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SEMIPRIME RINGS WITH NILPOTENT DERIVATIVES

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There has been a great deal of work recently concerning the relationship between the commutativity of a ring R and the existence of certain specified derivations of R. Bell, Herstein, Procesei, Schacher, Ligh, Martindale, Putcha, Wilson, and Yaqub [1, 2, 6, 8, 9, 10, 11, 12, 14] have studied conditions on commutators which imply the commutativity of rings. By noting that a commutator is simply the image of an element under an inner derivation, the present authors and A. N. Richoux [3, 4, 5] have generalized several earlier results by replacing inner derivations by certain (not necessarily inner) derivations. Recently in [8], Herstein claims that, for a prime ring R, if $x \in R$ and if there is a positive integer n such that $[x, y]^n = 0$ for all $y \in R$ then x is central in R. The purpose of this paper is to extend this result to semi-prime rings and, at the same time, to relax the hypothesis by replacing the commutator $[x, y]$ by ∂x for an arbitrary derivation ∂ of R.

THEOREM. *Let R be a semi-prime ring with a derivation ∂. Suppose there exists a positive integer n such that $(\partial x)^n = 0$ for all $x \in R$ and suppose R is $(n-1)!$-torsion free. Then $\partial = 0$.*

Let us first establish the following:

LEMMA 1. *Let R be an $m!$-torsion free ring. Suppose $y_1, y_2, \ldots, y_m \in R$ satisfy $\alpha y_1 + \alpha^2 y_2 + \cdots + \alpha^m y_m = 0$ for $\alpha = 1, 2, \ldots, m$. Then $y_i = 0$ for all i.*

Proof. *Let* A be the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^m \\ \cdots & \cdots & \cdots & \cdots \\ m & m^2 & \cdots & m^m \end{pmatrix}$$

Then, by our assumption,

$$A \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

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Premultiplying by the adjoint of A yields

$$(\det A)\begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$$

Since the determinant of A, $\det A$, known as a Vandermonde determinant, is equal to a product of positive integers, each of which is less than m, and since R is $m!$-torsion free, it follows immediately that $y_i = 0$ for all i.

Throughout the balance of this paper we assume R is a semi-prime ring with a derivation ∂. Assume n is a positive integer, R is $(n-1)!$-torsion free and $(\partial x)^n = 0$ for all $x \in R$. Moreover, Z denotes the ring of integers, ∂R denotes the set of all ∂x where $x \in R$.

LEMMA 2. *For all* $x, y \in R$,

$$\partial y(\partial x)^{n-1} + \partial x\, \partial y(\partial x)^{n-2} + \cdots + (\partial x)^{n-1}\, \partial y = 0. \tag{1}$$

Proof. Let $\alpha \in Z$ and $1 \leq \alpha \leq n-1$.

By expanding $(\partial(x + \alpha y))^n = 0$, we obtain

$$(\alpha x)^n + \alpha(\partial y(\partial x)^{n-1} + \partial x\, \partial y(\partial x)^{n-2} + \cdots + (\partial x)^{n-1}\, \partial y) + \alpha^2((\partial y)^2(\partial x)^{n-2} + \partial y\, \partial x\, \partial y(\partial x)^{n-3} + \partial x(\partial y)^2(\partial x)^{n-3} + \cdots + (\partial x)^{n-2}(\partial y)^2) + \cdots + \alpha^n(\partial y)^n = 0.$$

Since $(\partial x)^n = 0$ and $(\partial y)^n = 0$, it can be written abbreviately as

$$\alpha y_1 + \alpha^2 y_2 + \cdots + \alpha^{n-1} y_{n-1} = 0.$$

By Lemma 1, all $y_i = 0$ and, particularly,

$$y_1 = \partial y(\partial x)^{n-1} + \partial x\, \partial y(\partial x)^{n-2} + \cdots + (\partial x)^{n-1}\, \partial y = 0.$$

LEMMA 3. *For all* $x, y \in R$, *and* $k = 2, 3, 4, \ldots$,

$$\partial^k xy(\partial x)^{n-1} + \partial x\, \partial^k xy(\partial x)^{n-2} + \cdots + (\partial x)^{n-1}\, \partial^k xy = 0, \tag{2}$$

$$(\partial x)^{n-1} y\, \partial^k x + (\partial x)^{n-2} y\, \partial^k x . \partial x + \cdots + y\, \partial^k x(\partial x)^{n-1} = 0. \tag{2$'$}$$

