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Radical Ideals in Valuation Domains

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Abstract. An ideal I of a ring R is called a radical ideal if $I = \mathcal{R}(R)$ where $\mathcal{R}$ is a radical in the sense of Kurosh–Amitsur. The main theorem of this paper asserts that if R is a valuation domain, then a proper ideal I of R is a radical ideal if and only if I is a distinguished ideal of R (the latter property means that if J and K are ideals of R such that $J \subset I \subset K$ then we cannot have $I/J \cong K/I$ as rings) and that such an ideal is necessarily prime. Examples are exhibited which show that, unlike prime ideals, distinguished ideals are not characterizable in terms of a property of the underlying value group of the valuation domain.

1 Introduction

Our purpose in this paper is to describe the radical ideals of a valuation domain R. These are those ideals I of R of the form $I = \mathcal{R}(R)$ where $\mathcal{R}$ is a radical in the sense of Kurosh-Amitsur.

The problem of describing the radical ideals of principal ideal domains and, more generally, Dedekind domains, is addressed in [9] and [8]. The latter paper also considers natural extensions such as matrix rings and semigroup and polynomial rings. The following theorem due to Propes [9, Theorem 5] and McConnell [8, Theorem 1.1] provides a characterization of the radical ideals in a Dedekind domain.

Theorem *The following statements are equivalent for a nonzero proper ideal I of a Dedekind domain R:*

(i) *I is a radical ideal of R,* i.e., *$I = \mathcal{R}(R)$ for some radical $\mathcal{R}$;*

(ii) *there exist distinct prime ideals $P_1, P_2, \dots, P_n$ of R such that*
   - (a) $I = P_1P_2 \dots P_n$,
   - (b) *given any prime ideal Q of R and any $i \in \{1, 2, \dots, n\}$, $R/Q \cong R/P_i$ as rings implies $P_1P_2 \dots P_n \subseteq Q$.*

If R is a local Dedekind domain, or equivalently, a valuation domain whose value group equals the additive group of integers $\mathbb{Z}$, then by the above theorem, R will have a unique nonzero proper radical ideal and this coincides with the unique nonzero prime ideal of R.

A logical sequel to the above investigation is the study of radical ideals in Prüfer domains, for these are the natural generalization of Dedekind domains. We shall limit our investigation to valuation domains, these being precisely the local Prüfer domains.

Below is stated the main result of this paper.

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Main Theorem *The following statements are equivalent for a proper ideal I of a valuation domain R:*

(i) *I is a radical ideal of R;*
(ii) *I is a distinguished ideal of R;*
(iii) *I is a prime ideal of R and is distinguished in the set of all prime ideals of R.*

The tools we shall use are varied, comprising a combination of radical theoretic methods, localization in commutative rings, and a small measure of rudimentary module theory.

2 Preliminaries

The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets.

2.1 Rings

All rings are associative but do not necessarily possess an identity element. By an *ideal* of a ring R we always mean a two-sided ideal. We shall write $I \trianglelefteq R$ [resp., $I \triangleleft R$] to indicate that I is an ideal [resp., proper ideal] of R. We use $\operatorname{Id} R$ to denote the set of all ideals of R.

2.2 Kurosh–Amitsur Radicals

A class $\mathcal{R}$ of rings is called a *radical class* (in the sense of Kurosh–Amitsur) if

(i) $\mathcal{R}$ is closed under homomorphic images;
(ii) every ring R contains a unique maximal ideal denoted $\mathcal{R}(R)$ which belongs to $\mathcal{R}$;
(iii) $\mathcal{R}(R/\mathcal{R}(R)) = 0$ for all rings R.

We refer the reader to [3, 6] for background information on radicals.

2.3 Linearly Ordered Abelian Groups

A *linearly ordered abelian group* is a structure $\langle \Gamma; +; \leq \rangle$ where $\langle \Gamma; + \rangle$ is an abelian group and $\langle \Gamma; \leq \rangle$ a linearly ordered poset (*i.e.,* a chain) satisfying:

$$g_1 \leq g_2 \text{ and } h_1 \leq h_2 \text{ imply } g_1 + h_1 \leq g_2 + h_2 \text{ whenever } g_1, g_2, h_1, h_2 \in \Gamma.$$

Let Γ be a linearly ordered abelian group (written additively). We call

$$\Gamma^+ = \{g \in \Gamma \mid g \geq 0_\Gamma\}$$

the *positive cone* of Γ. A subset C of Γ is said to be *convex* if $g, h \in C$ and $g \leq x \leq h$ implies $x \in C$, and *symmetric* if for all $g \in \Gamma$, we have $g \in C$ if and only if $-g \in C$. We shall denote by $\operatorname{Co}\Gamma$ the set of all convex symmetric subsets of Γ. Observe that the linear ordering on Γ makes $\operatorname{Co}\Gamma$ a chain with respect to the inclusion relation $\subseteq$. The empty set $\varnothing$ and Γ are the smallest and largest elements of $\operatorname{Co}\Gamma$, respectively.

If Δ is a convex subgroup of Γ, then the factor group Γ/Δ is a linearly ordered abelian group under the ordering inherited from Γ: if $g_1 + \Delta, g_2 + \Delta \in \Gamma/\Delta$, then $g_1 + \Delta < g_2 + \Delta$ if and only if $g_2 - g_1 \in \Gamma^+ \backslash \Delta$ (thus $g_1 + \Delta \leq g_2 + \Delta$ if and only if $g_2 - g_1 \in \Gamma^+ \cup \Delta$). We refer the reader to [4] for a more detailed exposition on linearly ordered abelian groups.

A linearly ordered abelian group Γ is called *Archimedean* if Γ contains no proper nonzero convex subgroups. A classical result due to Hölder (a proof may be found in [4]) asserts that Γ is Archimedean if and only if Γ is order isomorphic to a subgroup of the additive group of real numbers.

Let Λ be a well-ordered set and $\{\Gamma_i \mid i \in \Lambda\}$ a family of linearly ordered abelian groups indexed by Λ. Then $\prod_{i\in\Lambda} \Gamma_i$ is a linearly ordered abelian group under the *lexicographic ordering* which is defined as follows:

$\{g_i\}_{i\in\Lambda} < \{h_i\}_{i\in\Lambda}$ if and only if $g_i < h_i$, where i is the smallest element in Λ for which $g_i \neq h_i$.

The lexicographic ordering defined above on a cartesian product with well-ordered index set shall be adequate for our purposes. We point out, however, that a more general type of linearly ordered abelian group called the *Hahn product* arises if the index set is allowed to be linearly rather than well-ordered (see [5, p. 8]).

The following result shows that the convex subgroups in a lexicographic ordering are easily identified.

