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THE SEIFERT FIBER SPACE CONJECTURE AND TORUS THEOREM FOR NONORIENTABLE 3-MANIFOLDS

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ABSTRACT. The Seifert-fiber-space conjecture for nonorientable 3-manifolds states that if M denotes a compact, irreducible, nonorientable 3-manifold that is not a fake $P^2 \times S^1$, if $\pi_1 M$ is infinite and does not contain $Z_2 * Z_2$ as a subgroup, and if $\pi_1 M$ does however contain a nontrivial, cyclic, normal subgroup, then M is a Seifert bundle. In this paper, we construct all compact, irreducible, nonorientable 3-manifolds (that do not contain a fake $P^2 \times I$) each of whose fundamental group contains $Z_2 * Z_2$ and an infinite cyclic, normal subgroup; none of these manifolds admits a Seifert fibration, but they satisfy Thurston's Geometrization Conjecture. We then reformulate the statement of the (nonorientable) SFS-conjecture and obtain a torus theorem for nonorientable manifolds.

1. **Introduction.** The proof of the Seifert fiber space conjecture (Theorem A) was recently completed by Casson and Jungreis [1] and, independently, by Gabai ([6], [7]). The nonorientable version (Theorem B) was given in [19].

THEOREM A. *Let M denote a compact, orientable, irreducible 3-manifold with infinite fundamental group. Then M is a Seifert fiber space if and only if $\pi_1 M$ contains a nontrivial, cyclic, normal subgroup.*

THEOREM B. *Let M denote a compact, irreducible, nonorientable 3-manifold with infinite fundamental group. Suppose that M is not a fake $P^2 \times S^1$, and that $\pi_1 M$ does not contain a subgroup isomorphic to $Z_2 * Z_2$. Then M is a Seifert bundle if and only if $\pi_1 M$ contains a nontrivial, cyclic, normal subgroup.*

REMARK. A 3-manifold is a *Seifert bundle* if it admits a decomposition into disjoint circles (*fibers*) each having a regular neighborhood that is either a fibered solid torus or a fibered solid Klein bottle. With this definition, a compact 3-manifold admits a Seifert fibration if and only if it can be foliated by circles ([3], [14]).

As mentioned in the abstract we construct all compact, irreducible, nonorientable 3-manifolds (not containing a fake $P^2 \times I$) that mimic Seifert bundles in the sense that they are not Seifert bundles even though the fundamental group of each of them contains a nontrivial (indeed, infinite), cyclic, normal subgroup. An example of such a manifold is the disk-connected sum $\mathbb{P}$ of $P^2 \times I$ with itself; notice that $\pi_1 \mathbb{P} \cong Z_2 * Z_2$. The remaining such manifolds contain at least one copy of $\mathbb{P}$ and are constructed by gluing together

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copies of $\mathbb{P}$ with Seifert bundles (with nonempty boundaries) in a certain way that we shall describe. We call each of these constructed manifolds a *Seifert bundle* mod $\mathbb{P}$ and point out here that each of them has at least two projective planes as boundary components (there can also be other types of boundary components).

TORUS THEOREM. *If M is an orientable, irreducible 3-manifold and $\mathbb{Z}\oplus\mathbb{Z} \subset \pi_1(M)$, then M contains an incompressible torus or M is a Seifert fiber space.*

For Haken manifolds this was announced by Waldhausen [18] and proved by Feustel [4], [5], Johannson [10], Jaco-Shalen [9]. For compact orientable 3-manifolds, Scott [15] shows that M either has an incompressible torus or $\pi_1(M)$ contains a cyclic normal subgroup. From this the Torus Theorem for compact orientable 3-manifolds follows by the Seifert fiber space conjecture. Gabai [7] also extends the theorem to the noncompact case.

Our final result is the following *torus theorem for nonorientable 3-manifolds.*

THEOREM. *Let M denote a nonorientable, irreducible 3-manifold. If $\mathbb{Z}\oplus\mathbb{Z} \subset \pi_1(M)$, then M contains an incompressible torus or Klein bottle.*

2. **Seifert bundles** mod $\mathbb{P}$. We first start with some examples.

Let $\mathbb{P} = P^2 \times I \triangle P^2 \times I$ be the disk connected sum of two copies of $P^2 \times I$ as in Figure 1 with $\partial\mathbb{P} = P_0^2 \cup P_1^2 \cup K$. Note that the simple closed curve $t = ab$ on the Klein bottle K generates a cyclic normal subgroup in $\pi_1(\mathbb{P}) = \langle a, b : a^2 = b^2 = 1\rangle$. An annulus A on K is *special* if A is parallel on K to a regular neighborhood of t in K.

