they investigated the conditions for the existence and uniqueness of the optimal reinsurance strategies, and derived the closed-form expressions for the optimal reinsurance strategy and the value function for the compound Poisson risk model as well as for the diffusion risk model. In this paper, under the mean–variance criterion, we study the optimal proportional reinsurance for the dependent compound Poisson risk model. Using stochastic LQ control theory [19] and the HJB equation, we derive the explicit expressions of the optimal reinsurance strategies and value function, and present the verification theorem within the framework of the viscosity solution. Furthermore, we extend the results in the LQ setting to the mean–variance problem, and obtain the explicit solutions for the efficient strategy and efficient frontier.

The rest of the paper is organized as follows. In Section 2 the model and the mean–variance problem are given. The main results and the explicit expressions for the optimal values are derived in Sections 3 and 4. In Section 5, some numerical examples are presented to illustrate the impact of some model parameters on the efficient frontier. The paper concludes with a summary in Section 6.

2. Model formulation

Suppose that an insurance company has two dependent classes of insurance business such as motor and life insurance. Let X_i be the claim size random variables for the first class with common distribution $Q_X(x)$, and Y_i be the claim size random variables for the second class with common distribution $Q_Y(y)$. Their moments are denoted by $\mu_{1X} = E(X_i)$, $\mu_{1Y} = E(Y_i)$, $\mu_{2X} = E(X_i^2)$ and $\mu_{2Y} = E(Y_i^2)$. Assume that $Q_X(x) = 0$ for $x \leq 0$, $Q_Y(y) = 0$ for $y \leq 0$, $0 < Q_X(x) \leq 1$ for $x > 0$, and $0 < Q_Y(y) \leq 1$ for $y > 0$. Then, the aggregate claim processes for the two classes are given by

$$S_t = S_1(t) + S_2(t)$$

with

$$S_1(t) = \sum_{i=1}^{M_1(t)} X_i \quad \text{and} \quad S_2(t) = \sum_{i=1}^{M_2(t)} Y_i,$$

where $M_i(t)$ is the claim number process for class i $(i = 1, 2)$. It is assumed that X_i and Y_i are independent claim size random variables, and that they are independent of $M_1(t)$ and $M_2(t)$. The two claim number processes are correlated such that

$$M_1(t) = N_1(t) + N(t) \quad \text{and} \quad M_2(t) = N_2(t) + N(t),$$

with $N_1(t)$, $N_2(t)$ and $N(t)$ being three independent Poisson processes with parameters λ_1, λ_2 and λ, respectively. It is obvious that the dependence of the two classes of business is due to a common shock governed by the counting process $N(t)$. This model has been studied extensively in the literature (see, for example, [17, 18]).

As usual, the risk reserve process is defined as $R_t = R_0 + ct - S_t$, where R_0 is the amount of initial risk reserve, and c is the premium rate. Moreover, we allow the insurance company to continuously reinsure a fraction of its claim with the retention levels $q_{1t} \geq 0$ and $q_{2t} \geq 0$ for X_i and Y_i, respectively. Note that $q_{it} \in [0, 1]$ corresponds to a reinsurance cover and $q_{it} > 1$ corresponds to acquiring new business (see, for example, [2]). A strategy $q_t = (q_{1t}, q_{2t})$ is said to be admissible if q_{1t} and q_{2t} are $\mathcal{F}_t$-predictable processes and satisfy $q_{1t} \geq 0$ and $q_{2t} \geq 0$ for all $t \geq 0$. We denote the set of all admissible strategies by U. Let the (re)insurance premium rate at time t be calculated by the expected value principle (see, for example, [8]), and $\{R_t^q, t \geq 0\}$ denote the wealth of the insurer at time t under the strategy $q_t = (q_{1t}, q_{2t})$. This process then yields

$$dR_t^q = c^q\,dt - q_{1t}\,dS_1(t) - q_{2t}\,dS_2(t), \tag{2.1}$$

with

$$c^q = (\lambda_1 + \lambda)\mu_{1X}[(1 + \eta_1)q_{1t} - (\eta_1 - \theta_1)] + (\lambda_2 + \lambda)\mu_{1Y}[(1 + \eta_2)q_{2t} - (\eta_2 - \theta_2)].$$

Here $\theta_i > 0$ $(i = 1, 2)$ and $\eta_i > 0$ $(i = 1, 2)$ are the corresponding safety loadings of the insurer and reinsurer, respectively. Without loss of generality, we assume that $\eta_i > \theta_i$, $i = 1, 2$.

The problem now is to find the reinsurance policy so that the expected terminal wealth satisfies $E[R_T^q] = b$, where b is a constant, while the risk, measured by the variance of the terminal wealth

$$\operatorname{Var} R_T^q = E(R_T^q - b)^2,$$

is minimized. Then the variance-minimizing problem can be formulated as the following optimization problem:

$$\begin{aligned} &\text{minimize} \quad \operatorname{Var} R_T^q = E(R_T^q - b)^2, \\ &\text{such that} \quad \begin{cases} E[R_T^q] = b, \\ q \in U, \\ (R, q) \quad \text{satisfy (2.1).} \end{cases} \end{aligned} \tag{2.2}$$

This is the so-called mean–variance problem, and can be dealt with by introducing a Lagrange multiplier $\beta \in R$, which means that problem (2.2) can be solved via the following optimal problem:

$$\begin{aligned} &\text{minimize} \quad E[(R_T^q - b)^2 + 2\beta(ER_T^q - b)], \\ &\text{such that} \quad \begin{cases} q \in U, \\ (R, q) \quad \text{satisfy (2.1),} \end{cases} \end{aligned} \tag{2.3}$$

where the factor 2 in front of β is introduced for convenience. Clearly, problem (2.3) is equivalent to

$$\begin{aligned} &\text{minimize} \quad E[R_T^q - (b - \beta)]^2, \\ &\text{such that} \quad \begin{cases} q \in U, \\ (R, q) \quad \text{satisfy (2.1)} \end{cases} \end{aligned} \tag{2.4}$$

for a fixed β.

