BULL. AUSTRAL. MATH. SOC. 53C40, 53A04
VOL. 56 (1997) [37–49]

A CHARACTERISATION OF HELICES AND CORNU SPIRALS IN REAL SPACE FORMS

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We classify unit speed curves contained in a real space form of arbitrary dimension $N^m(c)$, whose mean curvature vector is proper for the Laplacian. Then we use these results to classify Hopf cylinders of S^3 and semi-Riemannian Hopf cylinders of $H_1^3(-1)$ with proper mean curvature function.

1. INTRODUCTION

Given an isometric immersion of a Riemannian manifold into Euclidean space, $x : M^n \to \overline{M}^m$, we use H and Δ to denote the mean curvature vector and the Laplacian of (M, x) respectively. If $\overline{M}^m = E^m$, the Euclidean space, we have two interesting consequences of Beltrami's formula, $\Delta x = -nH$. The first one is that minimal submanifolds of E^m have harmonic position vector (*harmonic* submanifolds). By using again Beltrami's formula, we see that submanifolds satisfying $\Delta H = 0$ also verify $\Delta^2 x = 0$ and for this reason they were called *biharmonic* submanifolds [**6**]. Chen conjectured in [**6**] that the family of biharmonic immersions is the same as that of harmonic (minimal) immersions. The conjecture has been proved to be true in E^3, [**10**]. Other verifications have been made which provide further support for it : [**13, 11**]. As far as we know, the conjecture is still unsolved.

A second consequence is Takahashi's Theorem [**16**]: a Euclidean submanifold (M^n, x) satisfies $\Delta x = \lambda x$, if and only if, (M^n, x) is either minimal in E^m $(\lambda = 0)$ or minimal in a hypersphere of E^m $(\lambda > 0)$. One sees easily that condition $\Delta x = \lambda x$ implies $\Delta H = \lambda H$, where H is considered as an m-valued function on M. Unlike condition $\Delta x = \lambda x$ which makes no sense for a submanifold of any Riemannian manifold, condition $\Delta H = \lambda H$ can be considered in such a more general context as we see in Section 2. Thus, it makes sense to study the condition $\Delta H = \lambda H$ (and also $\Delta^D H = \lambda H$, see below) as a generalisation of Takahashi's one, for submanifolds of any Riemannian manifold. The study of submanifolds of Euclidean and pseudo-Euclidean spaces satisfying $\Delta H = \lambda H$ was initiated by Chen [**7, 8**], and it is related with the theory of submanifolds of finite

Received 1st August, 1996
Research partially supported by a DGICYT grant PB94-0750-C02-01 and by a grant from Gobierno Vasco PI95/95.

type introduced by the same author. (For details one may consult the excellent survey [**9**].) In Section 3, we see that we can reduce codimension to two, for curves in a real space form $N^m(c)$ satisfying $\Delta H = \lambda H$.

Moreover, they must be helices in a totally geodesic $N^3(c)$. In particular, the only curves with harmonic mean curvature vector in a $N^m(c)$ are the geodesics, and this extends a confirmation of Chen's conjecture due to Dimitric [**11**].

On the other hand, by looking at Chen's formula, [**5**, Lemma 4.1, p.271], one sees that the Laplacian in the normal bundle of $H, \Delta^D H$, is an ingredient of the normal part of ΔH to (M, x). This relation between ΔH and $\Delta^D H$ suggests that the condition $\Delta^D H = 0$ is less restrictive than $\Delta H = 0$ and, in fact, hypersurfaces of E^m satisfying $\Delta^D H = 0$ are precisely those hypersurfaces with harmonic mean curvature function. (This is a consequence of Chen's formula.) However, condition $\Delta H = \lambda H$ does not imply $\Delta^D H = \lambda H$. Thus one would expect that the family of curves in $N^m(c)$ with proper mean curvature vector in the normal bundle does not include helices other than geodesics and circles. This is true indeed, as it is proved in Section 4, where we classify unit speed curves in a real space form $N^m(c)$, with proper mean curvature in the normal bundle. We show first that we can reduce codimension for a curve of this type (namely $m \leqslant 3$) and then we integrate the differential equations which determine its curvature and torsion. As for the harmonic case, we extend the ideas used in [**3**] to identify them as members of a certain biparametric family of curves (Lemma 5). Then we use a different idea and what we call Cornu spirals in a surface to obtain their classification (of course for the case $N^m(c) = E^m$ our result becomes the one obtained in [**3**]). Also, some small errors concerning the parameters which appeared in [**3**] are modified here (see formula (14)) .