Proposition 2.1 *Let $\{\Gamma_i \mid i \in \Lambda\}$ be a family of nonzero linearly ordered abelian groups indexed by a well-ordered set Λ. The following statements are equivalent for a nonzero subgroup Δ of $\Gamma = \prod_{i\in\Lambda} \Gamma_i$:*

(i) *Δ is a convex subgroup;*

(ii) *there exists $\alpha \in \Lambda$ and a nonzero convex subgroup Δ_α of Γ_α such that $\Delta = 0 \times 0 \times \cdots \times \Delta_\alpha \times \Gamma_{\alpha+1} \times \cdots$.*

Proof (ii) $\Rightarrow$ (i) is easily shown to hold.

(i) $\Rightarrow$ (ii) For each $g = \{g_i\}_{i\in\Lambda} \in \Gamma$ define $\operatorname{supp} g = \{i \in \Lambda \mid g_i \neq 0\}$.

Put $X = \bigcup_{g\in\Delta} \operatorname{supp} g$. Note that $X \neq \varnothing$ because $\Delta \neq 0$. Let α be the smallest element in X. If $\{g_i\}_{i\in\Lambda} \in \Delta$, then $g_i = 0$ for all $i < \alpha$ whence $\{g_i\}_{i\in\Lambda} \in 0 \times 0 \times \cdots \times \Gamma_\alpha \times \Gamma_{\alpha+1} \times \cdots$. Thus $\Delta \subseteq 0 \times 0 \times \cdots \times \Gamma_\alpha \times \Gamma_{\alpha+1} \times \cdots$.

Choose $h = \{g_i\}_{i\in\Lambda} \in \Delta$ such that $g_\alpha > 0$. Since $-h < g < h$ for all $g \in 0 \times 0 \times \cdots \times 0 \times \Gamma_{\alpha+1} \times \cdots$ and Δ is convex, we must have $0 \times 0 \times \cdots \times 0 \times \Gamma_{\alpha+1} \times \cdots \subseteq \Delta$. It is easily shown that if Δ_α denotes the projection of Δ onto Γ_α, then Δ_α is a convex subgroup of Γ_α. We thus obtain $\Delta = 0 \times 0 \times \cdots \times \Delta_\alpha \times \Gamma_{\alpha+1} \times \cdots$, as required. ■

2.4 Valuation Domains

Let Γ be a linearly ordered abelian group. Adjoin to Γ a symbol ∞ to be regarded as larger than every element of Γ and set $g + \infty = \infty + g = \infty$ for all $g \in \Gamma$. Let F be a field. A *valuation on F* is a map $v\colon F \to \Gamma \cup \{\infty\}$ such that for all $a, b \in F$:

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(V1) $v(a) = \infty$ if and only if $a = 0$;
(V2) $v(ab) = v(a) + v(b)$;
(V3) $v(a + b) \geq \min\{v(a), v(b)\}$.

There is no loss of generality in assuming that v is onto since the image of $F\backslash\{0\}$ is necessarily a subgroup of Γ. Henceforth when we speak of a valuation map $v\colon F \to \Gamma \cup \{\infty\}$, F shall be understood to be a field and Γ a linearly ordered abelian group with adjoined symbol ∞.

We call the subring $R = \{q \in F \mid v(q) \geq 0\}$ of F the *valuation domain associated with* v. The linearly ordered abelian group Γ is referred to as the *value group* of R. Observe that F is the field of quotients of the subring R. If, on the other hand, R is an arbitrary commutative domain with field of quotients F, and R admits a map $v\colon R \to \Gamma^+ \cup \{\infty\}$ which satisfies properties (V1)–(V3) above, then v extends uniquely to a valuation map from F to $\Gamma \cup \{\infty\}$ (if $\frac{a}{b} \in F$ define $v(\frac{a}{b}) = v(a) - v(b)$). Moreover, in this situation, if $aR \subseteq bR$ whenever $v(a) \geq v(b)$, then R coincides with the valuation domain associated with $v\colon F \to \Gamma \cup \{\infty\}$. Consequently, in defining a valuation map v on a field F, it suffices to describe the action of v on any subring R of F for which F is the field of quotients of R.

We refer the reader to [5, 11, 12] as sources of information on valuation domains.

Remark 2.2 A classical theorem due to Krull (see [5, Theorem 3.4, p. 12] for a proof) shows that given an arbitrary linearly ordered abelian group Γ, it is possible to construct a field F and a valuation map $v\colon F \to \Gamma \cup \{\infty\}$. Thus every linearly ordered abelian group is the value group of some valuation domain R.

As the following classical result shows, the valuation map v establishes a correspondence between the convex symmetric subsets of the linearly ordered abelian group Γ, and the ideals of the associated valuation domain R. A proof may be found in [12, Theorem 15, p. 40].

Theorem 2.3 *Let $v\colon F \to \Gamma \cup \{\infty\}$ be a valuation map and R the associated valuation domain. The map from* $\operatorname{Id} R$ *to* $\operatorname{Co} \Gamma$ *defined by*

$$I \mapsto \Gamma_I \stackrel{\text{def}}{=} \bigl\{ g \in \Gamma \bigm| -v(r) < g < v(r)\, \forall r \in I\backslash\{0\} \bigr\} \quad (I \in \operatorname{Id} R)$$

and the map from $\operatorname{Co} \Gamma$ *to* $\operatorname{Id} R$ *defined by*

$$C \mapsto R_C \stackrel{\text{def}}{=} \{ r \in R \mid v(r) > c\, \forall c \in C \} \quad (C \in \operatorname{Co} \Gamma),$$

are mutually inverse order reversing bijections. Moreover, the aforementioned maps restrict to bijections between the sets of prime ideals of R and convex subgroups of Γ.

Remark 2.4

(1) Inasmuch as $\operatorname{Co} \Gamma$ is linearly ordered, it follows from the above theorem that $\operatorname{Id} R$ is linearly ordered. Thus every valuation domain is a commutative chain domain (this is a commutative domain whose ideals are linearly ordered by inclusion). Conversely, if R is an arbitrary commutative chain domain with field of quotients F, then

there exists a linearly ordered abelian group Γ and a valuation map $v\colon F \to \Gamma \cup \{\infty\}$ such that R is the valuation domain associated with v. (The positive cone of this linearly ordered abelian group Γ corresponds with the set of all principal ideals of R under the binary operation of ideal multiplication and the inclusion relation – for more details see [5, p. 11].) For this reason the valuation domains are precisely the commutative chain domains.

(2) It follows from the above theorem that R will have a unique nonzero prime ideal (and this must be the unique maximal proper ideal of R) if and only if the value group Γ of R contains no proper nonzero convex subgroups, that is to say, Γ is Archimedean.

Example 2.5 This example describes one of the classical prototypes of valuation domain.