To obtain the optimal value and optimal strategy of problem (2.2), we need to maximize the optimal value in (2.3) over $\beta \in \mathbb{R}$ according to the Lagrange duality theorem [3]. Since problems (2.3) and (2.4) have the same optimal control for fixed β, we maximize the optimal value in (2.4) over $\beta \in \mathbb{R}$. For further simplification, we set $x_t = R_t - (b - \beta)$; then our controlled stochastic differential equation (2.1) becomes

$$\begin{aligned} dx_t^q &= c^q\, dt - q_{1t}\, dS_1(t) - q_{2t}\, dS_2(t), \\ x_0 &= R_0 - (b - \beta), \end{aligned} \tag{2.5}$$

and the optimal problem (2.4) is equivalent to

$$\begin{aligned} &\text{minimize} \quad E[\tfrac{1}{2}(x_T^q)^2], \\ &\text{such that} \quad \begin{cases} q \in U, \\ (x, q) \quad \text{satisfy (2.5).} \end{cases} \end{aligned} \tag{2.6}$$

Now we define the objective function as

$$J^q(t, x) = E[\tfrac{1}{2}(x_T^q)^2 \mid x_t^q = x],$$

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and thus the corresponding value function is given by

$$V(t,x) = \inf_{q\in U} J^q(t,x) = \inf_{q\in U} E[\tfrac{1}{2}(x_T^q)^2 \mid x_t^q = x]$$

with the boundary condition $V(T,x) = x^2/2$. In Sections 3 and 4 we will show how to solve the mean–variance problem using the stochastic control theory and the HJB equation.

Remark 2.1. In this paper, we assume that continuous trading is allowed and that all assets are infinitely divisible. Also, we work on a complete probability space $(\Omega, \mathcal{F}, P)$ on which the process R_t^q is well-defined. The information at time t is given by the complete filtration $\mathcal{F}_t$ generated by R_t^q.

3. The HJB equation and verification theorem

Let $C^{1,2}([0,T]\times\mathbb{R})$ denote the space of $\phi(t,x)$ so that ϕ and its partial derivatives ϕ_t, ϕ_x, ϕ_{xx} are continuous on $[0,T]\times\mathbb{R}$. We use the dynamic programming approach described by Fleming and Soner [6] to solve problems (2.5)–(2.6). From the standard arguments, for $V\in C^{1,2}$, we obtain the HJB equation for problems (2.5)–(2.6) as follows:

$$\begin{cases} \inf\limits_{q\in U}\{V_t + c^q V_x + \lambda_1 E[V(t,x-q_1X) - V(t,x)] \\ + \lambda_2 E[V(t,x-q_2Y) - V(t,x)] \\ + \lambda E[V(t,x-q_1X-q_2Y) - V(t,x)]\} = 0, \\ V(T,x) = \tfrac{1}{2}x^2. \end{cases} \tag{3.1}$$

Suppose that the solution of the HJB equation (3.1) has the form

$$V(t,x) = \tfrac{1}{2}P(t)x^2 + Q(t)x + L(t); \tag{3.2}$$

then we have

$$V_t = \tfrac{1}{2}P_t x^2 + Q_t x + L_t, \quad V_x = P(t)x + Q(t), \quad V_{xx} = P(t), \tag{3.3}$$

where P_t, Q_t and L_t are the derivatives of $P(t)$, $Q(t)$ and $L(t)$, respectively.

For convenience, we denote

$$\begin{aligned} a_1 &= (\lambda_1+\lambda)\mu_{1X}, \quad a_2 = (\lambda_2+\lambda)\mu_{1Y}, \\ b_1^2 &= (\lambda_1+\lambda)\mu_{2X}, \quad b_2^2 = (\lambda_2+\lambda)\mu_{2Y}, \\ B &= a_1(\eta_1-\theta_1) + a_2(\eta_2-\theta_2). \end{aligned}$$

Substituting (3.3) into (3.1) yields

$$\begin{aligned} \inf_{q\in U}\{&\tfrac{1}{2}P_t x^2 + Q_t x + L_t + (a_1\eta_1 q_1 + a_2\eta_2 q_2 - B)(P(t)x + Q(t)) \\ &+ \tfrac{1}{2}P(t)(b_1^2q_1^2 + b_2^2q_2^2 + 2\rho b_1 b_1 q_1 q_2)\} = 0, \end{aligned} \tag{3.4}$$

where $\rho = \lambda\mu_{1X}\mu_{1Y}/b_1b_2$, $(0<\rho<1)$.

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Let

$$f(q) = (a_1\eta_1 q_1 + a_2\eta_2 q_2)(P(t)x + Q(t)) + \tfrac{1}{2}P(t)(b_1^2 q_1^2 + b_2^2 q_2^2 + 2\rho b_1 b_1 q_1 q_2). \quad (3.5)$$

We have

$$\frac{\partial f}{\partial q_1} = P(t)b_1^2 q_1 + (P(t)x + Q(t))a_1\eta_1 + q_2 P(t)\rho b_1 b_2,$$
$$\frac{\partial f}{\partial q_2} = P(t)b_2^2 q_2 + (P(t)x + Q(t))a_2\eta_2 + q_1 P(t)\rho b_1 b_2,$$
$$\frac{\partial^2 f}{\partial q_1^2} = P(t)b_1^2,$$
$$\frac{\partial^2 f}{\partial q_2^2} = P(t)b_2^2,$$
$$\frac{\partial^2 f}{\partial q_1 \partial q_2} = \frac{\partial^2 f}{\partial q_2 \partial q_1} = P(t)\rho b_1 b_2.$$

Since

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial q_1^2} & \dfrac{\partial^2 f}{\partial q_1 \partial q_2} \\ \dfrac{\partial^2 f}{\partial q_2 \partial q_1} & \dfrac{\partial^2 f}{\partial q_2^2} \end{vmatrix} = P^2(t)b_1^2 b_2^2(1 - \rho^2) > 0,$$

f is a convex function with respect to q_1(or q_2); therefore, without the restriction $q \in U$, the minimizer of $f(q)$ in (3.5) satisfies the equations

$$\begin{aligned} P(t)b_1^2 q_1 + (P(t)x + Q(t))a_1\eta_1 + q_2 P(t)\rho b_1 b_2 = 0, \\ P(t)b_2^2 q_2 + (P(t)x + Q(t))a_2\eta_2 + q_1 P(t)\rho b_1 b_2 = 0. \end{aligned} \quad (3.6)$$

Solving equations (3.6) yields

$$\bar{q}_1 = m_1\Big(x + \frac{Q(t)}{P(t)}\Big),$$
$$\bar{q}_2 = m_2\Big(x + \frac{Q(t)}{P(t)}\Big),$$

where

$$m_1 = -\frac{a_1\eta_1 b_2^2 - a_2\eta_2\rho b_1 b_2}{b_1^2 b_2^2(1 - \rho^2)} \quad \text{and} \quad m_2 = -\frac{a_2\eta_2 b_1^2 - a_1\eta_1\rho b_1 b_2}{b_1^2 b_2^2(1 - \rho^2)}. \quad (3.7)$$

