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ON THE AXIOMATIZABILITY OF C*-ALGEBRAS AS OPERATOR SYSTEMS

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Abstract. We show that the class of unital C*-algebras is an elementary class in the language of operator systems and that the algebra multiplication is a definable function in this language. Moreover, we prove a general model theoretic fact which implies that the aforementioned class is ∀∃∀-axiomatizable. We conclude by showing that this class is, however, neither ∀∃-axiomatizable nor ∃∀-axiomatizable.

1. Introduction. A C*-algebra is a self-adjoint subalgebra (from here on, $*$-subalgebra) of $\mathcal{B}(H)$, the $*$-algebra of bounded operators on a complex Hilbert space, that is, closed in the operator norm topology. In this note, we assume that all C*-algebras are unital, namely that they contain the identity operator. As shown in [**5**, Proposition 3.3], there is a natural (continuous) first-order language $\mathcal{L}_{\mathrm{C}^*}$ in which $\mathcal{K}_{\mathrm{C}^*}$, the class of $\mathcal{L}_{\mathrm{C}^*}$-structures that are unital C*-algebras, is an elementary class, meaning that there is a (universal) $\mathcal{L}_{\mathrm{C}^*}$-theory T_{C^*} for which $\mathcal{K}_{\mathrm{C}^*}$ is the class of models of T_{C^*}; in symbols, $\mathcal{K}_{\mathrm{C}^*} = \mathrm{Mod}(T_{\mathrm{C}^*})$. (The authors only treat not necessarily unital C*-algebras, but one just adds a constant to name the identity with no additional complications.)

An operator system is a $*$-closed subspace of $\mathcal{B}(H)$ that contains the unit and is closed in the operator norm topology, so every unital C*-algebra is an operator system but not vice versa. The appropriate morphisms between operator systems are the unital completely positive linear maps (see the appendix). There is a natural first-order language $\mathcal{L}_{\mathrm{os}}$ in which the class of operator systems is universally axiomatizable; see [**3**, Section 3.3] and [**7**, Appendix B]. Since the operator system structure on a C*-algebra is uniformly quantifier-free definable, we may assume that $\mathcal{L}_{\mathrm{os}} \subseteq \mathcal{L}_{\mathrm{C}^*}$. For a C*-algebra A, we let $A|\mathcal{L}_{\mathrm{os}}$ denote the reduct of A to $\mathcal{L}_{\mathrm{os}}$, which simply means that we view A merely as an operator system rather than as a C*-algebra. Set $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}} := \{A|\mathcal{L}_{\mathrm{os}} \, : \, A \in \mathcal{K}_{\mathrm{C}^*}\}$. In [**6**], the following question was raised: is $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ an elementary class? The main result of this note is to give an affirmative answer to this question.

2. The semantic approach. The following is the main result of this paper.

THEOREM 1. *$\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$ is an elementary class.*

To prove Theorem 1, we will use the semantic test for axiomatizability, that is, we show that $\mathcal{K}_{C^*}|\mathcal{L}_{os}$ is closed under isomorphism, ultraproduct, and ultraroot. (See [**1**, Proposition 5.14].) Closure under isomorphism and ultraproducts is clear. We thus only need to show that $\mathcal{K}_{C^*}|\mathcal{L}_{os}$ is closed under ultraroots. We will actually show something a priori more general, namely that $\mathcal{K}_{C^*}|\mathcal{L}_{os}$ is closed under elementary substructures. We first need a result, which is nearly identical to [**2**, Theorem 6.1]. Some notation: for a C*-algebra B and $x, y, z, b \in B$, let $\varphi(x, y, z, b)$ be the $\mathcal{L}_{os}$-formula

$$\left| \left\| \begin{bmatrix} 0 & y & 1_B & 0 \\ 2 \cdot 1_B & x & z & b \end{bmatrix} \right\|^2 - \left\| \begin{bmatrix} 2 \cdot 1_B & x & z & b \end{bmatrix} \right\|^2 \right|.$$