Let p be any positive prime integer and take Γ to be the additive group of integers $\mathbb{Z}$. The *p-adic valuation* $v_p\colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ is defined as follows: take $0 \neq r \in \mathbb{Q}$ and write $r = p^k \frac{m}{n}$ where $k, m, n \in \mathbb{Z}$, $n > 0$ and $(p, m) = 1 = (p, n)$. Then

$$v_p(r) \stackrel{\text{def}}{=} k.$$

The valuation domain associated with v_p is

$$\mathbb{Z}_{(p)} = \left\{ \frac{m}{n} \in \mathbb{Q} \mid m, n \in \mathbb{Z}, (n, p) = 1 \right\}.$$

The ring $\mathbb{Z}_{(p)}$ has a unique nonzero prime ideal

$$P = p\mathbb{Z}_{(p)} = \{r \in \mathbb{Q} \mid v_p(r) \geq 1\},$$

and every proper nonzero ideal of $\mathbb{Z}_{(p)}$ is of the form P^n for some $n \in \mathbb{N}$. Observe that the value group of $\mathbb{Z}_{(p)}$ is $\mathbb{Z}$. Valuation domains R with this property are Dedekind domains, so by the theorem of Propes and McConnell (stated in the introduction), the unique maximal proper ideal P of R is the unique proper nonzero radical ideal of R. Note that P corresponds with the Jacobson Radical of R.

A proof of the following standard result may be found in [11, Ch. C, Proposition 2, p. 60] and [12, p. 43].

Proposition 2.6 *Let $v\colon F \to \Gamma \cup \{\infty\}$ be a valuation map and R the associated valuation domain. Let P be a prime ideal of R so that $P = R_\Delta$ for some convex subgroup Δ of Γ. Then the map $\overline{v}\colon R/P \to \Delta^+ \cup \{\infty\}$ defined by*

$$\overline{v}(r + P) = \begin{cases} \infty & \text{if } r \in P, \\ v(r) & \text{if } r \notin P, \end{cases}$$

is onto and satisfies (V1)–(V3). *Hence R/P is a valuation domain with value group Δ.*

3 The Main Theorem

Following Puczyłowski [10] we call an ideal I of a ring R *distinguished* [resp., *distinguished in the set of prime ideals of* R] if R contains no ideals [resp. prime ideals] J and K such that $J \subset I \subset K$ and $I/J \cong K/I$.

Certainly, every radical ideal of an arbitrary ring is distinguished and this is implication (i) $\Rightarrow$(ii) in the Main Theorem.

In [10, Proposition 3] Puczyłowski proves that in general a distinguished ideal need not be a radical ideal but that the two notions do coincide in a ring all of whose ideals are idempotent. (The latter is deduced from [10, Proposition 2].) The equivalence (i) $\Leftrightarrow$ (ii) in the Main Theorem tells us that the notions of radical ideal and distinguished ideal also coincide in any valuation domain.

Our next objective is to prove the implication (ii) $\Rightarrow$ (iii) in the Main Theorem. It is obvious from the definition that if I is a distinguished ideal of a ring R, then I will be distinguished in the set of prime ideals of R. Thus, to establish the implication (ii) $\Rightarrow$ (iii), it suffices to prove that every proper distinguished ideal of a valuation domain is prime. This is achieved in Proposition 3.4. We shall require a number of preparatory results.

If I is a nonzero ideal of a valuation domain R, we define

$$I^{\#} = \{r \in R \mid rI \subset I\}.$$

A proof of the following simple lemma may be found in [5, Lemma 4.5, p. 15].

Lemma 3.1 *Let I be a nonzero ideal of a valuation domain R. Then*

(i) *$I^{\#}$ is a prime ideal containing I;*
(ii) *if J is any ideal of R such that $I \cong J$ as right R-modules, then $I^{\#} = J^{\#}$;*
(iii) *$I^{\#} = I$ if I is prime.*

Remark 3.2 If $v \colon F \to \Gamma \cup \{\infty\}$ is a valuation map with associated valuation domain R and I is a nonzero ideal of R, then it can be shown that $I^{\#} = R_{\Delta}$ where Δ is the largest convex subgroup of Γ such that $\Delta + v[I] = v[I]$.

We call a ring A (necessarily without an identity element) *square-zero* if $A^2 = 0$. Observe that the ideal structure of such a ring is determined entirely by its underlying abelian group structure.

Lemma 3.3 *Let I be a proper distinguished ideal of a valuation domain R. If $b \in R$ and $b^2 I \supseteq I^2$, then $bI = I$.*

Proof Suppose $b \in R$ and $b^2 I \supseteq I^2$. Note first that we cannot have $b \in I$ for then $b^{-1}I \supseteq R$ and so $(b^{-1}I)^2 = b^{-2}I^2 \supseteq R$. The hypothesis of the lemma yields $I \supseteq b^{-2}I^2$, whence $I \supseteq R$, a contradiction. Since $b \notin I$, $bR \supseteq I$, *i.e.*, $b^{-1}I \subseteq R$. We thus have $bI, I, b^{-1}I$ ideals of R with $bI \subseteq I \subseteq b^{-1}I$. Since $b^2 I \supseteq I^2$, $I \supseteq b^{-2}I^2 = (b^{-1}I)^2$, so the factor ring $b^{-1}I/I$ is square-zero. Inasmuch as $bI \supseteq I^2$, the factor ring I/bI is also square-zero.

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Consider the map $f_b : b^{-1}I/I \to I/bI$ defined by

$$b^{-1}a + I \overset{f_b}{\longmapsto} a + bI, \quad a \in I.$$

It is easily shown that f_b is an isomorphism of right R-modules. Since $b^{-1}I/I$ and I/bI are square-zero rings, f_b is a ring isomorphism. But I is a distinguished ideal of R, so we must have $b^{-1}I/I \cong I/bI = 0$, *i.e.*, $bI = I$. ■

Proposition 3.4 *Every proper distinguished ideal of a valuation domain is prime.*

Proof Let I be a proper distinguished ideal of a valuation domain R. Observe that $I^{\#}$ is a prime ideal of R containing I by Lemma 3.1(i). We shall demonstrate that $(I^{\#})^2 \subseteq I$. Suppose $b \in I^{\#}$ and $b^2 \notin I$. Then $b^2R \supset I$, whence $b^2I \supseteq I^2$. This implies, by the previous lemma that $bI = I$, *i.e.*, $b \notin I^{\#}$, a contradiction. This shows that $(I^{\#})^2 \subseteq I$. Suppose, contrary to the statement of the proposition, that I is not prime so that $I \subset I^{\#}$ and pick $b \in I^{\#} \backslash I$. It is easily checked that left multiplication by b constitutes a right R-module isomorphism from $b^{-1}I$ onto I and so by Lemma 3.1(ii), we must have $(b^{-1}I)^{\#} = I^{\#}$. Hence $b^{-1}I \subseteq (b^{-1}I)^{\#} = I^{\#}$ and so $(b^{-1}I)^2 = b^{-2}I^2 \subseteq (I^{\#})^2 \subseteq I$, *i.e.*, $I^2 \subseteq b^2I$. By the previous lemma, $bI = I$, *i.e.*, $b \notin I^{\#}$, a contradiction. We conclude that $I = I^{\#}$ and so I is a prime ideal of R. ■

Remark 3.5 It is shown in Example 4.1 that the converse of Proposition 3.4 is not true.