Because of the constraints of $(q_1^*, q_2^*) \in U$ and the result

$$\frac{(a_2 b_1/a_1\rho b_2)\eta_2}{(a_2\rho b_1/a_1 b_2)\eta_2} = \frac{1}{\rho^2} > 1,$$

we need to discuss the following five cases:

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1. $\eta_1 < (a_2\rho b_1/a_1b_2)\eta_2$ (that is, $m_1 > 0$, $m_2 < 0$);
2. $\eta_1 = (a_2\rho b_1/a_1b_2)\eta_2$ (that is, $m_1 = 0$, $m_2 < 0$);
3. $\eta_1 > (a_2b_1/a_1\rho b_2)\eta_2$ (that is, $m_1 < 0$, $m_2 > 0$);
4. $\eta_1 = (a_2b_1/a_1\rho b_2)\eta_2$ (that is, $m_1 < 0$, $m_2 = 0$);
5. $(a_2\rho b_1/a_1b_2)\eta_2 < \eta_1 < (a_2b_1/a_1\rho b_2)\eta_2$ (that is, $m_1 < 0$, $m_2 < 0$).

Case 1. Let $\eta_1 < (a_2\rho b_1/a_1b_2)\eta_2$.

In this case, $m_1 > 0$ and $m_2 < 0$. If $x + Q(t)/P(t) > 0$, then $\bar{q}_1 > 0$ and $\bar{q}_2 < 0$, and because of the restriction of $q^* \in U$, we have to choose $q_2^* = 0$. Inserting q_2^* into (3.5) with $\partial f(q)/\partial q_1 = 0$, we obtain $q_1 = -(a_1\eta_1/b_1^2)\{x + Q(t)/P(t)\} < 0$; then we get $q_1^* = 0$. Thus, the optimal strategy to minimize the left-hand side of equation (3.4) is $q^* = (q_1^*, q_2^*) = (0, 0)$. Substituting $q^* = (0, 0)$ back into (3.4) and grouping terms according to the powers of x leads to

$$\tfrac{1}{2}P_t = 0, \quad Q_t - BP(t) = 0 \quad \text{and} \quad L_t - BQ(t) = 0,$$

with the boundary conditions $P(T) = 1$, $Q(T) = 0$ and $L(T) = 0$. It is not difficult to get

$$P(t) = 1, \quad Q(t) = -B(T - t) \quad \text{and} \quad L(t) = \tfrac{1}{2}B^2(T - t)^2.$$

Substituting the expressions for $P(t)$, $Q(t)$, $L(t)$ into (3.2) and rearranging, we obtain

$$V(t, x) = \tfrac{1}{2}[x - B(T - t)]^2.$$

If $x + Q(t)/P(t) \le 0$, then $\bar{q}_1 \le 0$ and $\bar{q}_2 \ge 0$. For the restriction $q^* \in U$, we have to choose $q_1^* = 0$; then in the same manner as above, we get $q_2 = -(a_2\eta_2/b_2^2)\{x + Q(t)/P(t)\} \ge 0$. Therefore, the optimal strategy is $q^* = (q_1^*, q_2^*)$ with $q_1^* = 0$ and $q_2^* = -(a_2\eta_2/b_2^2)\{x + Q(t)/P(t)\}$. Substituting q^* back into (3.4) and grouping terms with like powers of x yields

$$\begin{cases} P_t - A_2P(t) = 0, \\ Q_t - A_2Q(t) - BP(t) = 0, \\ L_t - \dfrac{1}{2}A_2\dfrac{Q(t)^2}{P(t)} - BQ(t) = 0, \end{cases}$$

with the boundary conditions $P(T) = 1$, $Q(T) = 0$, $L(T) = 0$, where $A_2 = (a_2\eta_2/b_2)^2$. Solving the above differential equations, we derive

$$\begin{cases} P(t) = e^{-A_2(T-t)}, \\ Q(t) = -B(T - t)e^{-A_2(T-t)}, \\ L(t) = \tfrac{1}{2}B^2(T - t)^2e^{-A_2(T-t)}. \end{cases}$$

Substituting the expressions for $P(t)$, $Q(t)$, $L(t)$ into (3.2) and rearranging yields

$$V(t, x) = \tfrac{1}{2}e^{-A_2(T-t)}[x - B(T - t)]^2.$$

Along the same lines, we can get the optimal results for the other four cases as follows.

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Case 2. Let $\eta_1 = (a_2\rho b_1/a_1 b_2)\eta_2$.
The minimum of the left-hand side of the equation (3.1) is attained at

$$q_t^* = \begin{cases} (0,0), & x > B(T-t), \\ \Big(0, -\dfrac{a_2\eta_2}{b_2^2}[x - B(T-t)]\Big), & x \le B(T-t), \end{cases}$$

and the solution of equation (3.1) is

$$V(t,x) = \begin{cases} \frac{1}{2}[x - B(T-t)]^2, & x > B(T-t), \\ \frac{1}{2}e^{-A_2(T-t)}[x - B(T-t)]^2, & x \le B(T-t). \end{cases}$$

Case 3. Let $\eta_1 > (a_2 b_1/a_1\rho b_2)\eta_2$.
The minimum of the left-hand side of the equation (3.1) is attained at

$$q_t^* = \begin{cases} (0,0), & x > B(T-t), \\ \Big(-\dfrac{a_1\eta_1}{b_1^2}[x - B(T-t)], 0\Big), & x \le B(T-t), \end{cases}$$

and the solution of equation (3.1) is

$$V(t,x) = \begin{cases} \frac{1}{2}[x - B(T-t)]^2, & x > B(T-t), \\ \frac{1}{2}e^{-A_1(T-t)}[x - B(T-t)]^2, & x \le B(T-t), \end{cases}$$

where $A_1 = (a_1\eta_1/b_1)^2$.

Case 4. Let $\eta_1 = (a_2 b_1/a_1\rho b_2)\eta_2$.
The minimum of the left-hand side of the equation (3.1) is attained at

$$q_t^* = \begin{cases} (0,0), & x > B(T-t), \\ \Big(-\dfrac{a_1\eta_1}{b_1^2}[x - B(T-t)], 0\Big), & x \le B(T-t), \end{cases}$$

and the solution of equation (3.1) is

$$V(t,x) = \begin{cases} \frac{1}{2}[x - B(T-t)]^2, & x > B(T-t), \\ \frac{1}{2}e^{-A_1(T-t)}[x - B(T-t)]^2, & x \le B(T-t). \end{cases}$$

Case 5. Let $(a_2\rho b_1/a_1 b_2)\eta_2 < \eta_1 < (a_2 b_1/a_1\rho b_2)\eta_2$.
The minimum of the left-hand side of the equation (3.1) is attained at

$$q_t^* = \begin{cases} (0,0), & x > B(T-t), \\ (m_1[x - B(T-t)], m_2[x - B(T-t)]), & x \le B(T-t), \end{cases}$$

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and the solution of equation (3.1) is

$$V(t,x)=\begin{cases}\frac{1}{2}[x-B(T-t)]^2, & x>B(T-t),\\ \frac{1}{2}e^{A(T-t)}[x-B(T-t)]^2, & x\le B(T-t),\end{cases}$$

where $A=2(a_1m_1\eta_1+a_2m_2\eta_2)+m_1^2b_1^2+m_2^2b_2^2+2m_1m_2\rho b_1b_2$.