PROPOSITION 2. *Suppose that A is a operator subsystem of the unital* C*-*algebra B. Then, A is closed under products if and only if we have*

$$\sup_{x,y\in A_1} \inf_{z\in A_1} \sup_{b\in B_2} \varphi(x, y, z, b) = 0.$$

Proof. We first assume that the displayed equation holds and prove that A is closed under products. Towards this end, fix $\epsilon \in (0, 1)$ and $x, y \in A_1$. Choose $z \in A_1$ such that

$$\sup_{b\in B_2} \varphi(x, y, z, b) < \epsilon,$$

and let b be the square root of $\|xx^* + zz^*\| \cdot 1_B - xx^* - zz^* \in B$ so that $\|b\|^2 = \|b^2\| \leq \|xx^* + zz^*\| \leq 2$. Multiplying $\begin{bmatrix} 2 \cdot 1_B & x & z & b \end{bmatrix}$ by its adjoint have that

$$\left\| \begin{bmatrix} 2 \cdot 1_B & x & z & b \end{bmatrix} \right\|^2 = 4 \cdot 1_B + xx^* + zz^* + bb^* = (4 + \|xx^* + zz^*\|) \cdot 1_B.$$

Similarly, it holds that

$$\left\| \begin{bmatrix} 0 & y & 1_B & 0 \\ 2 \cdot 1_B & x & z & b \end{bmatrix} \right\|^2 = \left\| \begin{bmatrix} 1_B + yy^* & yx^* + z^* \\ xy^* + z & (4 + \|xx^* + zz^*\|) \cdot 1_B \end{bmatrix} \right\|.$$

Examining the second row, it follows that the norm of the right side is at least $(\|xy^* + z\|^2 + (4 + \|xx^* + zz^*\|)^2)^{1/2}$, whence $\|xy^* + z\| \leq 4\sqrt{\epsilon}$. As A is complete, it follows that $xy^* \in A$.

Conversely, the above calculations show that if A is closed under multiplication, then setting $z := -xy^*$ suffices. □

As mentioned above, the following proposition completes the proof of Theorem 1.

PROPOSITION 3. *Let B be a* C*-*algebra. If $A \subset B$ is a operator subsystem of B, which is an elementary substructure in the language of operator systems, then A is a* C*-*subalgebra of B.*

Proof. Since A inherits the unit of B and A is self-adjoint, we need only check that A is closed under products, that is, we need only verify the condition of the previous proposition. Fixing $x, y \in A_1$, we have

$$\inf_{z\in B_1} \sup_{b\in B_2} \varphi(x, y, z, b) = 0.$$

Since A is an elementary substructure of B, we have

$$\inf_{z\in A_1}\sup_{b\in A_2}\varphi(x,y,z,b)=0.$$

Fix $\epsilon>0$ and take $z\in A_1$ such that

$$\sup_{b\in A_2}\varphi(x,y,z,b)\leq\epsilon.$$

By elementarity again, we have

$$\sup_{b\in B_2}\varphi(x,y,z,b)\leq\epsilon,$$

whence we have

$$\inf_{z\in A_1}\sup_{b\in B_2}\varphi(x,y,z,b)=0,$$

which is what we desired. □

Now that Theorem 1 has been established, we let $T_{\mathrm{C}^*,\mathrm{os}}$ denote the $\mathcal{L}_{\mathrm{os}}$-theory axiomatizing $\mathcal{K}_{\mathrm{C}^*}|\mathcal{L}_{\mathrm{os}}$.

3. The syntactic approach and quantifier upper bounds. We begin this section by proving a general model-theoretic fact. First, we recall that, given a symbol f in a language, we let Δ_f denote the modulus of uniform continuity for f as provided by the language. (We use analogous notation for predicate symbols.) In what follows we need to make the following innocuous technical assumption: for every symbol and every $\epsilon<\epsilon'$, we have $\Delta(\epsilon)<\Delta(\epsilon')$.