In Section 5 we start by proving a technical result on totally geodesic semi-Riemannian submersions (Proposition 7). Proceeding from the proofs in the earlier section and with the aid of this technical result, we give then the classification of Hopf cylinders in $S^3(1)$ and semi-Riemannian Hopf cylinders in $H_1^3(-1)$ with proper mean curvature function. The spherical case corresponding to the harmonic mean curvature vector was treated in [**3**], but here instead of considering the Hopf map as in [P], we consider it as a Riemannian submersion in order to offer a unified analysis based on Proposition 7 of both spherical and hyperbolic cases.

2. Preliminaries

We consider an arclength parameterised curve $\beta = \beta(s) : I \subset R \longrightarrow N^m(c)$ in a real space form $N^m(c)$ with constant sectional curvature c and dimension m. We denote by $T = T(s) = \xi_1$ the unit tangent vector field of β and by $k_1(s) = \left\|\overline{\nabla}_T T\right\|$. If $k_1(s) = 0$, then β is a geodesic. If it is not zero, we can define a unit vector ξ_2 perpendicular to $T = \xi_1$ such that

$$\overline{\nabla}_T T(s) = k_1(s)\xi_2(s) \tag{1}$$

https://doi.org/10.1017/S0004972700030719 Published online by Cambridge University Press

where $\overline{\nabla}$ denotes the Levi-Civita connection of $N^m(c)$. Now we consider $\xi_2' = \overline{\nabla}_T\xi_2$, which can be decomposed as $\xi_2' = -k_1T + \delta$ with $\delta \in$ Span $\{\xi_1, \xi_2\}^\perp$. If δ it identically zero, then by using [**12**], we can reduce codimension and β is "plane", that is it is contained in a totally geodesic surface $N^2(c)$ of $N^m(c)$. If δ is not zero, then we can find a unit vector ξ_3 perpendicular to $\xi_1 = T$ and ξ_2 and a positive function $k_2(s)$, such that

$$\overline{\nabla}_T\xi_2(s) = -k_1(s)T(s) + k_2\xi_3(s). \tag{2}$$

Proceeding in the same way we find that β is fully contained in a totally geodesic (d+1)-dimensional submanifold $N^d(c)$, $d \leqslant m$, and we have $d-1$ positive functions $k_1, k_2, \ldots, k_{d-1} : I \longrightarrow R$ satisfying

$$\begin{aligned} \overline{\nabla}_T T(s) &= k_1(s)\xi_2(s) \\ \overline{\nabla}_T\xi_i(s) &= -k_{i-1}(s)\xi_{i-1}(s) + k_i(s)\xi_{i+1}(s),\ 2 \leqslant i \leqslant d \\ \overline{\nabla}_T\xi_d(s) &= -k_{d-1}(s)\xi_{d-1}(s). \end{aligned} \tag{3}$$

The functions $k_i > 0$ are called the *Frenet ith-curvatures* of β ($k_1 = k$, and $k_2 = -\tau$ are called simply curvature and torsion of β, respectively, if $d \leqslant 3$).

A unit speed curve β is a *helix* if it has constant Frenet curvatures. A helix is called a *circle* if k_1 is a non-zero constant (of course $k_1 = 0$ corresponds to the geodesics of $N^m(c)$) and $k_2 = 0$. A unit speed curve which lies in a simply connected real space form $N^m(c)$, $\beta : I \to N^m(c)$ is said to be a *general helix* if there exist a Killing vector field $V(s)$ with constant length along β, such that the angle between V and β' is a non-zero constant along β. The vector V is called the *axis* of the general helix. This definition generalises that of general helices in R^3 (Böschungslinien) and includes the above defined helices (with constant curvatures). For details see [**1**] and Remark 2.

A unit speed curve which lies in a surface S of $N^m(c)$, $\beta : I \to S \subset N^m(c)$ is said to be a *Cornu Spiral* if its curvature if in S is a non-constant linear function of the natural parameter s, that is if $\varphi(s) = \mu s + \varepsilon$ with $\mu \neq 0$. The classical Cornu spiral in E^2 was studied by J. Bernoulli and it appears in diffraction theory.