We summarize all the above results in the following theorem.

THEOREM 3.1. *Let m_1, m_2 be given as in (3.7). Then for any $t\in[0,T]$, the minimizer of the left-hand side of the equation (3.1) is*

$$q_t^*=\begin{cases}(0,0), & x>B(T-t),\\ (q_1^*(t,x),\,q_2^*(t,x)), & x\le B(T-t),\end{cases}\tag{3.8}$$

where

$$(q_1^*(t,x),\,q_2^*(t,x))=\begin{cases}\Big(0,-\dfrac{a_2\eta_2}{b_2^2}[x-B(T-t)]\Big), & \eta_1\le\dfrac{a_2\rho b_1}{a_1b_2}\eta_2,\\ (m_1[x-B(T-t)],m_2[x-B(T-t)]), & \dfrac{a_2\rho b_1}{a_1b_2}\eta_2<\eta_1<\dfrac{a_2b_1}{a_1\rho b_2}\eta_2,\\ \Big(-\dfrac{a_1\eta_1}{b_1^2}[x-B(T-t)],0\Big), & \eta_1\ge\dfrac{a_2b_1}{a_1\rho b_2}\eta_2.\end{cases}$$

Moreover, the solution of the HJB equation (3.1) is given by

$$V(t,x)=\begin{cases}\frac{1}{2}[x-B(T-t)]^2, & x>B(T-t),\\ V_1(t,x), & x\le B(T-t),\end{cases}\tag{3.9}$$

where

$$V_1(t,x)=\begin{cases}\frac{1}{2}e^{-A_2(T-t)}[x-B(T-t)]^2, & \eta_1\le\dfrac{a_2\rho b_1}{a_1b_2}\eta_2,\\ \frac{1}{2}e^{A(T-t)}[x-B(T-t)]^2, & \dfrac{a_2\rho b_1}{a_1b_2}\eta_2<\eta_1<\dfrac{a_2b_1}{a_1\rho b_2}\eta_2,\\ \frac{1}{2}e^{-A_1(T-t)}[x-B(T-t)]^2, & \eta_1\ge\dfrac{a_2b_1}{a_1\rho b_2}\eta_2,\end{cases}$$

with $A_1=(a_1\eta_1/b_1)^2$, $A_2=(a_2\eta_2/b_2)^2$ and $A=2(a_1m_1\eta_1+a_2m_2\eta_2)+m_1^2b_1^2+m_2^2b_2^2+2m_1m_2\rho b_1b_2$.

At the end of this section, we verify that the solution of the HJB equation (3.1) given in (3.9) is, indeed, the value function of our stochastic control problem (2.5)–(2.6). Since $V_{xx}(t,x)$ does not exist at the point $x=B(T-t)$, this means that $V(t,x)$ does not possess the necessary smoothness properties to qualify as a classical solution of the HJB equation (3.1). By the definition of viscosity solution and the same method as given by Bi and Guo [4], we can also show that $V(t,x)$ given in (3.9) is a viscosity solution of the HJB equation (3.1). Then the verification theorem within the framework of the viscosity solution is given as follows.

THEOREM 3.2 (Verification theorem). *Let q^* and $V(t,x)$ be given as in (3.8) and (3.9). Then the value function of stochastic control problem (2.5)–(2.6) is $V(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$, and for $t \le s \le T$, the optimal strategy is given by*

$$q_s^* = \begin{cases} (0,0), & x > B(T-t), \\ (q_1^*(s,x_s^*),\ q_2^*(s,x_s^*)), & x \le B(T-t), \end{cases}$$

where x_s^* is the risk reserve with optimal strategy q_s^*,

$$\begin{aligned} &(q_1^*(s,x_s^*), q_2^*(s,x_s^*)) \\ &= \begin{cases} \Big(0, -\dfrac{a_2\eta_2}{b_2^2}[x_s^* - B(T-s)]\Big), & \eta_1 \le \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2, \\ (m_1[x_s^* - B(T-s)], m_2[x_s^* - B(T-s)]), & \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2 < \eta_1 < \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \\ \Big(-\dfrac{a_1\eta_1}{b_1^2}[x_s^* - B(T-s)], 0\Big), & \eta_1 \ge \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2 \end{cases} \end{aligned}$$

for $t \le s < T \wedge \tau_{q^*}$ and

$$(q_1^*(s,x_s^*), q_2^*(s,x_s^*)) = (0,0) \quad \textit{for } T \wedge \tau_{q^*} \le s \le T,$$

in which

$$\tau_{q^*} = \inf\{s \ge t,\ x_s^* - B(T-s) > 0\}.$$

PROOF. The proof comprises the following three cases:

Case (1) $\eta_1 \le (a_2\rho b_1/a_1 b_2)\eta_2$,

Case (2) $(a_2\rho b_1/a_1 b_2)\eta_2 < \eta_1 < (a_2 b_1/a_1\rho b_2)\eta_2$,

Case (3) $\eta_1 \ge (a_2 b_1/a_1\rho b_2)\eta_2$.

Since the proofs of cases (1) and (3) are similar to case (2), we present only the proof of case (2) in detail.