Proof. We proceed by induction on k. In (1) we replace y by ∂xy. We obtain

$$[\partial^2 xy(\partial x)^{n-1} + \partial x\, \partial^2 xy(\partial x)^{n-2} + \cdots + (\partial x)^{n-1}\, \partial^2 xy] + [\partial x\, \partial y(\partial x)^{n-1} + (\partial x)^2\, \partial y(\partial x)^{n-2} + \cdots + (\partial x)^n\, \partial y] = 0.$$

The second bracket is zero by (1) and hence (2) holds for $k = 2$.

Now, we assume (2) holds for $k=m-1$. In (1), replacing y by $\partial^{m-1}xy$ yields

$$[\partial^m xy(\partial x)^{n-1}+\partial x\,\partial^m xy(\partial x)^{n-2}+\cdots+(\partial x)^{n-1}\,\partial^m xy] + [\partial^{m-1}x\,\partial y(\partial x)^{n-1}+\partial x\,\partial^{m-1}x\,\partial y(\partial x)^{n-2}+\cdots+(\partial x)^{n-1}\,\partial^{m-1}x\,\partial y]=0.$$

The second bracket again is zero by the induction hypothesis and thus (2) holds for $k=m$.

Similarly, in (1) replacing y by $y\,\partial x$ and $y\,\partial^{k-1}x$ respectively yield (2)′.

Lemma 4. *For all $x\in R$ and $k=2,3,4,\ldots,$*

$$(\partial x)^{n-1}\,\partial^k x=0, \tag{3}$$

and

$$\partial^k x(\partial x)^{n-1}=0. \tag{3′}$$

Proof. (3) can be obtained from (2) by premultiplying by $(\partial x)^{n-1}$ and by the semi-primeness of R, (3)′ can be obtained from (2)′ similarly.

Lemma 5. *For all x, $y\in R$ and positive integer k.*

$$\partial^k y(\partial x)^{n-1}+\partial^k x(\partial y(\partial x)^{n-2}+\partial x\,\partial y(\partial x)^{n-3}+\cdots+(\partial x)^{n-2}\,\partial y)=0, \tag{4}$$

and

$$(\partial x)^{n-1}\,\partial^k y+(\partial y(\partial x)^{n-2}+\partial x\,\partial y(\partial x)^{n-3}+\cdots+(\partial x)^{n-2}\,\partial y)\,\partial^k x=0. \tag{4′}$$

Proof. From Lemma 2, (4) and (4)′ both hold for $k=1$. Now we assume $k\geq 2$. We replace x by $x+\alpha y$ in (3)′, where $\alpha\in Z$ and $1\leq\alpha\leq n-1$ and then expand it. The identity (4) follows immediately from Lemma 1. Likewise (4)′ can be obtained from the identity (3).

Lemma 6. *For all $x\in R$,*

$$(\partial x)^{n-2}\partial^2 x=\partial^2 x(\partial x)^{n-2}=0. \tag{5}$$

Proof. In the identity (4) for $k=2$, replacing y by $y\,\partial x$ yields

$$(\partial^2 y\,\partial x+\partial y\,\partial^2 x+y\,\partial^3 x)(\partial x)^{n-1}+\partial^2 x[(\partial y\,\partial x+y\,\partial^2 x)(\partial x)^{n-2} + \partial x(\partial y\,\partial x+y\,\partial^2 x)(\partial x)^{n-3}+\cdots+(\partial x)^{n-2}(\partial y\,\partial x+y\,\partial^2 x)]=0$$

or

$$[\partial^2 y)(\partial x)^{n-1}+\partial^2 x(\partial y(\partial x)^{n-2}+\partial x\,\partial y(\partial x)^{n-3}+\cdots+(\partial x)^{n-2}\,\partial y)]\partial x + [\partial y\,\partial^2 x(\partial x)^{n-1}+y\,\partial^3 x(\partial x)^{n-1}]+\partial^2 x[y\,\partial^2 x(\partial x)^{n-2} + \partial xy\,\partial^2 x(\partial x)^{n-3}+\cdots+(\partial x)^{n-2}y\,\partial^2 x]=0.$$