Consider now an isometric immersion of a Riemannian manifold into Euclidean space $x : M^n \to E^m$. Let us denote by Δ the Laplacian of (M^n, x) and extend it in the natural way so that it acts on m-valued functions. Then we have the following Beltrami's formula : $\Delta x = -nH$, where H is the mean curvature vector of (M^n, x). Chen proved the following identity [**5**]:

$$\Delta H = -\sum_{i=1}^{n}\left(\overline{\nabla}_{E_i}\overline{\nabla}_{E_i}H - \overline{\nabla}_{\nabla_{E_i}E_i}H\right) \tag{4}$$

where $\overline{\nabla}$ is the connection on E^m, ∇ is the Riemannian connection of (M^n, x) and $\{E_i\}_{i=1}^n$ is a local orthonormal basis tangent to (M^n, x). For an isometric immersion in any Riemannian manifold $x : M^n \to \overline{M}^m$ we can define (see [**15**]) the Laplacian of the

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mean curvature vector formally as in (4), where this time $\overline{\nabla}$ is the Riemannian connection on $\overline{M}$. Also, following [**15**], we define the Laplacian of the mean curvature vector in the normal bundle by

$$\Delta^D H = -\sum_{i=1}^{n} \left(D_{E_i} D_{E_i} H - D_{\nabla_{E_i} E_i} H \right) \tag{5}$$

where D is the connection in the normal bundle.

3. Curves with proper mean curvature vector

The purpose of this section is to study arclength parameterised curves $\beta = \beta(s) : I \subset R \longrightarrow N^m(c)$ in a real space form $N^m(c)$ with constant sectional curvature c and dimension m, whose mean curvature vector satisfies

$$\Delta H = \lambda H. \tag{6}$$

The mean curvature vector field H of β is given by

$$H(s) = k_1(s)\xi_2(s). \tag{7}$$

If $k_1 = 0$, then β would be a geodesic. If $k_1 \neq 0$ and $k_2 = 0$, then β lies in a totally geodesic surface $N^2(c)$ of $N^m(c)$. Moreover, by using (3), (4) and (7) we see β satisfies (6) if and only if $k_1^2 = \lambda$. Thus β is a circle. Assume then that $k_1, k_2 \neq 0$, therefore we may suppose $m > 2$, so we can define a 2-dimensional subbundle, say ν, of the normal bundle Λ of β into $N^m(c)$ as follows:

$$\nu(s) = \text{Span}\{\xi_2, \xi_3\}(s) \tag{8}$$

where ξ_2 and ξ_3 are unit normal vector fields to β defined in (3) . Let us denote by ν' the orthogonal complementary subbundle of ν into Λ. Certainly the fibers of ν' have dimension $m - 3$. Therefore we have from (3)

$$\overline{\nabla}_T \xi_3(s) = -k_2(s)\xi_2(s) + \delta \tag{9}$$

where $\delta(s) \in \nu'(s)$ everywhere. Hence, we have from (3) and (4)

$$\Delta H = \left(\frac{3}{2}\left(k_1^2\right)'\right) T + \left(-k_1'' + k_1^3 + k_1 k_2^2\right) \xi_2 - \left(k_1' k_2 + (k_1 k_2)'\right) \xi_3 - k_1 k_2 \delta \tag{10}$$

so that β satisfies (6) if and only if

$$\begin{aligned}
\left(k_1^2\right)' &= 0 && (11)\\
-k_1'' + k_1^3 + k_1 k_2^2 &= \lambda k && (12)\\
k_1' k_2 + (k_1 k_2)' &= 0 && (13)\\
k_1 k_2 \delta &= 0. && (14)
\end{aligned}$$

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From (11), we see that k_1 is non-zero constant, then (12) gives that k_2 is also a non-zero constant. Therefore, (14) means that δ vanishes identically and so ν (and also ν') is a parallel subbundle in Λ. Furthermore ν' is formed by totally geodesic directions and consequently we can reduce codimension (see [**12**] for details) to get some $N^3(c)$, totally geodesic in $N^m(c)$, in which β lies. Hence we have

Lemma 1. *Let β be an unit speed curve fully immersed in a real space form $N^m(c)$. If β has proper mean curvature vector field, then $m \leqslant 3$.*

Thus, we conclude

Proposition 2. *Let $\beta : I \subset R \longrightarrow N^m(c)$ be a unit speed curve in a m-dimensional real space form of curvature c and denote by H its mean curvature vector. Then it satisfies $\Delta H = \lambda H$ if and only if β is a helix of a totally geodesic submanifold $N^2(c)$ or $N^3(c)$ of $N^m(c)$.*