It remains to prove the implication (iii) ⇒ (i) in the Main Theorem. We again require several preparatory results. The proof of the following result owes much to [2, Proposition 3.3] for inspiration.

Proposition 3.6 *Let A be an ideal of a ring B (necessarily without identity) such that B/A is a nil ring. Let R be any valuation domain and let $f\colon A \to R$ be a ring homomorphism. Then there exists a unique ring homomorphism $\overline{f}\colon B \to R$ which extends f.*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[dl, "\overline{f}"] \\ R & \end{tikzcd}$$

Proof Let F be the field of quotients of R. If $f = 0$, there is nothing to prove. Suppose $f \neq 0$ and take $a \in A$ such that $f(a) \neq 0$. Define a map $\overline{f}\colon B \to F$ by $\overline{f}(r) = \frac{f(ar)}{f(a)} \in F$ for all $r \in B$. The map $\overline{f}$ is clearly additive. Moreover, if $r, s \in B$, then

$$\overline{f}(rs) = \frac{f(ars)}{f(a)} = \frac{f(ars)f(a)}{f(a)f(a)} = \frac{f(ar)f(sa)}{f(a)f(a)}.$$

We claim that $f(sa) = f(as)$. Indeed, $f(a)f(sa) = f(as)f(a) = f(a)f(as)$, and so $f(a)[f(as) - f(sa)] = 0$, whence $f(as) - f(sa) = 0$, *i.e.*, $f(sa) = f(as)$, as

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claimed. It follows that $\overline{f}(rs) = \frac{f(ar)f(as)}{f(a)f(a)} = \overline{f}(r)\overline{f}(s)$. We conclude that $\overline{f}$ is a ring homomorphism. It is clear that $\overline{f}$ extends f for if $b \in A$ then $\overline{f}(b) = \frac{f(ab)}{f(a)} = \frac{f(a)f(b)}{f(a)} = f(b)$.

We now show that $\overline{f}[B] \subseteq R$. Take $b \in B$ and suppose $\overline{f}(b) \notin R$. Let $v\colon F \to \Gamma \cup \{\infty\}$ be the valuation map which gives rise to R. Since $\overline{f}(b) \notin R$, we must have $v(\overline{f}(b)) < 0$. Since B/A is nil, $b^n \in A$ for some $n \in \mathbb{N}$. Then $\overline{f}(b^n) = \overline{f}(b)^n \in \overline{f}[A] = f[A] \subseteq R$, so $v(\overline{f}(b)^n) = nv(\overline{f}(b)) \geq 0$. But this contradicts the fact that $v(\overline{f}(b)) < 0$. We conclude that $\overline{f}[B] \subseteq R$, as required.

It remains to establish uniqueness. Suppose $g\colon B \to R$ also extends f. If $r \in B$ then $g(r) = \frac{g(r)g(a)}{g(a)} = \frac{g(ra)}{g(a)} = \frac{f(ra)}{f(a)} = \overline{f}(r)$ (because g extends f and $a, ra \in A$). We conclude that $g = \overline{f}$. ■

If I is a proper ideal of an arbitrary ring R, we define the ideal $\sqrt{I}$ by

$$\sqrt{I}/I = \bigcap\{P \in \operatorname{Id} R \mid P \text{ is prime and } P \supseteq I\}.$$

Observe that $\sqrt{I}/I$ is precisely the prime radical of the factor ring R/I and that $\sqrt{I}$, being an intersection of prime ideals of R, is a semiprime ideal of R. If the ring R is commutative, then $\sqrt{I}/I$ coincides with the set of all nilpotent elements of the ring R/I [7, Theorem VIII.2.6, p. 379]. If R is a valuation domain, then $\sqrt{I}/I$ is always a prime ideal of R, for every semiprime ideal of a valuation domain is prime.

Andrunakievič's Lemma [1, Lemma 4] asserts that if R is an arbitrary ring, $K \trianglelefteq A \trianglelefteq R$ and $\langle K \rangle$ denotes the ideal of R generated by K, then $\langle K \rangle^3 \subseteq K$. We shall need to generalize this lemma.

Recall that a subring K of a ring R is called an *accessible subring* if there exists a chain

$$K = A_0 \trianglelefteq A_1 \trianglelefteq A_2 \trianglelefteq \cdots \trianglelefteq A_n = R.$$

We associate with the above chain, a chain of subrings of R,

$$K^{[1]} \subseteq K^{[2]} \subseteq \cdots \subseteq K^{[n]}$$

where, for each $i \in \{1, 2, \dots, n\}$, $K^{[i]}$ is the ideal of A_i generated by K. Observe that $K^{[1]} = K$ and $K^{[i]} \subseteq K^{[i+1]}$ because $A_i \subseteq A_{i+1}$ for each $i \in \{1, 2, \dots, n-1\}$. Since $K \subseteq A_{i-1} \trianglelefteq A_i$ and $K^{[i]}$ is the smallest ideal of A_i containing K, we must have $K^{[i]} \subseteq A_{i-1}$ for each $i \in \{1, 2, \dots, n\}$. Inasmuch as $K^{[i]} \subseteq K^{[i+1]} \subseteq A_i$ and $K^{[i]} \trianglelefteq A_i$, we must have that $K^{[i]} \trianglelefteq K^{[i+1]}$ for each $i \in \{1, 2, \dots, n-1\}$. It is also clear from the definition of the subrings $K^{[i]}$ that $K^{[i+1]}$ is the smallest ideal of A_{i+1} containing $K^{[i]}$ for each $i \in \{1, 2, \dots, n-1\}$. Since $K^{[i]} \trianglelefteq A_i \trianglelefteq A_{i+1}$ it follows from Andrunakievič's Lemma that $(K^{[i+1]})^3 \subseteq K^{[i]}$ for all $i \in \{1, 2, \dots, n-1\}$. We have thus proved the following extension of Andrunakievič's Lemma.

Proposition 3.7 *Let R be an arbitrary ring. If K is an accessible subring of R and $\langle K \rangle$ denotes the ideal of R generated by K, then there exists a chain of subrings*

$$K = B_1 \trianglelefteq B_2 \trianglelefteq \cdots \trianglelefteq B_n = \langle K \rangle$$

of R such that $B_{i+1}^3 \subseteq B_i$ for all $i \in \{1, 2, \dots, n-1\}$.