This first bracket is zero by Lemma 5 while the second bracket is zero by Lemma 4. Hence we have $\partial^2 x[y\,\partial^2 x(\partial x)^{n-2}+\partial xy\,\partial^2 x(\partial x)^{n-3}+\cdots+(\partial x)^{n-2}y\,\partial^2 x]=0$. Now we postmultiply by $(\partial x)^{n-2}$ and use Lemma 4. We arrive

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that $(\partial x)^{n-2}y\,\partial^2x(\partial x)^{n-2}=0$. Since y is arbitrary and R is semi-prime, $\partial^2x(\partial x)^{n-2}=0$ as we desired. Similarly $(\partial x)^{n-2}\,\partial^2x=0$.

LEMMA 7. *For all* $x\in R$,

$$\partial^3x(\partial x)^{n-2}=(\partial x)^{n-2}\,\partial^3x=0. \tag{6}$$

Proof. In $\partial^2x(\partial x)^{n-2}=0$, by replacing x by $x+\alpha y$, by expanding and by using Lemma 1, we obtain

$$\partial^2y(\partial x)^{n-2}+\partial^2x[(\partial x)^{n-2}\,\partial y+(\partial x)^{n-3}\,\partial y\,\partial x+\cdots+\partial y(\partial x)^{n-2}]=0.$$

Replacing y by $y\,\partial x$ and applying (5) yield

$$y\,\partial^3x(\partial x)^{n-2}+\partial^2x[(\partial x)^{n-2}y\,\partial^2x+(\partial x)^{n-3}y\,\partial^2x\,\partial x+\cdots+y\,\partial^2x(\partial x)^{n-2}]=0. \tag{7}$$

Now, we premultiply by $(\partial x)^{n-2}$ and use (5). It follows that $(\partial x)^{n-2}y\,\partial^3x(\partial x)^{n-2}=0$. The semi-primeness of R implies $\partial^3x(\partial x)^{n-2}=0$. Likewise, $(\partial x)^{n-2}\,\partial^3x=0$.

LEMMA 8. *For all* $x\in R$,

$$(\partial x)^2\,\partial^2x=\partial^2x(\partial x)^2=0. \tag{8}$$

Proof. For $n<4$, it is trivial by Lemma 6. Now we assume that $n\geq 4$. From (7), using (6) and (5) we obtain

$$\partial^2x[(\partial x)^{n-3}y\,\partial^2x\,\partial x+(\partial x)^{n-4}y\,\partial^2x(\partial x)^2+\cdots+\partial xy\,\partial^2x(\partial x)^{n-3}]=0. \tag{9}$$

Postmultiplying by $(\partial x)^{n-4}$ yields $\partial^2x(\partial x)^{n-3}y\,\partial^2x(\partial x)^{n-4}=0$ which, by the semi-primeness of R, implies $\partial^2x(\partial x)^{n-3}=0$. Likewise, $(\partial x)^{n-3}\,\partial^2x=0$. So we are done if $n=4$ or 5. Suppose $n>5$. The identity (9) becomes $\partial^2x[(\partial x)^{n-4}y\,\partial^2x(\partial x)^2+\cdots+(\partial x)^2y\,\partial^2x(\partial x)^{n-4}]=0$. Postmultiplying by $(\partial x)^{n-6}$ yields $\partial^2x(\partial x)^{n-4}y\,\partial^2x(\partial x)^{n-4}=0$. Again by the semi-primeness of R, $\partial^2x(\partial x)^{n-4}=0$. Continuing this process if necessary, we obtain $\partial^2x(\partial x)^2=0$ and, likewise, $(\partial x)^2\,\partial^2x=0$.

LEMMA 9. *For all* $x\in R$,

$$\partial x\,\partial^2x=\partial^2x\,\partial x=0. \tag{10}$$

Proof. From $\partial^2x(\partial x)^2=0$, we replace x by $x+y$. After expansion we get

$$\partial^2y(\partial x)^2+\partial^2x(\partial x\,\partial y+\partial y\,\partial x)=0. \tag{11}$$

Replacing y by $y\,\partial x$ yields

$$(\partial^2y\,\partial x+\partial y\,\partial^2x+y\,\partial^3x)(\partial x)^2+\partial^2x(\partial x(\partial y\,\partial x+y\,\partial^2x)+(\partial y\,\partial x+y\,\partial^2x)\partial x)=0.$$