Remark 1. From Takahashi's Theorem for $n = 1$ and using (3) and [**12**] as in the proof of the above Proposition, one can easily check that a unit speed curve $\beta : I \to E^m$ satisfies $\Delta\beta = \lambda\beta$ for some real number λ if and only if it is either a geodesic of E^2 (and then $\lambda = 0$), or a geodesic of a 2-dimensional sphere S^2 of E^3, that is a circle (and then $\lambda > 0$). (As we said before, condition $\Delta\beta = \lambda\beta$ implies $\Delta H = \lambda H$ for Euclidean submanifolds). Also we see that a unit curve in E^3 is a helix if and only if the position vector of its tangent spherical image is proper for the Laplacian. For curves in Euclidean space, Proposition 2 is basically due to Chen and Dimitric (see [**9**] for details).

Remark 2. Helices in E^3 have different characterisations. For instance, they can be seen as the path followed by the motion of an electron in a constant magnetic field. General helices were characterised in [**1**]. In fact, it was proved there, extending a theorem of Lancret first proved by Saint Venant in 1845, that a unit "non-plane" curve in a 3-dimensional simply connected space form $\beta : I \to N^3(c)$ is a general helix if and only if (1) it satisfies $\tau = bk$ when $c = 0$, or (2) it satisfies $\tau = bk \pm c$, when $c > 0$; or (3) it is a helix (constant curvatures) when $c < 0$. Thus in the 3-dimensional hyperbolic case, the only general helices are the helices. The above result characterises helices in $N^3(c)$ in terms of the spectral behaviour of its mean curvature vector. Observe that helices of E^3 are geodesics of 2-dimensional circular cylinders and none of them (but the circles) are closed. On the other hand, helices of the 3-dimensional sphere S^3 are also geodesics of 2-dimensional *hopf cylinders* shaped on circles of S^2 and there are many of them which are closed [**1**].

4. Curves with proper mean curvature vector in the normal bundle

As it was said in the introduction, one easily finds examples of non-minimal submanifolds of E^m with harmonic mean curvature vector in the normal bundle: it is enough to consider cylinders of E^3 constructed over Cornu spirals of E^2 or hypercylinders of E^m built on Cornu spirals of S^2, [**3**]. More examples are given in Remark 4.

Now we study curves in real space forms with proper mean curvature vector field in the normal bundle, that is satisfying.

$$\Delta^D H = \lambda H. \tag{15}$$

As before, we consider an arclength parameterised curve $\beta = \beta(s) : I \subset R \longrightarrow N^m(c)$ in a real space form $N^m(c)$ with constant sectional curvature c and dimension m, and use the same notation as in the previous sections. If $k_1 = 0$, then β is a geodesic. If $k_1 \neq 0$, and $k_2 = 0$, then β lies in a totally geodesic surface $N^2(c)$ of $N^m(c)$ and using (3), (5) and (7) we have $\Delta^D H = -k''\xi_2$. Then β satisfies (15) if and only if $-k_1'' = \lambda k_1$. Therefore, β is either a Cornu spiral in $N^2(c)$ if $\lambda = 0$, or

$$k_1 = c_1 \cos\left(\sqrt{\lambda} s\right) + c_2 \sin\left(\sqrt{\lambda} s\right) \tag{16}$$

if $\lambda < 0$, and

$$k_1 = c_1 \exp\left(\sqrt{-\lambda} s\right) + c_2 \exp\left(\sqrt{-\lambda} s\right) \tag{17}$$

if $\lambda > 0$, where $c_1, c_2 \in R$. Attending to the shape of curves in R^2 satisfying (16) and (17), we use the terms *curl curve* and *generalised Nielsen spiral* to denote curves in $N^2(c)$ satisfying (16) and (17) respectively. Assume then that $k_1, k_2 \neq 0$, therefore we may suppose $m > 2$, so we can define subbundles, say ν, ν', of the normal bundle Λ of β into $N^m(c)$ as in (8). From (3) and (7) we have that the normal connection D of β into $N^m(c)$ behaves on ν as follows :

$$D_T\xi_2 = -\tau\xi_3 \tag{18}$$

$$D_T\xi_3 = \tau\xi_2 + \delta \tag{19}$$

with $\delta \in \nu'$. Therefore combining (3), (5), (7), (18) and (19), we obtain

$$\Delta^D H = \left(-k'' + k\tau^2\right)\xi_2 + (2k'\tau + k\tau')\xi_3 + k\tau\delta. \tag{20}$$