Corollary 3.8 *Let A be an arbitrary ring and K an accessible subring of A. Let $\langle K\rangle$ denote the ideal of A generated by K and let I be an ideal of A such that $I \supseteq K$ and $I/\langle K\rangle$ is nil. If R is any valuation domain and $f\colon K \to R$ a ring homomorphism, then f extends uniquely to a ring homomorphism $\overline{f}\colon I \to R$.*

Proof By the previous proposition there exists a chain of subrings,

$$K = B_1 \trianglelefteq B_2 \trianglelefteq \cdots \trianglelefteq B_n = \langle K\rangle \trianglelefteq I$$

of A such that B_{i+1}/B_i is a nil ring for all $i \in \{1, 2, \dots, n-1\}$. The map f extends uniquely to a ring homomorphism $\overline{f}\colon I \to R$ by a repeated application of Proposition 3.6. ■

Corollary 3.9 *Let R and T be valuation domains with Q a prime ideal of R and K an accessible subring of T. If $Q \cong K$ as rings, then K is a prime ideal of T.*

Proof Let $f\colon K \to Q$ be a ring isomorphism. Let $\langle K\rangle$ denote the ideal of T generated by K. Since T is a commutative ring, $\sqrt{\langle K\rangle}/\langle K\rangle$ is nil. By Corollary 3.8, f extends uniquely to a ring homomorphism $\overline{f}\colon \sqrt{\langle K\rangle} \to R$. Inasmuch as $\sqrt{\langle K\rangle}$ is nil over $\langle K\rangle$ and $\langle K\rangle$ nil over K (by Proposition 3.7), we must have $\sqrt{\langle K\rangle}$ nil over K. It follows that $\overline{f}[\sqrt{\langle K\rangle}]$ is nil over $\overline{f}[K] = f[K] = Q$. But Q is a prime ideal of R, so this entails $\overline{f}[\sqrt{\langle K\rangle}] = Q$.

Consider the ring homomorphism $f^{-1}\circ\overline{f}\colon \sqrt{\langle K\rangle} \to \sqrt{\langle K\rangle}$. Observe that $f^{-1}\circ\overline{f}$ extends the identity map on K. Inasmuch as the identity map on $\sqrt{\langle K\rangle}$ also extends the identity map on K, it follows from the uniqueness of $f^{-1}\circ\overline{f}$ (established in Proposition 3.6) that $f^{-1}\circ\overline{f}$ coincides with the identity map on $\sqrt{\langle K\rangle}$. Hence $K = \sqrt{\langle K\rangle}$. Since T is a valuation domain, $K = \sqrt{\langle K\rangle}$ is prime. ■

We require one further preparatory result. The following proposition is due to Puczyłowski [10, Proposition 1].

Proposition 3.10 *The following statements are equivalent for an ideal I of an arbitrary ring R.*

(i) *I is a radical ideal of R;*
(ii) *if $J \trianglelefteq I \trianglelefteq K$, K is an accessible subring of R and $I/J \cong K/I$ as rings, then $J = I = K$.*

Recall that a nonempty class $\mathcal{M}$ of prime rings is said to be a *special class* if (i) $\mathcal{M}$ is hereditary (meaning, if $R \in \mathcal{M}$ and $I \trianglelefteq R$, then $I \in \mathcal{M}$), and (ii) whenever J is a large ideal of a ring R (meaning, $J \cap I \neq 0$ whenever $0 \neq I \trianglelefteq R$) and $J \in \mathcal{M}$, we have $R \in \mathcal{M}$. In the proof below we make use of the fact that the class of all prime rings is special. (A proof may be found in [3, pp. 140–155].)

We are finally in a position to prove implication (iii) ⇒ (i) in the Main Theorem.

Proof of Main Theorem (iii) $\Rightarrow$ (i). Suppose R is a valuation domain and I an ideal of R which satisfies (iii). We shall use the previous proposition to prove that I is a radical ideal. Suppose $J \trianglelefteq I \trianglelefteq K$ where K is an accessible subring of R and $I/J \cong K/I$ as rings. We need to show that $J = I = K$.

Since I is, by hypothesis, a prime ideal of R, R/I is a prime ring. Since the class of all prime rings is hereditary and K/I is an accessible subring of R/I, we must have that K/I and hence I/J, is a prime ring. Let $\langle J\rangle$ denote the ideal of R generated by J. Note that $J \subseteq \langle J\rangle \subseteq I$. By Andrunakievič's Lemma, $\langle J\rangle^3 \subseteq J$. But I/J is a prime ring and so we must have $\langle J\rangle = J$. Thus $J \trianglelefteq R$.

Since the ideals of the ring R/J are linearly ordered, every nonzero ideal of R/J is large. In particular, I/J is large. Since I/J is a prime ring and, as noted above, the class of all prime rings is special, we must have that R/J is a prime ring. We conclude that J is a prime ideal of R.

Observe that I/J is a prime ideal of the valuation domain R/J and I/J is isomorphic, as a ring, to the accessible subring K/I of the valuation domain R/I. It follows from Corollary 3.9 that K/I is a prime ideal of R/I and so K is a prime ideal of R. We thus have that J, I, K are prime ideals of R with $I/J \cong K/I$. By hypothesis, I is distinguished in the set of all prime ideals of R, so we must have $J = I = K$, as required. ■

4 Distinguished Ideals

In the light of the Main Theorem, the problem of determining the radical ideals of a valuation domain R reduces to locating the distinguished ideals of R. This task is by no means straightforward. Every proper distinguished ideal of a valuation domain is prime. But, whereas prime ideals are characterizable in terms of a property of the value group of the valuation domain, the same is not true of distinguished ideals. This fact is made evident in Example 4.4 and Proposition 4.6. The former exhibits a valuation domain R with value group $\mathbb{Z} \times \mathbb{Z}$ and prime ideals $0 \subset Q \subset P$ such that $Q \cong P/Q$, which implies that Q is not distinguished. Proposition 4.6, on the other hand, establishes the existence of a valuation domain R with the same value group $\mathbb{Z} \times \mathbb{Z}$ and prime ideals $0 \subset Q \subset P$ such that $Q \ncong P/Q$, and this implies that Q is distinguished.

Our first task shall be to construct a valuation domain which contains a prime ideal which is not distinguished thus providing a counterexample to the converse of Proposition 3.4.

Let F be an arbitrary field and Γ a linearly ordered (additively written) abelian group. Let $F[\Gamma]$ denote the group ring over F, *i.e.*,

$$F[\Gamma] = \left\{ a_0x^{g_0} + a_1x^{g_1} + \cdots + a_nx^{g_n} \mid x \text{ is an indeterminate}, \right.$$
$$\left. a_0, a_1, \ldots, a_n \in F \text{ and } g_0 < g_1 < \cdots < g_n \in \Gamma \right\}.$$

We define $F[\Gamma^+]$ to be the subring of $F[\Gamma]$ consisting of all those elements $a_0x^{g_0} + a_1x^{g_1} + \cdots + a_nx^{g_n} \in F[\Gamma]$ for which all the g_i belong to Γ^+. Observe that every nonzero element in the field of quotients $F(\Gamma)$ of $F[\Gamma]$, is uniquely expressible in the

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form

$$x^g\left(\frac{a_0 + a_1x^{g_1} + \cdots + a_nx^{g_n}}{b_0 + b_1x^{h_1} + \cdots + b_mx^{h_m}}\right), \tag{1}$$

where $0 \neq a_0, a_1, \ldots, a_n \in F$, $0 \neq b_0, b_1, \ldots, b_m \in F$, $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m \in \Gamma^+$ and $g \in \Gamma$. It follows that $F[\Gamma^+]$ and $F[\Gamma]$ both have $F(\Gamma)$ as their field of quotients.