THEOREM 4. *Suppose that T is an $\mathcal{L}$-theory and $\mathcal{L}_0$ is a sublanguage of $\mathcal{L}$. Suppose that the following holds:*
- *for every predicate $P\in\mathcal{L}\setminus\mathcal{L}_0$, there is an $\mathcal{L}_0$-formula $\psi_P(x)$ such that, for every $\mathfrak{M}\models T$ and $a\in M$, we have $P^{\mathfrak{M}}(a)=\psi_P^{\mathfrak{M}}(a)$; and*
- *for every function symbol $f\in\mathcal{L}\setminus\mathcal{L}_0$, there is an $\mathcal{L}_0$-formula $\psi_f(x,y)$ such that, for every $\mathfrak{M}\models T$ and $a,b\in\mathfrak{M}$, we have $f^{\mathfrak{M}}(a)=b$ if and only if $\psi_f(a,b)^{\mathfrak{M}}=0$.*

Then, the class of $\mathcal{L}_0$-reducts of models of T is an elementary class.

Proof. Given a quantifier-free $\mathcal{L}$-formula $\varphi(x_1,\dots,x_n)$, define an $\mathcal{L}_0$-formula $\tilde{\varphi}(x_1,\dots,x_n,z_1)$ by setting $\varphi'(x_1,\dots,x_n,z_1)$ to be the formula one obtains by replacing an innermost occurrence of some function symbol $f(x_{i_1},\dots,x_{i_n})$ that does not belong to $\mathcal{L}_0$ by some fresh variable z_1 and then setting

$$\tilde{\varphi}(x_1,\dots,x_n,z_1):=\max(\psi_f(x_{i_1},\dots,x_{i_k},z_1),\varphi'(x_1,\dots,x_n,z_1)).$$

By continuing this process until there are no occurrences of any function symbols in $\mathcal{L}\setminus\mathcal{L}_0$, and by replacing every predicate symbol $P\in\mathcal{L}\setminus\mathcal{L}_0$ by ψ_P, we obtain an $\mathcal{L}_0$-formula $\varphi^{\#}(\vec{x},\vec{z})$.

For example, suppose that $\varphi(x_1,x_2,x_3):=P(g(f(x_1,x_2),x_2,x_3),x_1)$. Then, we would have

$$\varphi^{\#}(x_1,x_2,z_1,z_2):=\max(\psi_f(x_1,x_2,z_1),\psi_g(z_1,x_2,x_3),\psi_P(z_2,x_1)).$$

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Next, given an axiom $\sigma = 0$ from T, where

$$\sigma := Q_1 x_1 \cdots Q_n x_n \varphi(x_1, \ldots, x_n)$$

with φ quantifier-free, consider the closed $\mathcal{L}_0$-condition $\sigma^\# = 0$, where

$$\sigma^\# := Q_1 x_1 \cdots Q_n x_n \inf_{\vec{z}} \varphi^\#(\vec{x}, \vec{z}).$$

Now observe that, for any function symbol f and any $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$T \models \sup_{\vec{x}} \sup_{y_1, y_2} \min\left(\min_{i=1,2}(\delta \dot{-} \psi_f(\vec{x}, y_i)), d(y_1, y_2) \dot{-} \epsilon\right) = 0. \quad (\dagger)$$

Indeed, if this were not the case, then we would have $\mathfrak{M} \models T$ and $\vec{a}, b_1, b_2 \in M$ such that, for $i = 1, 2$, $\psi_f(\vec{a}, b_i) = 0$ (whence $f^{\mathfrak{M}}(\vec{a}) = b_i$) and yet $d(b_1, b_2) > \epsilon$, yielding a contradiction.