Let us suppose that β has proper mean curvature vector field in Λ, that is $\Delta^D H = \lambda H$. Then

$$k'' - k\tau^2 = -\lambda k \tag{21}$$

$$\tau k^2 = a \tag{22}$$

$$\delta = 0 \tag{23}$$

where we have used $k_1 = k$, $k_2 = -\tau$ and a is some constant which we may assume to be non-zero ($m > 2$). One can proceed in just the same way as in the proof of Lemma 1 to obtain

LEMMA 3. *Let β be an unit speed curve fully immersed in a real space form $N^m(c)$. If β has proper mean curvature vector field in the normal bundle, then $m \leqslant 3$.*

Assume first that $\lambda \neq 0$. Then one can integrate the above equations to obtain

$$k^2 = \frac{b}{2\lambda} + c_1 \cos\left(2\sqrt{\lambda}s\right) + c_2 \sin\left(2\sqrt{\lambda}s\right) \tag{24}$$

if $\lambda > 0$, and

$$k^2 = \frac{b}{2\lambda} + c_1 \exp\left(2\sqrt{-\lambda}s\right) + c_2 \exp\left(2\sqrt{-\lambda}s\right) \tag{25}$$

if $\lambda < 0$, where $b, c_1, c_2 \in R$. Hence, from the fundamental theorem of curves, we have:

PROPOSITION 4. *Let $\beta : I \subset R \longrightarrow N^m(c)$ be a unit speed curve in a m-dimensional real space form of curvature c and denote by H its mean curvature vector. Assume it satisfies $\Delta^D H = \lambda H$, $\lambda \neq 0$. Then either*

1. $m = 2$ and it is a curl curve or a generalised Nielsen spiral in $N^2(c)$, or
2. $m = 3$ and its curvature and torsion are given either by (22) and (24), or by (22) and (25) respectively.

It is easy to check that curves given in 2. of Proposition 4 never lie in a totally umbilical surface of $N^3(c)$. This fact should be compared with Proposition 6. We now consider the harmonic case, that is $\lambda = 0$. The following Lemma 5 is obtained by using arguments similar to those in [**3**] for the Euclidean case. We repeat it here for a better understanding of the proof of Proposition 5. The case $\tau = 0$ has been treated at the beginning of this section. If $m = 3$, one can use standard arguments to integrate equations (21) and (22) with $\lambda = 0$ to obtain

$$k(s) = \sqrt{b(s-d)^2 + \frac{a^2}{b}} \tag{26}$$

$$\tau = \frac{ab}{b^2(s-d)^2 + a^2} \tag{27}$$

where $a \neq 0$, $b > 0$ and $d \in R$. The parameter d is not essential in the sense that it depends on the origin we use to measure the arclength function of β. The fundamental theorem of curves says that there exists a curve in $N^3(c)$ (unique up to isometries in $N^3(c)$) whose curvature and torsion functions are given by (26) and (27) respectively. Also the class of curves obtained in this way can be *a priori* parameterised into $R^+ \times (R - \{0\})$, according to the values of b and a respectively. But from (27), the sign of a is determined by the orientation of β. Therefore we have a family of curves in $N^3(c)$ parameterised into $R^+ \times R^+$. Thus if we take $e \in R^+ - \{0\}$ with $e^2 = b$ and denote by Ω^c_{ae} the set of unit speed curves in $N^3(c)$ whose curvature and torsion are given by (26) and (27) respectively, we have:

LEMMA 5. *Let β be an unit speed curve fully immersed in a real space form $N^m(c)$. Then β has harmonic mean curvature vector field in the normal bundle if and only if one of the two following cases occurs:*

(i) *β is either a Cornu spiral or a circle in a totally geodesic surface $N^2(c)$ of $N^m(c)$; or*

(ii) *β belongs to Ω^c_{ae} for some constants $a \in R^+, e \in R^+ - \{0\}$.*

Let us suppose that $N^3(c)$ is complete and simply connected, so it is R^3, $S^3 \subset R^4$ or $H^3(c) \subset R^4_1$ according to $c = 0$, $c > 0$ or $c < 0$ respectively. We shall denote by P the spaces R^3, R^4 or R^4_1 in order to unify the three cases. Now let us study the family Ω^c_{ae}. We use (26) and (27) to obtain

$$\frac{1}{k^2} + \frac{(k')^2}{\tau^2 k^4} = \frac{b}{a^2} = \frac{e^2}{a^2}. \tag{28}$$