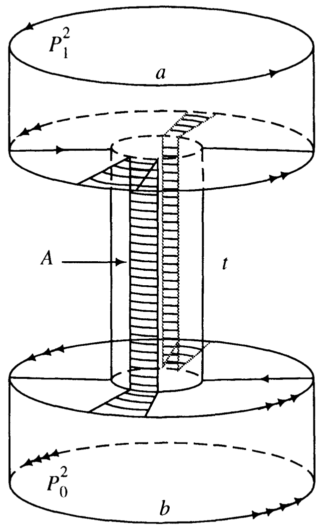

FIGURE 1

EXAMPLE 1. Let M be either a Seifert bundle or a copy of $\mathbb{P}$ and let $A'_1, \ldots, A'_m$ be disjoint annuli on ∂M (not necessarily in the same boundary component). If M is a bundle assume that each A'_i is fibered; if M is a copy of $\mathbb{P}$ assume the A'_i's are special. Let $A_1, \ldots, A_m$ be disjoint parallel copies of A on K. Let W be obtained from M and $\mathbb{P}$ by gluing A'_i to A_i by some homeomorphism for $i = 1, \ldots, n$. Note that $\pi_1(W)$ has a cyclic normal subgroup generated by t.

EXAMPLE 2. Note that the Klein bottle K can be fibered over S^1 with fiber t. Let M be either a Seifert bundle that contains a fibered (without exceptional fibers) Klein bottle K' in its boundary or a copy of $\mathbb{P}$. Let W be obtained from M and $\mathbb{P}$ by identifying K' and K by a fiber preserving homeomorphism.

It is not hard to see that the infinite cyclic normal subgroup of $\pi_1(M)$ corresponding to the fiber remains infinite cyclic in $\pi_1(W)$.

We wish to define a 3-manifold W to be a *Seifert bundle* mod $\mathbb{P}$ if W is obtained by applying the process of Examples 1 and 2 a finite number of times. Note that Example 1 can be thought of as being obtained from example 2 as follows. Let $K \times I$ be a collar of $K = K \times 0$ in $\mathbb{P}$ and if M is a copy of $\mathbb{P}$ let $K' \times I$ be a collar of the Klein bottle boundary $K' = K' \times 0$ of the copy of $\mathbb{P}$. If in Example 1, M is a Seifert bundle, then $K_1 = K \times 1$ splits W into $\mathbb{P}$ and a Seifert bundle. If M is a copy of $\mathbb{P}$, then K_1 and $K'_1 = K' \times 1$ split W into two copies of $\mathbb{P}$ and a Seifert fiber space.

DEFINITION. A 3-manifold W is a *Seifert bundle* mod $\mathbb{P}$ if there is a collection of mutually disjoint Klein bottles $K_1, \ldots, K_s$ in Int(W) that splits W into 3-manifolds $\mathbb{P}_1, \ldots, \mathbb{P}_n$, $M_1, \ldots, M_m$ where each $\mathbb{P}_i$ is homeomorphic to $\mathbb{P}$ and each M_i is a Seifert bundle. Furthermore, each K_i is a boundary component of some $\mathbb{P}_j$ and is fibered (without exceptional fibers) such that a fibered annulus is special in $\mathbb{P}_j$. If K_i lies in $\partial \mathbb{P}_j$ and ∂M_k then this fibering of K_i agrees with the fibering of K_i induced by the fibration of M_k. If K_i lies in $\partial \mathbb{P}_j$ and $\partial \mathbb{P}_k$ then a fibered annulus on K_i is special in $\mathbb{P}_j$ and in $\mathbb{P}_k$.
We allow $n = 0$ or $m = 0$, so that $\mathbb{P}$ and Seifert bundles are also Seifert bundles mod $\mathbb{P}$.

Note that a Seifert bundle mod $\mathbb{P}$ contains an even number (possibly 0) of projective planes in its boundary and every P^2 in this manifold is parallel to the boundary.

LEMMA 1. *Let M be a compact, irreducible 3-manifold which does not contain a fake $P^2 \times I$ and suppose $\pi_1(M)$ contains an infinite cyclic normal subgroup N. If M contains a 2-sided P^2 then either $M = P^2 \times S^1$ or P^2 is parallel to a component of ∂M.*

PROOF. Let P^2_* be a 2-sided P^2 in M. Then P^2_* lifts to an incompressible sphere S^2_* in the 2-fold orientable cover $\tilde{M}$ of M and $\tilde{N} = N \cap \pi_1(\tilde{M})$ is an infinite cyclic normal subgroup of $\pi_1(\tilde{M})$.