We define a valuation map $v\colon F(\Gamma) \to \Gamma \cup \{\infty\}$, which we shall henceforth call the *canonical valuation*, as follows:

$$\text{if } r = x^g\left(\frac{a_0 + a_1x^{g_1} + \cdots + a_nx^{g_n}}{b_0 + b_1x^{h_1} + \cdots + b_mx^{h_m}}\right), \text{ then } v(r) = g.$$

It is easily checked that v is indeed a valuation map on $F(\Gamma)$. Note that the valuation domain associated with v is the collection of all elements of the form (1) with $g \in \Gamma^+$.

Let Δ be a convex subgroup of Γ. Observe that v restricts to a valuation from the subfield $F(\Delta)$ of $F(\Gamma)$ onto Δ. For notational convenience we shall identify v with its restriction to $F(\Delta)$. Let R and R' denote the valuation domains associated with v in $F(\Gamma)$ and $F(\Delta)$, respectively. Note that $F[\Gamma^+] \subseteq R \subseteq F(\Gamma)$ and $F[\Delta^+] \subseteq R' \subseteq F(\Delta)$. Let $f\colon F[\Gamma^+] \to F[\Delta^+]$ be the canonical ring epimorphism induced by the mapping which sends all terms of the form x^g with $g \in \Gamma^+ \backslash \Delta$ to zero. The epimorphism f extends naturally to an epimorphism

$$\hat{f}\colon R \to R' \tag{2}$$

with

$$\ker \hat{f} = \{r \in R \mid v(r) \in \Gamma^+ \cup \{\infty\} \backslash \Delta\} = R_\Delta. \tag{3}$$

Furthermore, note that if C is any convex symmetric subset of Γ contained in Δ, then

$$\hat{f}[R_C] = R'_C. \tag{4}$$

Example 4.1 Let $\Gamma = \prod_\omega \mathbb{Z}$ (ordered lexicographically) with convex subgroup $\Delta = 0 \times \mathbb{Z} \times \mathbb{Z} \times \cdots$. The "shift" map $\sigma\colon \Delta \to \Gamma$ defined by $(0, n_1, n_2, \ldots) \overset{\sigma}{\mapsto} (n_1, n_2, \ldots)$ is an isomorphism of linearly ordered abelian groups. Let F be a field, $v\colon F(\Gamma) \to \Gamma \cup \{\infty\}$ the canonical valuation and let R and R' denote the valuation domains associated with v in $F(\Gamma)$ and $F(\Delta)$, respectively. The isomorphism σ induces a ring isomorphism $\hat{\sigma}\colon F(\Delta) \to F(\Gamma)$ defined by

$$x^g\left(\frac{a_0 + a_1x^{g_1} + \cdots + a_nx^{g_n}}{b_0 + b_1x^{h_1} + \cdots + b_mx^{h_m}}\right) \overset{\hat{\sigma}}{\longmapsto} x^{\sigma(g)}\left(\frac{a_0 + a_1x^{\sigma(g_1)} + \cdots + a_nx^{\sigma(g_n)}}{b_0 + b_1x^{\sigma(h_1)} + \cdots + b_mx^{\sigma(h_m)}}\right) \tag{5}$$

where $g_1, g_2, \ldots, g_n, h_1, h_2, \ldots, h_m \in \Delta^+$ and $g \in \Delta$.

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If C is any convex symmetric subset of Δ, then

$$\begin{aligned} 0 \neq r \in R'_C &\Leftrightarrow \nu(r) \in \Delta^+ \backslash C; \\ &\Leftrightarrow \sigma(\nu(r)) \in \sigma[\Delta^+] \backslash \sigma[C] = \Gamma^+ \backslash \sigma[C]; \\ &\Leftrightarrow \nu(\hat{\sigma}(r)) \in \Gamma^+ \backslash \sigma[C] \quad [\text{because } \sigma(\nu(r)) = \nu(\hat{\sigma}(r))]; \\ &\Leftrightarrow 0 \neq \hat{\sigma}(r) \in R_{\sigma[C]}. \end{aligned}$$

Therefore

$$\hat{\sigma}[R'_C] = R_{\sigma[C]}. \tag{6}$$

Taking $C = \varnothing$ in (6), we get that the restriction of $\hat{\sigma}$ to R' (and for convenience we shall identify this restriction map with $\hat{\sigma}$) yields a ring isomorphism

$$\hat{\sigma} \colon R' \to R. \tag{7}$$

Composing the maps in (2) and (7), we obtain the ring epimorphism

$$\hat{\sigma}\hat{f} \colon R \to R \tag{8}$$

with

$$\ker \hat{\sigma}\hat{f} = R_\Delta \tag{9}$$

by (3).

Now let Δ' be the convex subgroup of Γ defined by $\Delta' = 0 \times 0 \times \mathbb{Z} \times \mathbb{Z} \times \cdots \subset \Delta$. By (4) and (6), $\hat{\sigma}\hat{f}[R_{\Delta'}] = \hat{\sigma}[R'_{\Delta'}] = R_{\sigma[\Delta']}$. Observe, moreover, that $\sigma[\Delta'] = \Delta$, so

$$\hat{\sigma}\hat{f}[R_{\Delta'}] = R_\Delta. \tag{10}$$

Put $Q = R_\Delta$ and $P = R_{\Delta'}$. Inasmuch as Δ and Δ' are convex subgroups, Q and P are prime ideals of R. By (8) and (9) there exists an isomorphism $R/Q \cong R$ which restricts by (10) to an isomorphism $P/Q \cong Q$. This clearly implies that Q is not distinguished.

The valuation domain R in the above example has value group $\prod_\omega \mathbb{Z}$. Our next objective is to refine the method of construction used in the above example and show that the ring R can be chosen to be a valuation domain with the much smaller value group $\mathbb{Z} \times \mathbb{Z}$. Our method involves localizing the ring R at the prime ideal P. We require two preparatory results.

If R is a commutative ring with multiplicatively closed subset S, we shall denote by RS^{-1} the ring of quotients of R with respect to S. If, in particular, $S = R \backslash P$ for some prime ideal P of R, we write R_P in place of RS^{-1}.

Lemma 4.2 *Let R be a valuation domain with prime ideal P and let $S = R\backslash P$. If Q is any prime ideal of R such that $Q \cap S = \varnothing$,* i.e., *$Q \subseteq P$, then $QS^{-1} = Q$.*

Proof It clearly suffices to show that $Q \subseteq Qs$ for each $s \in S$. Take $q \in Q$ and $s \in S$. Since R is a valuation domain and $s \notin Q$, we must have $Q \subset Rs$, so $q = rs$ for some $r \in R$. Inasmuch as Q is prime, $r \in Q$. Thus $q \in Qs$, as required. ∎

A proof of the following proposition may be found in [11, Ch. C, Theorem 1, p. 56] and [12, p. 43].