We define the curve

$$\alpha(s) = \beta(s) + \frac{1}{k}\xi_2(s) - \frac{1}{\tau}\left(\frac{1}{k}\right)' \xi_3(s) \tag{29}$$

and denote by ∇ and σ the Levi-Civita connection of $N^3(c)$ and its second fundamental form in P. Notice that if $c = 0$, then $\sigma = 0$ because $N^3(0) = R^3 = \mathsf{P}$ and in the other two cases $N^3(c)$ is totally umbilical in P. In particular $\sigma(T, \xi_2) = \sigma(T, \xi_3) = 0$ and then

$$\alpha'(s) = \left[\left(\frac{k'}{\tau k^2}\right)' - \frac{\tau}{k}\right]\xi_3.$$

Consequently we use (28) to show that α' vanishes identically which proves that $\alpha(s)$ is some point, say $p_o \in \mathsf{P}$. Therefore

$$\beta(s) - p_o = -\frac{1}{k}\xi_2 - \frac{k'}{\tau k^2}\xi_3 \tag{30}$$

If $\langle , \rangle$ denotes the usual inner product in P, then

$$\langle \beta(s) - p_o, \beta(s) - p_o \rangle = \frac{b}{a^2} = \frac{e^2}{a^2} \tag{31}$$

which proves that β is contained in some totally umbilical surface, say S of $N^3(c)$. Moreover the geodesic curvature of β in S is

$$\rho(s) = r\frac{k'(s)}{\tau(s)k(s)} \tag{32}$$

with $r = a/e$ if $N^m(c) = E^3$ and r depending on e, a, c and $\|p_o\|$ if $N^m(c) \neq E^3$. Then from (26) and (27), we see that $\rho(s)$ is a linear function. Thus β is a Cornu spiral in S.

Conversely, if β is a Cornu spiral in a totally umbilical surface S of $N^3(c)$, then the geodesic curvature $\rho(s)$ of β in S is a linear function $\rho(s) = \mu(s) + \varepsilon$ and also $\rho(s)$ is given by (32) for some constant $r \in R$ satisfying $r^2 + \rho^2 = \kappa^2$. Using these facts one gets $\tau\kappa^2 = \mu r$ which is nothing but (22). By differentiating this formula and (32) and using $r^2 + \rho^2 = \kappa^2$ we obtain (21). Hence $\beta \in \Omega^c_{ae}$ for some constants $a, e \in R^+$. Therefore we have proved the following:

Proposition 6. *Let β be an unit speed curve immersed in a complete, simply connected real space form $N^m(c)$ of constant sectional curvature c and dimension m. Then β has harmonic mean curvature vector field in the normal bundle if and only if β is a Cornu spiral in some totally umbilical surface S (which can be totally geodesic and β could have constant curvature in such a case) of $N^m(c)$.*

Remark 3. From the above proof, we see that β is a Cornu spiral in a totally umbilical surface S of $N^3(c)$ if and only if $\beta \in \Omega^c_{ae}$ for some constants $a \in R^+, e \in R^+ - \{0\}$.

5. Hopf cylinders with proper mean curvature function

We begin this section by proving a result which will be useful later. Let $\Pi : P \longrightarrow M$ be a harmonic submersion between semi-Riemannian manifolds P and M, that is to say, Π is a semi-Riemannian submersion and the fibers $\Pi^{-1}(x)$, $x \in M$ are minimal submanifolds of P. Given a vector field X on M, there exists a unique horizontal lift $\overline{X}$ on P (that is, orthogonal to the fibers) and Π-related to X. For any regular curve $\gamma : I \subset R \longrightarrow M$ (we shall assume γ arc-length parameterised) we consider the submanifold $N = \Pi^{-1}(\gamma)$ on P. If $X = \gamma'$ is the unit tangent vector field of γ and $\overline{X}$ its horizontal lift, let us denote by V_p the tangent space to the fiber through p. Then for any point $p \in N$, $T_pN = \mathrm{Span}\{\overline{X}(p), V_p\}$, and the normal space, T'_pN, of N in P at p is a horizontal subspace. Denote by $\overline{\nabla}$ and ∇ the Levi-Civita connections of P and M respectively associated with their semi-Riemannian metrics $\langle , \rangle$ and $\ll , \gg$ respectively. Let us denote by σ and h the second fundamental forms of N and of the fibers in P respectively and, as usual, by $\mathcal{H}$ the horizontal orthogonal projection in P. Then for any $\overline{\xi} \in T^{\perp}_pN$ we have