By noting that $\partial^2x(\partial x)^2=0$, we get

$$[\partial^2y(\partial x)^3+\partial^2x\,\partial x\,\partial y\,\partial x+\partial^2x\,\partial y(\partial x)^2]+y\,\partial^3x(\partial x)^2+\partial^2x(\partial xy\,\partial^2x+y\,\partial^2x\,\partial x)=0.$$

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The first bracket is zero according to (11). So

$$y\,\partial^3 x(\partial x)^2+\partial^2 x(\partial xy\,\partial^2 x+y\,\partial^2 x\,\partial x)=0. \tag{12}$$

Now premultiplying by $(\partial x)^2$ yields $(\partial x)^2 y\,\partial^3 x(\partial x)^2=0$ which, by semi-primeness of R, implies $\partial^3 x(\partial x)^2=0$. Thus the identity (12) becomes $\partial^2 x(\partial xy\,\partial^2 x+y\,\partial^2 x\,\partial x)=0$. Postmultiplying by ∂x yields $\partial^2 x\,\partial xy\,\partial^2 x\,\partial x=0$, and hence by the semi-primeness of R, $\partial^2 x\,\partial x=0$. That $\partial x\,\partial^2 x=0$ can be obtained analogously.

Lemma 10. *For all* $x\in R$, $\partial^3 x=0$.

Proof. By Lemma 9, for all $x, y\in R$, $\partial(x+y)\,\partial^2(x+y)=0$ which implies $\partial y\,\partial^2 x+\partial x\,\partial^2 y=0$. Premultiplying by $\partial^2 x$ yields

$$\partial^2 x\,\partial y\,\partial^2 x=0. \tag{13}$$

Now we replace y by $y\,\partial^2 xz$.

It follows $\partial^2 x(\partial y\,\partial^2 xz+y\,\partial^3 xz+y\,\partial^2 x\,\partial z)\,\partial^2 x=0$ or, by (13), $\partial^2 xy\,\partial^3 xz\,\partial^2 x=0$. By the semi-primenesss of R, $\partial^2 xy\,\partial^3 x=0$. Replacing x by $x+z$ yields $\partial^2 zy\,\partial^3 x+\partial^2 xy\,\partial^3 z=0$. Now by premultiplying by $\partial^3 z$ and by noting that $\partial^3 z\,\partial^2 z=0$ (Lemma 9), we obtain $\partial^3 z\,\partial^2 xy\,\partial^3 z=0$. Consequently,

$$\partial^3 z\,\partial^2 x=0 \quad \text{for all } x, z\in R. \tag{14}$$

Replacing x by ∂xy yields.

$$\partial^3 z\,\partial x\,\partial^2 y=0 \quad \text{for all } x, y, z\in R \tag{15}$$

by using (14). On the other hand, in (14), replacing x by $x\,\partial y$ yields $\partial^3 x(\partial^2 x\,\partial y+\partial x\,\partial^2 y+x\,\partial^3 y)=0$ which, by (14) and (15), implies $\partial^3 zx\,\partial^3 y=0$ for all $x, y, z\in R$. The semi-primeness of R gives $\partial^3 y=0$ for all $y\in R$.

It is perhaps worth noting that for an arbitrary ring A $\partial^3 x=0$ for all $x\in A$ does not imply $\partial=0$ or the commutativity of A.

Example. Let A be the 3 by 3 matrix ring over a division ring and ∂ be the inner derivation of A defined by

$$\partial\begin{pmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix}=\left[\begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \begin{pmatrix}x_{11} & x_{12} & x_{13}\\ x_{21} & x_{22} & x_{23}\\ x_{31} & x_{32} & x_{33}\end{pmatrix}\right]=\begin{pmatrix}x_{31} & x_{32} & x_{33}-x_{11}\\ 0 & 0 & -x_{21}\\ 0 & 0 & -x_{31}\end{pmatrix}.$$

It can be seen easily that $\partial^3 x=0$ for all $x\in A$. However, A is not commutative. Now we are in a position to prove our main theorem.