Proposition 4.3 *Let $v\colon F \to \Gamma \cup \{\infty\}$ be a valuation map and R the associated valuation domain. Let Δ' be a proper convex subgroup of Γ and $\pi\colon \Gamma \cup \{\infty\} \to (\Gamma/\Delta') \cup \{\infty\}$ the canonical projection. Then*

(i) *the composition $\pi v\colon F \to (\Gamma/\Delta') \cup \{\infty\}$ is a valuation map on F;*
(ii) *if $P = R_{\Delta'}$, then R_P is the valuation domain associated with πv. Hence R_P has value group Γ/Δ'.*

Example 4.4 Let R, P and Q be as in Example 4.1. In that example it was shown that $P/Q \cong Q$ as rings. Consider the localization R_P. By Lemma 4.2, P and Q are ideals of R_P. Thus Q is a prime, but not distinguished, ideal of the valuation domain R_P. Since $P = R_{\Delta'}$, it follows from Proposition 4.3(ii) that R_P has value group Γ/Δ'. Inasmuch as $\Gamma = \prod_\omega \mathbb{Z}$ and $\Delta' = 0 \times 0 \times \mathbb{Z} \times \mathbb{Z} \times \cdots$, $\Gamma/\Delta' \cong \mathbb{Z} \times \mathbb{Z}$.

The following result, which has been extracted from [2, Corollary 3.2 and Proposition 3.3], is needed to prove Proposition 4.6.

Proposition 4.5 *The following statements are equivalent for a commutative noetherian domain R.*

(i) *R is integrally closed.*
(ii) *Every ring monomorphism $f\colon A \to R$ where A is a nonzero accessible subring of a commutative domain T, extends uniquely to a monomorphism $\overline{f}\colon T \to R$ defined by $\overline{f}(r) = \frac{f(ar)}{f(a)}$ where a is any fixed element of A for which $0 \neq f(a) \in R$.*

If R is a valuation domain with unique maximal proper ideal P, we call the factor ring R/P the *residue field* of R.

Proposition 4.6 *Let R be a valuation domain with value group $\mathbb{Z}\times\mathbb{Z}$ and with a finite residue field. Then the following statements are equivalent for a proper ideal Q of R:*

(i) *Q is a prime ideal of R;*
(ii) *Q is a distinguished ideal of R.*

Proof (ii) $\Rightarrow$ (i) is a consequence of the Main Theorem (ii) $\Rightarrow$ (iii).

(i) $\Rightarrow$ (ii) By Proposition 2.1, 0 and $0 \times \mathbb{Z}$ are the only proper convex subgroups of $\mathbb{Z} \times \mathbb{Z}$, so by Theorem 2.3, the prime ideals associated with 0 and $0 \times \mathbb{Z}$, say P and Q, are the only nonzero prime ideals of R. Note that P is the maximal proper ideal of

R and thus the Jacobson Radical of R and this is distinguished by the Main Theorem (i) $\Rightarrow$ (ii). It remains to show that Q is distinguished.

By the Main Theorem (iii) $\Rightarrow$ (ii), it suffices to show that Q is distinguished in the set of all prime ideals of R. This entails showing that $Q \not\cong P/Q$ as rings. Suppose, on the contrary, that $f\colon Q \to P/Q$ is a ring isomorphism. Consider the factor ring R/Q. It follows from Proposition 2.6 that R/Q is a valuation domain with value group $0 \times \mathbb{Z} \cong \mathbb{Z}$. This implies that R/Q is a Dedekind domain and so R/Q is integrally closed (see [7, Theorem VIII.6.10, p. 405]). By Proposition 4.5, the isomorphism $f\colon Q \to P/Q$ extends uniquely to a ring monomorphism $\overline{f}\colon R \to R/Q$. Let A/Q denote the image of R in R/Q. Then

$$R/Q \cong \overline{f}[R]/\overline{f}[Q] = (A/Q)/(P/Q) \cong A/P \subseteq R/P.$$

Thus R/Q embeds in the finite ring R/P, an impossibility. ∎

Remark 4.7 Proposition 4.6 shows that there exist valuation domains which have value group $\mathbb{Z} \times \mathbb{Z}$ and which are such that all their prime ideals are distinguished. This result, in conjunction with Example 4.4, shows that the distinguished ideals of a valuation domain are not characterizable in terms of a property of the associated value group.

Notwithstanding the above remark, the value group of a valuation domain does encode important information about the possible location of distinguished ideals. This is shown in Theorem 4.10 which is our next main result.

The proof method used in Proposition 4.8 below has been borrowed from Proposition 3.6.

Proposition 4.8 *Let R and R' be valuation domains with unique maximal proper ideals P and P', respectively. If $P \cong P'$ as rings, then $R \cong R'$ as rings.*

Proof Let $f\colon P \to P'$ be a ring isomorphism and let F be the field of quotients of R'. Choose $0 \neq a \in P$ and define a map $\overline{f}\colon R \to F$ by $\overline{f}(r) = \frac{f(ar)}{f(a)} \in F$ for all $r \in R$. An argument identical to that used in the proof of Proposition 3.6 shows that $\overline{f}$ is a well-defined ring homomorphism which extends f and that $\overline{f}$ is unique with this property. Moreover, $\overline{f}$ is monic since f is monic. It remains to show that $\overline{f}[R] = R'$.

Take $r \in R\backslash P$. Note that r is a unit of R because P is the unique maximal proper ideal of R. Suppose $\overline{f}(r) \notin R'$. Let $v\colon F \to \Gamma \cup \{\infty\}$ be the valuation map which gives rise to R'. Since $\overline{f}(r) \notin R'$, we must have $v(\overline{f}(r)) < 0$. Inasmuch as $v(\overline{f}(r)) + v(\overline{f}(r^{-1})) = v(\overline{f}(1)) = v(1) = 0$, we must have $v(\overline{f}(r^{-1})) > 0$, so $\overline{f}(r^{-1}) \in P'$, *i.e.*, $r^{-1} \in P$, a contradiction. This shows that $\overline{f}[R] \subseteq R'$. A symmetrical argument shows that the isomorphism $f^{-1}\colon P' \to P$ extends uniquely to a ring monomorphism $\overline{f^{-1}}$ from R' into R. It follows from uniqueness that $\overline{f}$ and $\overline{f^{-1}}$ are mutually inverse ring isomorphisms between R and R'. ∎

Theorem 4.9 *Let $v\colon F \to \Gamma \cup \{\infty\}$ and $v'\colon F' \to \Gamma' \cup \{\infty\}$ be valuation maps with respective associated valuation domains R and R'. Let Δ and Δ' be convex subgroups*

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of Γ and Γ', respectively. Put $Q = R_\Delta$ and $Q' = R'_{\Delta'}$. If $Q \cong Q'$ as rings, then $\Gamma/\Delta \cong \Gamma'/\Delta'$ as linearly ordered abelian groups.