When $x > B(T-t)$ at the initial time t, we define $q_s^* = (q_1^*(s,x_s), q_2^*(s,x_s)) = (0,0)$ for any $s \in (t,T]$, and the corresponding dynamical reserve process is given as

$$dx_s^{q^*} = -B\,ds, \quad t < s \le T.$$

We prove that q^* is the optimal strategy. Note that if $x_s^{q^*} = x - B(s-t)$, then

$$x_T^{q^*} = x - B(T-t) > 0.$$

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Also, note that

$$x_s^{q^*} - B(T-s) = x - B(T-t) > 0.$$

For any admissible strategy $q \in U$,

$$\begin{aligned} dx_s^q &= c^q\,ds - \Big(q_{1s} d\sum_{i=1}^{M_1(s)} X_i + q_{2s} d\sum_{i=1}^{M_2(s)} Y_i\Big) \\ &= [(\lambda+\lambda_1)\mu_{1X}(1+\eta_1)q_{1s} + (\lambda+\lambda_2)\mu_{1Y}(1+\eta_2)q_{2s} - B]\,ds \\ &\quad - \Big(q_{1s} d\sum_{i=1}^{M_1(s)} X_i + q_{2s} d\sum_{i=1}^{M_2(s)} Y_i\Big). \end{aligned}$$

It is not difficult to find that

$$Ex_T^q > Ex_T^{q^*} = x_T^{q^*} > 0$$

and, thus,

$$E(x_T^q)^2 \ge (Ex_T^q)^2 > (Ex_T^{q^*})^2 = E[(x_T^{q^*})^2],$$

which implies that q^* is optimal. The optimal value is

$$\begin{aligned} J^{q^*}(t,x) &= \tfrac{1}{2}E[(x_T^{q^*})^2 \mid x_t^{q^*} = x] \\ &= \tfrac{1}{2}[x_T^{q^*}]^2 \\ &= \tfrac{1}{2}[x - B(T-t)]^2 \\ &= V(t,x). \end{aligned}$$

When $x \le B(T-t)$ at the initial time t, let q be any admissible strategy. We define

$$\hat{q} = (\hat{q}_1, \hat{q}_2) = \begin{cases} q, & t \le s < T \wedge \tau_q, \\ (0,0), & T \wedge \tau_q \le s \le T, \end{cases}$$

in which

$$\tau_q = \inf\{s \ge t \mid x_s^q - B(T-s) > 0\},$$

and

$$T \wedge \tau_q = \begin{cases} \tau_q, & \tau_q \le T, \\ T, & \tau_q > T. \end{cases}$$

By the definition of $\hat{q}$ and the same analysis as in the former part of this proof, we get

$$E[(x_T^q)^2 \mid x_t^q = x] \ge E[(x_T^{\hat{q}})^2 \mid x_t^{\hat{q}} = x].$$

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Thus, the optimal problem can be restricted to the class of the strategy $\hat{q}$ such that when $T \wedge \tau_{\hat{q}} \le s \le T$, we have $\hat{q}_s = (0, 0)$, where

$$\tau_{\hat{q}} = \inf\{s \ge t \mid x_s^{\hat{q}} - B(T-s) > 0\}.$$

All such strategies can be denoted by the set $U' \subset U$. For arbitrary $\hat{q} \in U'$, applying Itô's lemma (see, for example, [14]) to $V(t, x)$ yields

$$\begin{aligned}
V(T \wedge \tau_{\hat{q}}, x^{\hat{q}}_{T\wedge\tau_{\hat{q}}}) = V(t, x) &+ \int_t^{T\wedge\tau_{\hat{q}}} (V_t + c^{\hat{q}} V_x)\, ds \\
&+ \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x) - V(s, x^{\hat{q}}_{s^-}) N_1(ds, dx) \\
&+ \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-}) N_2(ds, dy) \\
&+ \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-}) N(ds, dx, dy),
\end{aligned}$$

where $N(dt, dx)$, $N(dt, dy)$ and $N(dt, dx, dy)$ are Poisson random measures. Let

$$\begin{aligned}
M_1(dt, dx) &= N_1(dt, dx) - \lambda_1\, dt Q_X(dx), \\
M_2(dt, dy) &= N_2(dt, dy) - \lambda_2\, dt Q_Y(dy), \\
M(dt, dx, dy) &= N(dt, dx, dy) - \lambda\, dt Q_X(dx) Q_Y(dy).
\end{aligned}$$

Then $M_1(dt, dx)$, $M_2(dt, dy)$ and $M(dt, dx, dy)$ are the compensated Poisson random measures.

Note that

$$\begin{aligned}
V(T \wedge \tau_{\hat{q}}, x^{\hat{q}}_{T\wedge\tau_{\hat{q}}}) = V(t, x) &+ \int_t^{T\wedge\tau_{\hat{q}}} (V_t + c^{\hat{q}} V_x)\, ds \\
&+ \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x) - V(s, x^{\hat{q}}_{s^-}) M_1(ds, dx) \\
&+ \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-}) M_2(ds, dy) \\
&+ \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-}) M(ds, dx, dy) \\
&+ \lambda_1 \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x) - V(s, x^{\hat{q}}_{s^-}) Q_X(dx)\, ds \\
&+ \lambda_2 \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-}) Q_Y(dy)\, ds \\
&+ \lambda \int_t^{T\wedge\tau_{\hat{q}}} \int_0^\infty \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x - \hat{q}_{2s} y) \\
&- V(s, x^{\hat{q}}_{s^-}) Q_X(dx) Q_Y(dy)\, ds.
\end{aligned}$$

Since $V(t, x)$ satisfies the HJB equation, we have

$$
\begin{aligned}
V(T \wedge \tau_{\hat{q}}, x^{\hat{q}}_{T \wedge \tau_{\hat{q}}}) \geq V(t, x) &+ \int_t^{T \wedge \tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x) - V(s, x^{\hat{q}}_{s^-}) M_1(ds, dx) \\
&+ \int_t^{T \wedge \tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-}) M_2(ds, dy) \\
&+ \int_t^{T \wedge \tau_{\hat{q}}} \int_0^\infty V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x - \hat{q}_{2s} y) \\
&- V(s, x^{\hat{q}}_{s^-}) M(ds, dx, dy).
\end{aligned}
\tag{3.10}
$$

The equality is obtained when the policy $q = q^*$. Note that

$$
E\Big[\int_t^{T \wedge \tau_{\hat{q}}} \int_0^\infty |V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x) - V(s, x^{\hat{q}}_{s^-})| Q_X(dx)\, ds\Big] < \infty,
$$

$$
E\Big[\int_t^{T \wedge \tau_{\hat{q}}} \int_0^\infty |V(s, x^{\hat{q}}_{s^-} - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-})| Q_Y(dy)\, ds\Big] < \infty
$$

and

$$
E\Big[\int_t^{T \wedge \tau_{\hat{q}}} \int_0^\infty \int_0^\infty |V(s, x^{\hat{q}}_{s^-} - \hat{q}_{1s} x - \hat{q}_{2s} y) - V(s, x^{\hat{q}}_{s^-})| Q_X(dx) Q_Y(dy)\, ds\Big] < \infty,
$$

and thus these integrals are martingales. Taking the conditional expectation on both sides of (3.10) yields

$$
E[\tfrac{1}{2}(x^{\hat{q}}_T)^2 \mid x^{\hat{q}}_t = x] \geq V(t, x),
$$

when $\hat{q} = q^*$, therefore the equality is obtained, which completes the proof of the theorem. □