$$\begin{aligned} \langle \sigma\big(\overline{X}, \overline{X}\big), \overline{\xi} \rangle &= \langle \overline{\nabla}_{\overline{X}} \overline{X}, \overline{\xi} \rangle = \langle \mathcal{H} \overline{\nabla}_{\overline{X}} \overline{X}, \overline{\xi} \rangle \\ &= \langle \overline{\nabla_X X}, \overline{\xi} \rangle = \ll \nabla_X X, \xi \gg = \lambda_1 \end{aligned}$$

where $d\Pi\big(\overline{\xi}\big) = \xi \in T_{\Pi(p)}M$. On the other hand, if the semi-Riemannian submersion is totally geodesic and of codimension one, then N is a surface which can be parameterised in such a way that the parametric curves are precisely the horizontal lifts of γ and the fibers (which are geodesics in P). Moreover

$$\langle \sigma(\eta, \eta), \overline{\xi} \rangle = 0$$

therefore the Weingarten map with respect to $\overline{\xi}$ can be written in the natural parameterisation for N as

$$\begin{pmatrix} \lambda_1 & \lambda \\ \lambda_1 & 0 \end{pmatrix}.$$

Thus, choosing an orthonormal frame $\{\xi_i\}_{i=2}^m$ of $T^{\perp}_{\pi(p)}(\gamma')$ in $T_{\pi(p)}M$ and denoting by $\overline{\xi_i}$ their horizontal lifts, we have that the mean curvature function of N in P satisfies

$$\alpha^2 = \frac{1}{4} \sum_{i=2}^{n} \left(tr A_{\overline{\xi}_i}\right)^2 = \frac{1}{4} \sum_{i=2}^{n} \ll \nabla_X X, \xi_i \gg^2 = \frac{1}{4} \left\| \nabla_X X \right\|^2 = \frac{1}{4} \rho^2, \tag{33}$$

ρ being the curvature function of γ in M and n the dimension of M. Hence we have proved:

PROPOSITION 7. *Let $\Pi : P \to M$ be a totally geodesic semi-Riemannian submersion of codimension one. For any curve γ in M we denote by N the surface of P defined by $N = \pi^{-1}(\gamma)$. Then the square of the mean curvature of N in P coincides, up to a constant, with the square of the curvature of γ in M.*

One can use the previous results to classify Hopf cylinders in the 3-dimensional sphere and in the 3-dimensional hyperbolic space with proper mean curvature function. As we said earlier, the harmonic case in the sphere was treated in [**3**], but in order to unify both the spherical and hyperbolic cases, we give here a proof based on Proposition 7. In order to do so, we need to describe the Hopf map as a Riemannian submersion.

Let us denote by $S^m(r)$ the m-dimensional sphere of radius r. The Hopf map $\pi : S^3(1) \to S^2(1/2)$ is considered as the restriction to the unit 3-sphere of the usual projection of $C^2 - \{0\}$ on CP^1. Then π is a Riemannian submersion (more precisely, a circle bundle). Also if we consider a unit speed curve $\gamma : I \to S^2(1/2)$ and pull it back via π, we obtain its total horizontal lift $M_\gamma \subset S^3(1)$ which is isometric to $[0, L] \times S^1$, L being the length of γ, [**14**]. M_γ is a flat surface which is called a Hopf cylinder over γ. Moreover, if γ is a closed curve of length L and enclosing oriented area A in $S^2(1/2)$, then M_γ is isometric to the flat torus R^2/Λ where Λ is the lattice spanned by $(2A, L)$ and $(2\pi, 0)$. Then, we have:

PROPOSITION 8. *A Hopf cylinder M_γ has proper mean curvature function if and only if γ has proper mean curvature vector for the Laplacian in the normal bundle. In particular, M_γ has harmonic mean curvature function if and only if either γ has constant curvature in $S^2(1/2)$ or γ is a Cornu spiral in $S^2(1/2)$. It has proper non-harmonic mean curvature function if and only if either γ is a curl curve or a generalised Nielsen spiral in $S^2(1/2)$.*

PROOF: Let us denote by ρ the geodesic curvature of γ in $S^2(1/2)$ and by α the mean curvature function of M_γ in $S^3(1)$. By using (33) and a suitable parameterisation of the flat surface M_γ we see that α is proper, that is $\Delta\alpha = \lambda\alpha$, Δ being the Laplacian of M_γ if and only if $\rho'' = \lambda\rho$. This means that the mean curvature vector of the curve is proper for Δ^D (see comments before (16)). This finishes the proof. ∎

The following corollary is a consequence of Remark 3 and it was proven in [**3**] for Hopf cylinders over curves γ of $S^2(1)$.