In a similar manner, there is $\eta = \eta(\epsilon) > 0$ such that

$$T \models \sup_{\vec{x}_1, \vec{x}_2, y_1, y_2} \min\left(\Delta_f(\epsilon) \dot{-} d(\vec{x}_1, \vec{x}_2), \eta \dot{-} \max_{i=1,2} \psi_f(\vec{x}_i, y_i), d(y_1, y_2) \dot{-} \epsilon\right) = 0. \quad (\dagger\dagger)$$

Indeed, if this were not the case, then we would have $\mathfrak{M} \models T$ and $\vec{a}_1, \vec{a}_2, b_1, b_2 \in M$ such that, for $i = 1, 2$, we have $d(\vec{a}_1, \vec{a}_2) \leq \Delta_f(\epsilon)$, $\psi_f(\vec{a}_i, b_i) = 0$ (so $f(\vec{a}_i) = b_i$) but $d(b_1, b_2) > \epsilon$. Take $\epsilon' > \epsilon$ such that $d(b_1, b_2) > \epsilon'$. Then, by assumption, $d(\vec{a}_1, \vec{a}_2) \leq \Delta_f(\epsilon) < \Delta_f(\epsilon')$, whence $d(b_1, b_2) \leq \epsilon'$, a contradiction.

We now let T_0 denote the following $\mathcal{L}_0$-theory:

(1) $\sigma^\# = 0$ whenever $\sigma = 0$ is an axiom of T;
(2) $\sup_{\vec{x}} \inf_y \psi_f(\vec{x}, y) = 0$ whenever $f \in \mathcal{L} \setminus \mathcal{L}_0$;
(3) for every $f \in \mathcal{L} \setminus \mathcal{L}_0$ and every $\epsilon > 0$, the axiom appearing in ($\dagger$);
(4) for every $f \in \mathcal{L} \setminus \mathcal{L}_0$ and every $\epsilon > 0$, the axiom appearing in ($\dagger\dagger$);
(5) for every $P \in \mathcal{L} \setminus \mathcal{L}_0$ and every $\epsilon > 0$, the axiom

$$\sup_{\vec{x}, \vec{x}'} \min\left(\Delta_P(\epsilon) \dot{-} d(\vec{x}, \vec{x}'), |\psi_P(\vec{x}) - \psi_P(\vec{x}')| \dot{-} \epsilon\right) = 0.$$

We claim that the $\mathcal{L}_0$-structures that model T_0 are exactly the $\mathcal{L}_0$-reducts of models of T. It is clear that the $\mathcal{L}_0$-reduct of any model of T is a model of T_0. Conversely, suppose that $\mathfrak{M}$ is an $\mathcal{L}_0$-structure and $\mathfrak{M} \models T_0$. We first note that the zeroset of ψ_f is the graph of a function. Indeed, fix $\vec{a} \in M$ and use axiom (2) to find $b_n \in M$ such that $\psi_f(\vec{a}, b) \leq \frac{1}{n}$. By axiom group (3), the sequence (b_n) is Cauchy whence converges to some $b \in M$. Moreover, b is unique by axiom group (3) again. We may thus define $f^{\mathfrak{M}}(\vec{a})$ to be this unique b. Next, note that axiom group (4) shows that $f^{\mathfrak{M}}$ has Δ_f as its modulus of uniform continuity. Finally, note that we may set $P^{\mathfrak{M}} := \psi_P^{\mathfrak{M}}$ and that $P^{\mathfrak{M}}$ has Δ_P as its modulus of uniform continuity by axiom group (5).