4. The efficient strategy and efficient frontier

In this section we apply the results in Section 3 to solve the mean–variance problem, and derive the efficient strategy and efficient frontier of problem (2.2). Our primitive mean–variance problem refers to finding the optimal reinsurance strategy such that the expected terminal wealth satisfies $ER^q_T = b$, where b is a constant, while the risk measured by the variance of the terminal wealth $\mathrm{Var}\, R^q_T = E(R^q_T - ER^q_T)^2 = E(R^q_T - b)^2$ is minimized. If we let b be a variable, then our mean–variance problem (2.2) can be changed into a multi-objective optimization problem that maximizes the expected terminal wealth ER^q_T, and at the same time minimizes the variance of the terminal wealth $\mathrm{Var}\, R^q_T$ over $q \in U$. The optimal reinsurance strategy for the multi-objective optimization problem is called a *variance-minimizing strategy* corresponding to a fixed b, and the set of all points $(\mathrm{Var}\, R^*_T, b)$ is called the *variance-minimizing frontier*. When $b \geq R_0 - BT$, the optimal reinsurance strategy for the multi-objective optimization problem is called an *efficient strategy*, the corresponding $(\mathrm{Var}\, R^*_T, b)$ is an *efficient point*, and the set of all efficient points when b runs over $[R_0 - BT, \infty)$ is called the *efficient frontier*.

Since we have set $x_t = R_t - (b - \beta)$, we get

$$E[\tfrac{1}{2}(x_T^q)^2] = \tfrac{1}{2}E[(R_T^q - b)^2 + 2\beta(ER_T^q - b) + \beta^2].$$

Therefore, for every fixed β, we have

$$\begin{aligned}\min_{q\in U} E[(R_T^q - b)^2 + 2\beta(ER_T^q - b)] \\ = \begin{cases} [R_0 - (b-\beta) - BT]^2 - \beta^2, & R_0 > (b-\beta) + BT, \\ 2V_1(0, x_0) - \beta^2, & R_0 \le (b-\beta) + BT, \end{cases}\end{aligned} \tag{4.1}$$

where

$$2V_1(0, x_0) - \beta^2 = \begin{cases} e^{-A_2 T}[R_0 - (b-\beta) - BT]^2 - \beta^2, & \eta_1 \le \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2, \\ e^{AT}[R_0 - (b-\beta) - BT]^2 - \beta^2, & \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2 < \eta_1 < \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \\ e^{-A_1 T}[R_0 - (b-\beta) - BT]^2 - \beta^2, & \eta_1 \ge \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2. \end{cases}$$

Furthermore, when $R_0 > (b - \beta) + BT$, the variance-minimizing strategy is

$$q_t^* = (0, 0), \quad 0 \le t < T.$$

When $R_0 \le (b - \beta) + BT$, let $G(t, R_t^*) = R_t^* - (b - \beta) - B(T - t)$; then the variance-minimizing strategy is

$$q_t^* = (q_1^*(t, R_t^*), q_2^*(t, R_t^*)), \tag{4.2}$$

where

$$(q_1^*(t, R_t^*), q_2^*(t, R_t^*)) = \begin{cases} \Big(0, -\dfrac{a_2\eta_2}{b_2^2}\cdot G(t, R_t^*)\Big), & \eta_1 \le \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2, \\ (m_1 \cdot G(t, R_t^*),\ m_2 \cdot G(t, R_t^*)), & \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2 < \eta_1 < \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \\ \Big(-\dfrac{a_1\eta_1}{b_1^2}\cdot G(t, R_t^*), 0\Big), & \eta_1 \ge \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \end{cases}$$

for $0 \le t < T \wedge \tau_{q^*}$, and

$$(q_1^*(t, R_t^*), q_2^*(t, R_t^*)) = (0, 0)$$

for $T \wedge \tau_{q^*} \le t < T$, in which

$$\tau_{q^*} = \inf\{t \ge 0, G(t, R_t^*) > 0\}.$$

Note that the above value still depends on the Lagrange multiplier β. Making use of the fact that $R_0 \le b + BT$, which ensures that $ER_T^q = b$ can be satisfied, we see that to obtain the minimum $\operatorname{Var} R_T^q$ and the optimal strategy for the original control problem (2.2), it is sufficient to maximize the value in (4.1) over $\beta \in \mathbb{R}$ by the Lagrange duality theorem (see, for example [3]). The above discussion leads to the following theorem.

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THEOREM 4.1. *Assume that $R_0 \le b + BT$. Then the efficient frontier of problem (2.2) is*

$$\operatorname{Var} R_T^* = \begin{cases} \dfrac{e^{-A_2T}(R_0 - b - BT)^2}{1 - e^{-A_2T}}, & \eta_1 \le \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2, \\ \dfrac{e^{AT}(R_0 - b - BT)^2}{1 - e^{A_T}}, & \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2 < \eta_1 < \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \\ \dfrac{e^{-A_1T}(R_0 - b - BT)^2}{1 - e^{-A_1T}}, & \eta_1 \ge \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2. \end{cases}$$

Moreover, the efficient strategy is $q_t^* = (q_1^*(t, R_t^*), q_2^*(t, R_t^*))$, where

$$(q_1^*(t, R_t^*), q_2^*(t, R_t^*)) = \begin{cases} \Big(0, -\dfrac{a_2\eta_2}{b_2^2} H(t, R_t^*, A_2)\Big), & \eta_1 \le \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2, \\ (m_1 H(t, R_t^*, A), m_2 H(t, R_t^*, A)), & \dfrac{a_2\rho b_1}{a_1 b_2}\eta_2 < \eta_1 < \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \\ \Big(-\dfrac{a_1\eta_1}{b_1^2} H(t, R_t^*, A_1), 0\Big), & \eta_1 \ge \dfrac{a_2 b_1}{a_1\rho b_2}\eta_2, \end{cases}$$

for $0 \le t < T \wedge \tau_{q^*}$, and

$$(q_1^*(t, R_t^*), q_2^*(t, R_t^*)) = (0, 0)$$

for $T \wedge \tau_{q^*} \le t < T$, in which

$$H(t, R_t^*, y) = R_t^* + \frac{e^{-yT}(R_0 - Bt) - B(T - t) - b}{1 - e^{-yT}}$$

and R_t^* is the risk reserve with optimal strategy q_t^*.