Proof Let R_Q and $R'_{Q'}$ be the localizations of R at Q and R' at Q'. By Lemma 4.2, Q and Q' are the unique maximal proper ideals of R_Q and $R'_{Q'}$, respectively. Since $Q \cong Q'$, it follows from Proposition 4.8 that $R_Q \cong R'_{Q'}$. This implies that the value groups of R_Q and $R'_{Q'}$ are isomorphic. Hence, by Proposition 4.3(ii), $\Gamma/\Delta \cong \Gamma'/\Delta'$. ■

We call a convex subgroup Δ of a linearly ordered abelian group Γ *distinguished* if Γ contains no convex subgroups Δ' and Δ'' such that $\Delta' \subset \Delta \subset \Delta''$ and $\Delta/\Delta' \cong \Delta''/\Delta$.

Theorem 4.10 *Let R be a valuation domain with value group Γ. If Δ is a distinguished convex subgroup of Γ, then R_Δ is a distinguished and hence radical ideal of R.*

Proof Put $Q = R_\Delta$. By the Main Theorem, it suffices to show that Q is distinguished in the set of prime ideals of R. Suppose then I and I' are prime ideals of R such that $I \subseteq Q \subseteq I'$ and $Q/I \cong I'/Q$. Let $I = R_\Theta$ and $I' = R_{\Theta'}$ with Θ and Θ' convex subgroups of Γ satisfying $\Theta' \subseteq \Delta \subseteq \Theta$. Consider the valuation domain R/I. By Proposition 2.6, the valuation map v induces a valuation map $\overline{v}\colon R/I \to \Theta^+ \cup \{\infty\}$ defined by:

$$\overline{v}(r+I) = \begin{cases} \infty & \text{if } r \in I, \\ v(r) & \text{if } r \notin I. \end{cases}$$

Note that

$$(R/I)_\Delta = Q/I. \tag{11}$$

Similarly, there exists a valuation map $\overline{\overline{v}}\colon R/Q \to \Delta^+ \cup \{\infty\}$ and

$$(R/Q)_{\Theta'} = I'/Q. \tag{12}$$

Since $Q/I \cong I'/Q$, it follows from (11), (12) and Theorem 4.9 that $\Theta/\Delta \cong \Delta/\Theta'$. Since Δ is distinguished, this entails $\Theta = \Delta = \Theta'$ whence $I = Q = I'$. This shows that Q is distinguished in the set of prime ideals of R, as required. That Q is a radical ideal of R is a consequence of the Main Theorem. ■

Remark 4.11 The converse of Theorem 4.10 is, of course, not valid. If R is a valuation domain satisfying the conditions of Proposition 4.6, then every prime ideal of R is distinguished yet $\mathbb{Z} \times \mathbb{Z}$, which is the value group of R, contains the convex subgroup $0 \times \mathbb{Z}$, which is not distinguished.

In the light of Theorem 4.10 our focus shifts to the determination of distinguished convex subgroups in linearly ordered abelian groups. We shall attempt this only in the case of a countable direct product of Archimedean linearly ordered abelian groups. This problem would appear to be both tractable and interesting with strong combinatorial overtones.

Let $\{\Gamma_n \mid n \in \mathbb{N}\}$ be a (countable) family of nonzero Archimedean linearly ordered abelian groups. Define

$$\Gamma = \prod_{n \in \mathbb{N}} \Gamma_n$$

ordered lexicographically. By Proposition 2.1, the nonzero convex subgroups of Γ are precisely those of the form

$$\Delta_n = \overbrace{0 \times 0 \times \cdots \times 0}^{n-1} \times \Gamma_n \times \Gamma_{n+1} \times \cdots$$

for each $n \in \mathbb{N}$. Every nonzero convex subgroup of Γ is thus a member of the descending chain

$$\Gamma = \Delta_1 \supset \Delta_2 \supset \cdots .$$

Suppose that for some $n \in \mathbb{N}$, Δ_n is not a distinguished convex subgroup of Γ. Then

$$\Delta_i/\Delta_n \cong \Gamma_i \times \Gamma_{i+1} \times \cdots \times \Gamma_{n-1} \cong \Delta_n/\Delta_j \cong \Gamma_n \times \Gamma_{n+1} \times \cdots \times \Gamma_{j-1}$$

for some $i, j \in \mathbb{N}$, $i < n < j$. Since each Γ_k is Archimedean, the number of factors appearing in the direct products $\Gamma_i \times \Gamma_{i+1} \times \cdots \times \Gamma_{n-1}$ and $\Gamma_n \times \Gamma_{n+1} \times \cdots \times \Gamma_{j-1}$ must be equal. Thus $n - i = j - n$. We have thus proved the following result.

Proposition 4.12 *Let $\{\Gamma_n \mid n \in \mathbb{N}\}$ be a (countable) family of nonzero Archimedean linearly ordered abelian groups. The following statements are equivalent for the convex subgroup*

$$\Delta_n = \overbrace{0 \times 0 \times \cdots \times 0}^{n-1} \times \Gamma_n \times \Gamma_{n+1} \times \cdots \quad \textit{of} \quad \Gamma = \prod_{n \in \mathbb{N}} \Gamma_n :$$

(i) Δ_n *is distinguished;*
(ii) $\Gamma_i \times \Gamma_{i+1} \times \cdots \times \Gamma_{n-1} \not\cong \Gamma_n \times \Gamma_{n+1} \times \cdots \times \Gamma_{2n-i-1}$ *for each* $i \in \{1, 2, \ldots, n-1\}$.

Remark 4.13 If, in Proposition 4.12, the Γ_i are all distinct (up to isomorphism), then clearly Δ_n will be distinguished for all $n \in \mathbb{N}$.

We conclude with two examples.

Example 4.14 Take two non-isomorphic nonzero Archimedean linearly ordered abelian groups, say $\mathbb{Z}$ and $\mathbb{Q}$.

(i) Define $\Gamma = \mathbb{Z} \times \mathbb{Q} \times \mathbb{Z} \times \mathbb{Q} \times \cdots$ ordered lexicographically. It is not difficult to see, using Proposition 4.12, that $\Delta_2 = 0 \times \mathbb{Q} \times \mathbb{Z} \times \mathbb{Q} \times \cdots$ is the only proper nonzero distinguished convex subgroup of Γ.

As the following shows, a rearrangement of the factors $\mathbb{Z}$ and $\mathbb{Q}$ in Γ can have a marked effect on the distribution of distinguished subgroups.

(ii) Define $\Gamma = \mathbb{Z} \times \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$ ordered lexicographically. Here, Proposition 4.12 tells us that Δ_2, Δ_3, Δ_5, Δ_7, Δ_{10}, Δ_{13}, etc., are all proper nonzero distinguished convex subgroups of Γ.

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