At this point, we have expanded $\mathfrak{M}$ to an $\mathcal{L}$-structure. It remains to show that this expanded structure is a model of T. However, this follows from the fact that, for any

$a_1, \ldots, a_n \in M$ and any quantifier-free $\mathcal{L}$-formula $\varphi(x_1, \ldots, x_n)$, we have

$$\varphi(a_1, \ldots, a_n)^{\mathfrak{M}} = 0 \Leftrightarrow \inf_{\vec{z}} \varphi^{\#}(\vec{a}, \vec{z})^{\mathfrak{M}} = 0.$$

Thus, the class of $\mathcal{L}_0$-reducts of models of T is seen to be elementary. □

We next remark that if one is given upper bounds on the quantifier-complexity of the axioms of T and the axioms of the formulae defining the symbols in $\mathcal{L} \setminus \mathcal{L}_0$, then we have upper bounds on the quantifier-complexity of the axioms of T_0. First, we call a formula in prenex normal form a $\forall_n$-formula if it has n alternations of quantifiers with the first group of quantifiers being sup quantifiers. We call a theory $\forall_n$ if it consists of closed conditions of the form $\sigma = 0$ with σ a $\forall_n$-sentence. Finally, let $p : \mathbb{N} \to \mathbb{N}$ be the function $p(n) = n \mod 2$. The proof of Theorem 4 shows the following.

COROLLARY 5. *Suppose that T, $\mathcal{L}$, and $\mathcal{L}_0$ are as in Theorem 4. Further suppose that T is a $\forall_m$-theory and that each symbol in $\mathcal{L} \setminus \mathcal{L}_0$ is defined by a $\forall_n$-formula in $\mathcal{L}_0$. Then, setting T_0 to be the $\mathcal{L}_0$-theory axiomatizing the class of $\mathcal{L}_0$-reducts of models of T, we have that $\mathcal{L}_0$ is a $\forall_{m+n+p(n)}$-theory.*

We now apply the above abstract results to the setting from the previous section. Thus, we set $\mathcal{L} := \mathcal{L}_{C^*}$, $\mathcal{L}_0 := \mathcal{L}_{os}$, and $T := T_{C^*}$. In this case, $\mathcal{L} \setminus \mathcal{L}_0$ consists of a single binary function symbol for multiplication. Let $\psi_{\mathrm{mult}(x,y,z)} := \sup_{\|b\| \leq 2} \varphi(-x, y^*, z, b)$. By Proposition 2, we have that the zeroset of ψ_{mult} defines multiplication in models of T. Note also that T is a $\forall_1$-theory and that ψ_{mult} is a $\forall_1$-formula. Consequently, by Corollary 5, we have the following.

COROLLARY 6. *$\mathcal{K}_{C^*}|\mathcal{L}_{os}$ is a $\forall_3$-axiomatizable class.*

4. Quantifier lower bounds. Although we have established that $T_{C^*}|\mathcal{L}_{os}$ is an $\forall_3$-axiomatizable $\mathcal{L}_{os}$-theory, it is a priori possible that it is in fact two quantifier axiomatizable. Our last two results show that this is not the case. Recall that if X is an operator system and $u \in X$, then u is called a unitary of X if u is a unitary of the C*-envelope $C^*_u(X)$, i.e., the universal unital C*-algebra generated by X.

PROPOSITION 7. *$T_{C^*}|\mathcal{L}_{os}$ is* not *$\forall_2$-axiomatizable.*

Proof. If $T_{C^*}|\mathcal{L}_{os}$ were $\forall_2$-axiomatizable, then there would be $A \in \mathcal{K}_{C^*}|\mathcal{L}_{os}$ that is existentially closed for $\mathcal{K}_{C^*}|\mathcal{L}_{os}$. However, in [**6**, Section 5], it was observed that if $\phi : X \to Y$ is a complete order embedding that is also existential, then ϕ maps unitaries to unitaries. Take a complete order embedding of A into a C*-algebra B that is not a $*$-homomorphism (see, for example, [**6**, Section 5]); since this embedding maps unitaries to unitaries (since A is existentially closed for $\mathcal{K}$), this contradicts a well-known consequence of Pisier's linearization trick. (For the convenience of the reader, we include a proof of this fact in the appendix.) □

One defines a $\exists_2$-axiomatizable theory in an analogous fashion using prenex normal formula that begin with inf quantifiers.