COROLLARY 9. *A Hopf cylinder M_γ has harmonic mean curvature function if and only if*

1. *γ is a (piece of) great circle in $S^2(1/2)$ and M_γ is a (piece of) the Clifford torus in $S^3(1)$; or*

2. *γ is a (piece of) a small circle in $S^2(1/2)$ and M_γ is a (piece of) the rectangular torus in $S^3(1)$; or*
3. *$\gamma \in \Omega^1_{ae}$ for suitable constants $a \in R^+, e \in R^+ - \{0\}$.*

REMARK 4. Take $\gamma \in \Omega^1_{ae}$ a curve lying in $S^2(1/2)$ and consider its associated Hopf cylinder M_γ in $S^3(1)$. One can check by direct computation that the cone of E^4 shaped on M_γ is a hypersurface having non-constant harmonic mean curvature and, therefore, it satisfies $\Delta^D H = 0$. This contrasts with the fact that every biharmonic hypersurface of E^4 is harmonic (minimal) [**13**].

Now we go to the hyperbolic case. Let R^4_2 be the 4-dimensional linear space R^4 endowed with the inner product of signature (2,2) given by $\langle x, y\rangle = -x_1y_1 - x_2y_2 + x_3y_3 + x_4y_4$ for $x, y \in R^4$. The space $H^3_1(-1)$ is the hypersurface of R^4_2 defined by $H^3_1(-1) = \{x \in R^4 : \langle x, x\rangle = -1\}$. Then $H^3_1(-1)$ with the restrictions of $\langle , \rangle$ is a Lorentzian manifold with constant sectional curvature -1 which is known as the 3-dimensional anti De Sitter space. We denote by C^2_1 the 2-dimensional complex linear space C^2 with the Hermitian form $(a, b) = -a_1\overline{b_1} + a_2\overline{b_2}$ with $a, b \in C^2$. Then $H^3_1(-1) = \{a \in C^2 : (a, a) = -1\}$. There is a natural action of S^1 on $H^3_1(-1)$ given by $(r(a_1, a_2)) = (ra_1, ra_2)$. Then the hyperbolic space $H^2(-1/4)$ with Gaussian curvature -4 is obtained as the orbit space. Thus we have a Hopf fibration $\pi : H^3_1(-1) \to H^2((-1)/4)$ with fibers S^1. Actually, π is a semi-Riemannian submersion.

Let β be a unit speed curve immersed in $H^2(-1/4)$. By pulling back β via π we obtain a total horizontal lift M_β of β which is an immersed flat surface in $H^3_1(-1)$ called the semi-Riemannian Hopf cylinder over β. M_β is a Lorentzian surface which can be described as a B-scroll of any horizontal lift $\overline{\beta}$ of β (see [**2**] for details).

By using Proposition 1, we can proceed as in the spherical case to prove:

PROPOSITION 10. *A semi-Riemannian Hopf cylinder M_β of $H^3_1(-1)$ has proper mean curvature function if and only if β has proper mean curvature vector for Δ^D. In particular if M_β has harmonic mean curvature function then β is a circle or a Cornu spiral in $H^2(-1/4)$ and it has proper non-harmonic mean curvature function when β is a curl curve or generalised Nielsen spiral in $H^2(-1/4)$*

Finally from Remark 3 and the description of curves of constant curvature in $H^2(-1/4)$, [**4**], we get:

COROLLARY 11. *A semi-Riemannian Hopf cylinder M_β of $H^3_1(-1)$ has harmonic mean curvature vector function if and only if*

1. *M_β is a minimal complex circle ($\rho = 0$); or*
2. *M_β is a non-minimal complex circle ($0 < \rho^2 < 4$); or*
3. *M_β is a Hopf cylinder over the horocycle ($\rho^2 = 4$); or*
4. *M_β is the semi-Riemannian product $H^1_1(-r^2) \times S^1(r^2 - 1)$ when ($\rho^2 > 4$); or*

5. *M_β is a B-scroll of any horizontal lift $\overline{\beta}$ of a curve $\beta \in \Omega_{ae}^{-1}$ for suitable constants $a \in R^+, e \in R^+ - \{0\}$.*

REMARK 5. Observe that the first four cases correspond with the constancy of the curvature ρ of β in $H^2(-1/4)$

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