PROOF. Suppose first that $R_0 \le BT + b - \beta$. When $\eta_1 \le (a_2\rho b_1/a_1 b_2)\eta_2$, from (4.1), we have

$$\begin{aligned} \min_{q\in U} E[(R_T^q - b)^2 + 2\beta(ER_T^q - b)] &= 2V(0, x_0) - \beta^2 \\ &= (e^{-A_2T} - 1)\beta^2 + 2e^{-A_2T}(R_0 - b - BT)\beta + e^{-A_2T}(R_0 - b - BT)^2. \end{aligned} \tag{4.3}$$

Maximizing expression (4.3) over $\beta \in \mathbb{R}$ yields

$$\beta^* = \frac{e^{-A_2T}(R_0 - b - BT)}{1 - e^{-A_2T}}.$$

Substituting β^* into (4.1) and (4.2), the efficient strategy of the mean–variance problem is given by

$$q_t^* = (q_1^*(t, R_t^*),\ q_2^*(t, R_t^*)),$$

where $q_1^*(t, R_t^*) = 0$, $0 \le t < T$, and

$$q_2^*(t, R_t^*) = \begin{cases} -\dfrac{a_2\eta_2}{b_2^2}\Big[R_t^* + \dfrac{e^{-}A_2T(R_0 - Bt) - B(T - t) - b}{1 - e^{-A_2T}}\Big], & 0 \le t < T \wedge \tau_{q^*}, \\ 0, & T \wedge \tau_{q^*} \le t < T. \end{cases}$$

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Note that the efficient frontier is

$$\operatorname{Var} R_T^* = \frac{e^{-A_2T}(R_0 - b - BT)^2}{1 - e^{-A_2T}}.$$

When $(a_2\rho b_1/a_1b_2)\eta_2 < \eta_1 < (a_2b_1/a_1\rho b_2)\eta_2$, from (4.1), we have

$$\begin{aligned}\min_{q\in U} E[(R_T^q - b)^2 + 2\beta(ER_T^q - b)] &= 2V_1(0, x_0) - \beta^2\\ &= (e^{AT} - 1)\beta^2 + 2e^{AT}(R_0 - b - BT)\beta + e^{AT}(R_0 - b - BT)^2.\end{aligned}$$

Since

$$\begin{aligned}A &= 2(a_1m_1\eta_1 + a_2m_2\eta_2) + m_1^2b_1^2 + m_2^2b_2^2 + 2m_1m_2\rho b_1b_2\\ &= -\frac{a_1^2b_2^2\eta_1^2 + a_2^2b_1^2\eta_2^2 - 2\rho a_1a_2b_1b_2\eta_1\eta_2}{b_1^2b_2^2(1 - \rho^2)}\end{aligned}$$

and

$$a_1^2b_2^2\eta_1^2 + a_2^2b_1^2\eta_2^2 - 2\rho a_1a_2b_1b_2\eta_1\eta_2 \geq 2(1 - \rho)a_1a_2b_1b_2\eta_1\eta_2 > 0,$$

we have $A < 0$, which means that the result of (4.4) is a concave function with respect to β.

Maximizing expression (4.4) over $\beta \in \mathbb{R}$ yields

$$\beta^* = -\frac{e^{AT}(R_0 - b - BT)}{e^{AT} - 1}.$$

Substituting β^* back into (4.1) and (4.2), we derive the efficient strategy of the mean–variance problem,

$$q_t^* = (q_1^*(t, R_t^*), q_2^*(t, R_t^*)),$$

where

$$q_1^*(t, R_t^*) = \begin{cases} m_1\Big[R_t^* - \dfrac{e^{AT}(R_0 - Bt) - B(T - t) - b}{e^{AT} - 1}\Big], & 0 \leq t < T \wedge \tau_{q^*},\\ 0, & T \wedge \tau_{q^*} \leq t < T,\end{cases}$$

and

$$q_2^*(t, R_t^*) = \begin{cases} m_2\Big[R_t^* - \dfrac{e^{AT}(R_0 - Bt) - B(T - t) - b}{e^{AT} - 1}\Big], & 0 \leq t < T \wedge \tau_{q^*},\\ 0, & T \wedge \tau_{q^*} \leq t < T.\end{cases}$$

The efficient frontier is

$$\operatorname{Var} R_T^* = \frac{e^{AT}(R_0 - b - BT)^2}{1 - e^{AT}}.$$

When $\eta_1 \geq (a_2b_1/a_1\rho b_2)\eta_2$, along the same lines as in the case of $\eta_1 \leq (a_2\rho b_1/a_1b_2)\eta_2$, we can also get the explicit expression for the efficient strategy and the efficient frontier. We omit the detailed proof here. Since $R_0 \leq b + BT - \beta^*$ is equivalent to $R_0 \leq b + BT$, the proof is now complete. □

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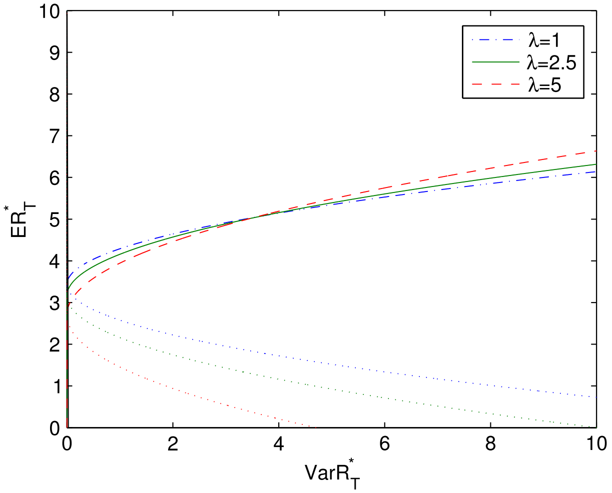

FIGURE 1. The efficient frontiers for different λ.

REMARK 4.2. Under the assumption of $R_0 \le b + BT$, the solution of the mean–variance problem consists of only one region, since once the boundary is reached, we apply the full reinsurance $q_t^* = (0, 0)$ for the remaining time and the risk reserve falls on the straight line $R_0 - Bt$ and reaches b at time T, which is the same as in [2].

REMARK 4.3. The efficient strategy and efficient frontier in [2] can be derived directly from Theorem 4.1 by setting $\eta_1 = \eta_2$, $\theta_1 = \theta_2$, $\alpha_1 = \alpha_2$, $\lambda_1 = \lambda_2$ and $\lambda = 0$.

5. Numerical examples

In this section, we assume that the claim sizes X_i and Y_i are exponentially distributed with parameters α_1 and α_2, respectively. Then we have $\mu_{1X} = 1/\alpha_1$, $\mu_{1Y} = 1/\alpha_2$, $\mu_{2X} = 2/\alpha_1^2$, $\mu_{2Y} = 2/\alpha_1^2$. Here we only take the case $(a_2\rho b_1/a_1b_2)\eta_2 < \eta_1 < (a_2b_1/a_1\rho b_2)\eta_2$ in Theorem 4.1 as an example to verify our outcomes in the foregoing. In the following examples, we show how the dependence between two classes of insurance business affects the efficient frontiers, and we present the impact of the parameters λ_1, α_1 and η on the efficient frontier.