PROPOSITION 8. *$T_{C^*}|\mathcal{L}_{os}$ is* not *$\exists_2$-axiomatizable.*

Proof. Fix a C*-algebra A and an operator system X that is not a C*-algebra with $A \subseteq X \subseteq A^{\mathcal{U}}$. Suppose, towards a contradiction, that $T_{C^*}|\mathcal{L}_{os}$ is $\exists_2$-axiomatizable and let $\sigma := \inf_x \sup_y \varphi(x, y)$ be such an axiom. Fix $\epsilon > 0$ and take $a \in A$ such that

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$(\sup_y \varphi(a, y))^A \leq \epsilon$. It follows that $(\sup_y \varphi(a, y))^{A^{\mathcal{U}}} \leq \epsilon$, whence $(\sup_y \varphi(a, y))^X \leq \epsilon$. Since $a \in X$ and $\epsilon > 0$ was arbitrary, we see that $\sigma^X = 0$. Since σ was an arbitrary axiom, we see that X is a C*-algebra, yielding a contradiction. □

Appendix on Pisier's linearization trick. If $E \subset \mathcal{B}(H)$ is an operator system, then $M_n(E) := E \otimes M_n(\mathbb{C})$ sits naturally as an operator system in $\mathcal{B}(H) \otimes M_n(\mathbb{C}) \cong \mathcal{B}(H^{\oplus n})$. Each $M_n(E)$ inherits a 'positive' cone from the cone of positive operators in $\mathcal{B}(H^{\oplus n})$. A $*$-linear map $\phi : E \to F$ between operator systems is said to be n-positive if $\phi_n := \phi \otimes \mathrm{id}_{M_n(\mathbb{C})} : M_n(E) \to M_n(F)$ maps positive elements to positive elements and completely positive if it is n-positive for all n.

The following facts are well-known.

FACT 9. *Suppose that $\phi : A \to B$ is a unital 2-positive map between unital C*-algebras. Then, for all $x, y \in A$, we have*

$$\phi(x)^*\phi(x) \leq \phi(x^*x)$$

and

$$\|\phi(y^*x) - \phi(y)^*\phi(x)\| \leq \|\phi(y^*y) - \phi(y)^*\phi(y)\|^{1/2}\|\phi(x^*x) - \phi(x)^*\phi(x)\|^{1/2}.$$

Proof. The first part follows from the fact that the matrix

$$\begin{pmatrix} 1 & x \\ x^* & x^*x \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

is positive in $M_2(A)$, whence so is $\begin{pmatrix} 1 & \phi(x) \\ \phi(x)^* & \phi(x^*x) \end{pmatrix}$ in $M_2(B)$. (The inequlity works exactly same in the general case as it does for a positive matrix in $M_2(\mathbb{C})$.) From the first part, we conclude that for any positive linear functional $\sigma \in B^*$, the bilinear form $(x, y) := \sigma(\phi(y^*x) - \phi(y)^*\phi(x))$ on A is positive semidefinite. The second part then follows from the Cauchy–Schwartz inequality and the fact that, for $b \in B$, $\|b\|$ is the supremum of $|\sigma(b)|$, where σ ranges over all contractive, positive linear functionals on B. □

COROLLARY 10. *Suppose that $\phi : A \to B$ is a unital, completely positive map between C*-algebras that maps unitaries to unitaries. Then, ϕ is a $*$-homomorphism.*

Proof. The previous fact shows that the set

$$M_\phi := \{a \in A : \phi(a^*)\phi(a) = \phi(a^*a),\ \phi(a)\phi(a^*) = \phi(aa^*)\}$$

is a C*-subalgebra of A on which ϕ is a $*$-homomorphism. By assumption, we have that $\mathcal{U}(A) \subset M_\phi$ whence $M_\phi = A$ as A is the linear span of its unitaries. □

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