Proof of the Theorem. By Lemma 9, $\partial(x+y)\,\partial^2(x+y)=0$ which implies

$$\partial y\,\partial^2 x+\partial x\,\partial^2 y=0. \tag{16}$$

On the other hand, by Lemma 10, $\partial^3(xy)=0$ which implies

(17) $$\partial^2 x\, \partial y + \partial x\, \partial^2 y = 0.$$

Thus, from (16) and (17),

(18) $$\partial^2 x\, \partial y = \partial y\, \partial^2 x, \quad \text{for all } x, y \in R.$$

In (18), replacing y by $y\,\partial x$ yields

(19) $$\partial^2 x(\partial y\, \partial x + y\, \partial^2 x) = (\partial y\, \partial x + y\, \partial^2 x)\,\partial^2 x,$$

while in (17), replacing x by ∂x yields

(20) $$\partial^2 x\, \partial^2 y = 0, \quad \text{for all } x, y \in R.$$

From (19), using (20) and (10), we obtain $\partial^2 x(\partial y\, \partial x + y\, \partial^2 x) = 0$. But $\partial^2 x\, \partial y\, \partial x = \partial y\, \partial^2 x\, \partial x = 0$ by (18) and (10). So we have

(21) $$\partial^2 x = 0 \quad \text{for all } x \in R.$$

Here to prove $\partial = 0$ we might use a result of Posner [13] which says that a product of two non-trivial derivations is not a derivation in a prime ring if the characteristic of the ring is not 2. However, for the sake of self containment we provide a direct and elementary proof. Indeed, from (21), for all $x, y \in R$, $\partial^2(xy)=0$. This implies $\partial x\, \partial y = 0$. Now by replacing y by yx we obtain $\partial xy\, \partial x = 0$ for all $x, y \in R$. Therefore, $\partial = 0$ by the semi-primeness of R.

As an immediate consequence of the theorem we have

Corollary. *Let R be a semi-prime ring and $x \in R$. If there exists a positive integer n such that $[x, y]^n = 0$ for all $y \in R$ and R is $(n-1)!$-torsion free, then x lies in the center of R.*

We conclude with some open problems:

1. It can be shown that for some small n, e.g. 2, 3, the theorem is true without assuming that R is $(n-1)!$-torsion free. Is it true for general n?

2. Does the theorem remain true if one weakens the assumption by assuming that n *depends* upon x?

3. Let R be a semi-prime ring with derivation ∂. If there exist positive integers n and k such that $(\partial^k x)^n$ is central for all $x \in R$, what kind of conclusion can be drawn on R and ∂? (cf. [10]).

References

1. H. Bell, *Duo rings, some applications to commutativity theorem*, Canad. Math. Bull. **11** (1968), 375–380.

2. ——, *Some commutativity results for periodic rings*, Acta Math. Sci. Hungaricae **28** (1976), 279–283.

3. L. O. Chung and Jiang Luh, *Derivations of higher order and commutativity of rings* (to appear).

4. L. O. Chung, Jiang Luh and A. N. Richoux, *Derivations and commutativity of rings*, Pac. J. Math **80** (1979), 77–89.

5. ——, *Derivations and commutativity of rings II*, Pac. J. Math. 85 (1979), 19–34

https://doi.org/10.4153/CMB-1981-064-9 Published online by Cambridge University Press

6. I. N. Herstein, *A condition for the commutativity of rings*, Canad. J. Math. **9** (1957), 583–586.
7. ——, *A note on derivations*, Canad. Math. Bull. **21** (1978), 369–370.
8. ——, *Center-like in prime rings*, Notices of ams 26 No. 3 (1979) p. A-329.
9. I. N. Herstein, C. Procesi and M. Schacher, *Algebraic valued functions on non-commutative rings*, J. of Algebra **36** (1975), 128–150.
10. M. Ikeda and C. Koc, *On the commutator ideal of certain rings*, Arch. Math. **25** (1974), 348–353.
11. S. Ligh, *The structure of certain classes of rings and near rings*, J. London Math. Soc., (2) **12** (1975), 27–31.
12. W. S. Martindale III, *The commutativity of a special class of rings*, Canad. J. Math. **12** (1960), 263–268.
13. E. C. Posner, *Derivations in prime rings*, Proc. AMS **8** (1957), 1093–1100.
14. M. S. Putcha, R. S. Wilson and A. Yaqub, *Structure of rings satisfying certain identities on commutators*, Proc. AMS **32** (1972), 57–62.

Added in proof.

The detailed proof of the results in reference [8] has appeared in J. Algebra 60 (1979), 567–574.

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