LIFTING SETS AND THE CALKIN ALGEBRA

by G. J. MURPHY

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H will denote a Hilbert space of infinite dimension, $\mathcal{B}(H)$ the algebra of bounded linear operators on H, and $\mathcal{K}(H)$ the ideal of compact operators on H. We let σ, σ_e and σ_ω denote the spectrum, essential spectrum and Weyl spectrum respectively. It is well known that for arbitrary $T \in \mathcal{B}(H)$ we have by [**5**]

$$\sigma_e(T) = \sigma(T + \mathcal{K}(H)) = \sigma(T) \backslash \{\lambda \in \sigma(T) : \lambda 1 - T \text{ is a Fredholm operator}\}$$

and

$$\sigma_\omega(T) = \bigcap_{K \in \mathcal{K}(H)} \sigma(T+K)$$
$$= \sigma(T) \backslash \{\lambda \in \sigma(T) : \lambda 1 - T \text{ is a Fredholm operator of index zero}\}$$

and

$$\sigma_e(T) \subseteq \sigma_\omega(T) \subseteq \sigma(T).$$

Δ will always denote a non-empty compact subset of the plane. We say Δ is a *lifting set* (for H) if for every $T \in \mathcal{B}(H)$ with $\sigma_e(T) = \Delta$ there exists $K \in \mathcal{K}(H)$ such that $\sigma(T+K) = \Delta$. (It is easily seen that there is always some $T \in \mathcal{B}(H)$ with $\sigma_e(T) = \Delta$.) We show that Δ is a lifting set if and only if it has no holes. The following theorem is stated for reference (see also [**1**]). The result is due to Stampfli [**6**].

THEOREM 1. *For any $T \in \mathcal{B}(H)$ there exists $K \in \mathcal{K}(H)$ such that $\sigma(T+K) = \sigma_\omega(T)$.*

The following result is immediate from this theorem.

LEMMA 2. *Δ is a lifting set if and only if $\sigma_e(T) = \sigma_\omega(T)$ for every $T \in \mathcal{B}(H)$ such that $\sigma_e(T) = \Delta$.*

THEOREM 3. *Δ is a lifting set if and only if Δ is polynomially convex.*

Proof. Suppose Δ is a lifting set, but is not polynomially convex. Let ω be any hole of Δ. It is a consequence of Berger and Shaw [**2**] that there exists $T \in \mathcal{B}(H)$ with $\sigma_e(T) = \Delta$ and $\text{ind}(\lambda I - T) = 1$ $(\lambda \in \omega)$. (Here ind denotes the Fredholm index.) Now since Δ is a lifting set, $\sigma_\omega(T) = \sigma_e(T)$ by the Lemma. But

$$\sigma_\omega(T) = \sigma_e(T) \cup \{\lambda : \lambda I - T \text{ is Fredholm and ind } (\lambda I - T) \neq 0\}.$$

Hence $\omega \subseteq \sigma_\omega(T) = \sigma_e(T) = \Delta$. This contradiction shows that lifting sets have no holes.

Conversely, suppose Δ is polynomially convex, and let $\sigma_e(T) = \Delta$. By Stampfli's theorem there exists $K \in \mathcal{K}(H)$ with $\sigma_\omega(T) = \sigma(T+K)$. Also, $\sigma_\omega(T)$ consists of $\sigma_e(T)$ with some of its holes [**4**], and so in this case $\Delta = \sigma_e(T) = \sigma_\omega(T) = \sigma(T+K)$. Thus Δ is a lifting set.

It is interesting to note that the well-known result of West [**7**] that a Riesz operator T can be written $T = Q + K$, where Q is quasinilpotent and $K \in \mathcal{K}(H)$, says precisely that $\{0\}$

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is a lifting set. (By the Ruston characterization of Riesz operators [**3** p. 43], T is Riesz if and only if $\sigma_e(T)=\{0\}$.) It is unknown whether the West decomposition of a Riesz operator holds in arbitrary Banach spaces.

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School of Mathematics
Trinity College,
Dublin

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