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ERRATUM

The following solution, which corrects a mistake in the solution to Problem 92.2.7 by Peter C.B. Phillips, was prepared by Marie Blanchet and Peter Burridge.

The DGP is the cointegrated VAR:

$$\begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix} = \begin{bmatrix} 0 & a_{12} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{1,t-1} \\ y_{2,t-1} \end{bmatrix} + \begin{bmatrix} u_{1t} \\ u_{2t} \end{bmatrix}, \tag{1}$$

with $\mathbf{u}_t$ $\text{IID}(0,\Sigma)$.

We require the limit distribution of $(\hat{a}_{11}, \hat{a}_{12})$ from the OLS estimation of

$$y_{1,t} = a_{11} y_{1,t-1} + a_{12} y_{2,t-1} + u_{1,t}; \text{ or more compactly, } y_1 = \mathbf{Z}a + \mathbf{u}_1. \tag{2}$$

To avoid the singularity of the limit of the second moment matrix of the regressors in (2), we follow Sims, Stock and Watson (1990) and transform the equation by using the cointegrating vector, $\alpha' = (1, -a_{12})$. Introduce the nonsingular matrix, $\mathbf{D} = \begin{bmatrix} 1 & 0 \\ -a_{12} & 1 \end{bmatrix}$, so that we may write (2) as:

$$y_1 = \mathbf{Z}\mathbf{D}\delta + \mathbf{u}_1, \tag{3}$$

with $\mathbf{D}\delta = a$. By denoting the transformed variables as $\mathbf{Z}^* = \mathbf{Z}\mathbf{D}$, we find, as in the previously published solution, that the following second moment convergence applies: $T^{-1}\mathbf{Z}_1^{*\prime}\mathbf{Z}_1^* \xrightarrow{P} \alpha'\Sigma\alpha$; $T^{-3/2}\mathbf{Z}_1^{*\prime}\mathbf{Z}_2^* \xrightarrow{P} 0$; $T^{-2}\mathbf{Z}_2^{*\prime}\mathbf{Z}_2^* \xrightarrow{D} \int B_2^2$, where B_2 is the Brownian motion on [0,1] to which normalized partial sums of y_2 converge weakly.

Similarly, $T^{-1/2}\mathbf{Z}_1^{*\prime}\mathbf{u}_1 \xrightarrow{D} N(0, \sigma_{11}, \alpha'\Sigma\alpha)$, and $T^{-1}\mathbf{Z}_2^{*\prime}\mathbf{u}_1 \xrightarrow{D} \int B_2 dB_1$. Thus, the limit behavior of $\hat{\delta}$ from (3) is found to be:

$$T^{1/2}(\hat{\delta}_1 - \delta_1) \xrightarrow{D} N(0, \sigma_{11}/\alpha'\Sigma\alpha); \; T(\hat{\delta}_2 - \delta_2) \xrightarrow{D} \left(\int B_2^2\right)^{-1} \int B_2 \, dB_1. \tag{4}$$

However, to recover the behavior of $\hat{a}$, we must transform back by using the identity, $\mathbf{D}\hat{\delta} \equiv \hat{a}$, which follows from the nonsingularity of $\mathbf{D}$. We find, therefore, that $\hat{a}_{11} = \hat{\delta}_1$, with limit distribution given by the first expression in (4), and $\hat{a}_{12} = -a_{12}\hat{\delta}_1 + \hat{\delta}_2$ with limit distribution given by

$$T^{1/2}(\hat{a}_{12} - a_{12}) \xrightarrow{D} -a_{12}T^{1/2}(\hat{\delta}_1 - \delta_1) + T^{-1/2}\left(\int B_2^2\right)^{-1} \int B_2 \, dB_1, \tag{5}$$

and because the second term on the right-hand side converges in probability to 0, we find that *both* $\hat{a}_{11}$ and $\hat{a}_{12}$ have $\sqrt{T}$ convergence to normal limit distributions. Notice also that their joint limit distribution is singular.

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Comments

The mistake in the previously published solution is this. Let $v_t = u_{1t} - a_{12}u_{2t}$; it is certainly true that *as data-generating mechanisms*, the two equations

$$y_{1t} = a_{11}y_{1,t-1} + a_{12}y_{2,t-1} + u_{1t} \tag{6}$$

and

$$y_{1t} = a_{11}v_{t-1} + a_{12}y_{2,t-1} + u_{1t} \tag{7}$$

are equivalent when $a_{11} = 0$. However, this would also be true if we replace v_{t-1} with, say, $t_{3,t-1}$, any variable (possibly a random walk independent of everything else in the problem); *thus, it is not true that OLS estimates of the coefficients in* (6) *and* (7) *have the same properties.* Algebraically, treating estimates from (6) and (7) as equivalent corresponds to failing to transform back from δ to a. Actually, (5) is not surprising when we realize that the redundant regressor in (2) is cointegrated with y_2; because of this, its presence in the estimated equation is not negligible asymptotically. However, because $\delta_2 = a_{12}(1 + a_{11})$, a superior estimate of a_{12}, which uses the prior information that $a_{11} = 0$, is obviously given by $\hat{\delta}_2$, which does indeed have the limit distribution given in the previously published solution. In view of this, the important comments about the presence of bias of order $1/T$ arising from correlation between B_2 and B_1 contained therein should be taken to apply to $\hat{\delta}_2$.

In a recent paper, Phillips (1995) gave a very general account of the application of fully modified least squares to vector autoregressions, which covers the model discussed above in a correct way and shows how the bias of order T^{-1} may be removed. The solution given above is a special case of the much more general results contained therein, where transformations to appropriate coordinate systems play a key role in developing the distribution theory.

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