EXAMPLE 5.1. Let $\lambda_2 = 5$, $T = 10$, $R_0 = 4$, $\alpha_1 = 1.5$, $\alpha_2 = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.12$, $\eta_1 = 0.12$, $\eta_2 = 0.125$. The results are shown in Figures 1 and 2.

From Figure 1 ($\lambda_1 = 1$) and Figure 2 ($\lambda = 4$), we see that when $\mathrm{Var}\, R_T^*$ is small enough, a greater value of λ (λ_1) yields a smaller value of ER_T^* with the same $\mathrm{Var}\, R_T^*$. On the other hand, when $\mathrm{Var}\, R_T^*$ is large enough, a greater value of λ (λ_1) yields a greater ER_T^* with the same $\mathrm{Var}\, R_T^*$. Besides, the top half of the parabola is the efficient frontier and the whole parabola is the variance-minimizing frontier. Similar results are shown in Figures 3 and 4.

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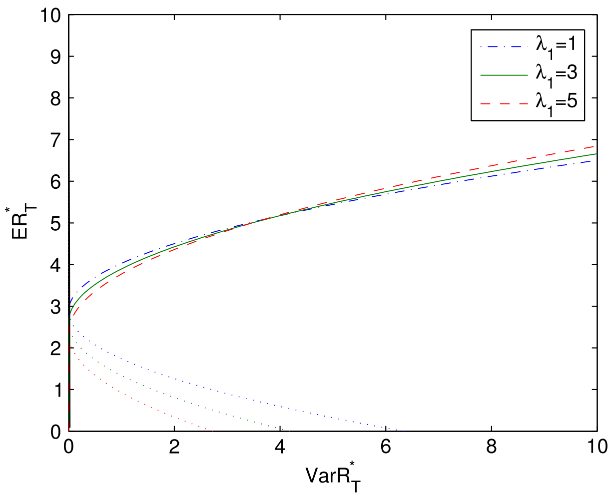

Figure 2. The efficient frontiers for different λ_1.

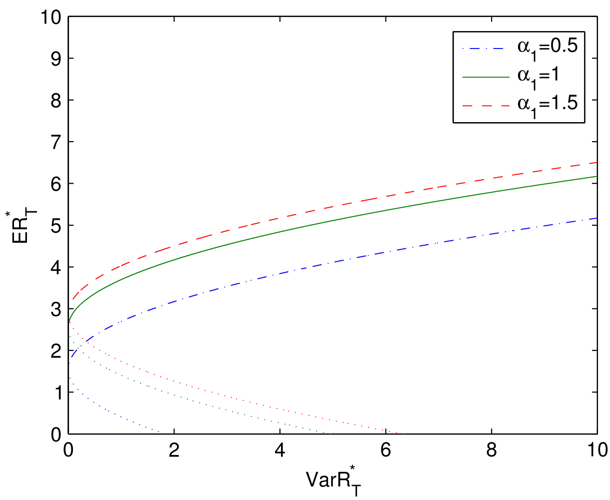

Figure 3. The efficient frontiers for different α_1.

Example 5.2. Let $\lambda_1 = 1$, $\lambda_2 = 5$, $\lambda = 4$, $T = 10$, $R_0 = 4$, $\alpha_2 = 1$, $\theta_1 = 0.1$, $\theta_2 = 0.12$, $\eta_2 = 0.125$. The results are given in Figures 3 and 4.

From Figure 3 ($\eta_1 = 0.11$) and Figure 4 ($\alpha_1 = 1.5$), we see that a greater value of α_1 gives a greater value of ER_T^* with the same value $\operatorname{Var} R_T^*$, whereas a greater η_1 gives a smaller ER_T^* with the same $\operatorname{Var} R_T^*$.

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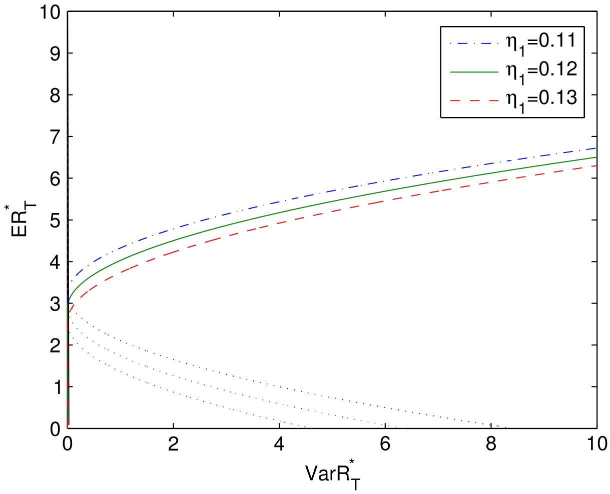

FIGURE 4. The efficient frontiers for different η_1.

6. Conclusion

Here we summarize the main results of the paper. From an insurer's point of view, we consider the optimal proportional reinsurance strategy in a compound Poisson risk model with two dependent classes of insurance business, where the two claim number processes are correlated. By the stochastic control theory and HJB equation, we derive the explicit expressions for the optimal reinsurance strategies and value function in the LQ setting, and present the verification theorem within the framework of the viscosity solution. Furthermore, we extend the results in the LQ setting to the mean–variance problem and obtain the efficient strategy and efficient frontier. Some numerical examples are given to show the impact of model parameters on the efficient frontier.

Later, we may extend our work to the case of a diffusion approximation risk model with two dependent classes of insurance business which has already been discussed in [1] and [11]. However, we find that under the mean–variance framework, the HJB equation for the diffusion approximation case is exactly the same as that in the compound Poisson risk model (see, for example, [8]); then the optimal strategies and value function are the same, which is very different from the other risk measures.

Although the literature on optimal reinsurance is increasing rapidly, very few of these contributions deal with the problem in relation to dependent risks. Therefore, there are still some interesting problems in this direction that can be further studied. For example, one may consider the optimal reinsurance with dependent risks under additional constraints on the probability of ruin, which is a very challenging problem, though some useful results have already been derived.

Acknowledgements

The authors would like to thank the anonymous referee for his/her careful reading and helpful comments on an earlier version of this paper, which led to a considerable improvement of the presentation of the work. This work was supported by the National Natural Science Foundation of China (Grant No. 11471165) and Jiangsu Natural Science Foundation (Grant No. BK20141442).

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