CASE (1). S^2_* separates $\tilde{M}$ into $\tilde{M}_1$ and $\tilde{M}_2$. Since P^2_* is 2-sided, the covering transformation c does not interchange the sides of S^2_* and hence P^2 separates M into M_1, M_2, where $\tilde{M}_i$ is the 2-fold orientable cover of M_i. If $\pi_1(\tilde{M}_i) \neq 1$ for $i = 1$ and 2, then since $\tilde{N}$ is a cyclic normal subgroup of $\pi_1(\tilde{M}_1) * \pi_1(\tilde{M}_2)$, we must have $\pi_1(\tilde{M}_i) = \mathbb{Z}_2$ for $i = 1$ and 2. But then $\pi_1(M_i)$ is finite and must also be $\mathbb{Z}_2$, by [2], a contradiction. Therefore

$\pi_1(\tilde{M}_1) = 1$, say. Then $\pi_1(M_1) = \mathbb{Z}_2$ and by [2], $M_1 = P^2 \times I$ with P^2_* as one of the boundary components. Therefore P^2_* is parallel to a boundary component of M.

CASE (2). S^2_* does not separate $\tilde{M}$. Then $\pi_1(\tilde{M}) = \pi_1(\tilde{M} \setminus S^2_*) * \mathbb{Z}$ and since $\tilde{N}$ is a cyclic normal subgroup it follows that $\pi_1(\tilde{M} \setminus S^2_*) = 1$, hence $\pi_1(M \setminus P^2_*) = \mathbb{Z}_2$. As before, $M \setminus P^2_* = P^2 \times I$ and therefore $M = P^2 \times S^1$. ■

LEMMA 2. *Let M be as in Lemma 1 and let $\hat{M}$ be obtained from the 2-fold orientable cover $\tilde{M}$ by capping off the boundary 2-spheres with 3-balls. Then $\hat{M}$ is either $S^2 \times S^1$ or irreducible.*

PROOF. Assuming that $\hat{M} \neq S^2 \times S^1$ it suffices to show that every S^2 in $\mathrm{Int}(\tilde{M})$ bounds a (punctured) ball in $\tilde{M}$ (where the punctures are components of $\partial\tilde{M}$).

Suppose S is a 2-sphere in $\mathrm{Int}(\tilde{M})$. Let $c: \tilde{M} \longrightarrow \tilde{M}$ be the non-trivial covering transformation. By an isotopy we can assume that either $c(S) = S$ or $c(S) \cap S$ consists of simple closed curves. If $c(S) = S$ then S covers a P^2 in M which by Lemma 1 is parallel to the boundary of M. But then S bounds the punctured ball $S^2 \times I$ in $\tilde{M}$. So assume S does not bound a punctured ball in $\tilde{M}$ and $c(S) \cap S$ consists of n simple closed curves, where S is chosen so that in addition n is minimal. If $n > 0$, let D be an innermost disk on $c(S)$. Then ∂D bounds a disk D' on S. Let $S_1 = S \setminus D' \cup D$ and $S_2 = D \cup D'$. By a small isotopy (see [16]), either $S_j = c(S_j)$ or $S_j \cap c(S_j)$ has fewer than n components. Moreover, at least one of S_1, S_2 does not bound a punctured ball in $\tilde{M}$, say S_1. As above, $S_1 \neq c(S_1)$, and so $S_1 \cap c(S_1)$ has fewer than n components. Hence for our original sphere S, we have $n = 0$ and $S \cap c(S) = \emptyset$. Therefore S covers a 2-sphere in M that bounds a 3-ball in M and so S bounds a 3-ball in $\tilde{M}$. Thus $\hat{M}$ is irreducible. ■

The next result is a reformulation of the *SFS*-conjecture (Theorem B) for nonorientable 3-manifolds.

THEOREM 1. *Let M be a compact, irreducible, nonorientable 3-manifold that does not contain a fake $P^2 \times I$. Then $\pi_1(M)$ contains a nontrivial cyclic normal subgroup iff M is either $P^2 \times I$ or a Seifert bundle* mod $\mathbb{P}$.

PROOF. If $\pi_1(M)$ is finite then $M = P^2 \times I$ (by [2]). So we assume that $\pi_1(M)$ is infinite. By [19, Proof of Theorem 1] we can also assume that the cyclic normal subgroup N is infinite. Let $p: \tilde{M} \longrightarrow M$ be the 2-fold orientable covering and $c: \tilde{M} \longrightarrow \tilde{M}$ the covering transformation. Then c extends to an involution $\hat{c}: \hat{M} \longrightarrow \hat{M}$ of the manifold $\hat{M}$ obtained from $\tilde{M}$ by filling in the boundary spheres with 3-balls, such that $\hat{c}$ has one isolated fixed point for each such 3-ball. Now $\tilde{N} = p_*^{-1}(N)$ is an infinite cyclic normal subgroup of $\pi_1(\hat{M})$ and since $\hat{M}$ is irreducible (by Lemma 2) it follows from Theorem A that $\hat{M}$ is a Seifert fiber space. By the argument in the proof of Theorem 1 of [19], $\hat{c}_*$ leaves the subgroup of $\pi_1(\hat{M})$ that is generated by a fiber H invariant and therefore it follows from [12] that $\hat{M}$ has an $\hat{c}$-invariant Seifert fibration. (If $\hat{M}$ is different from a Seifert fiber space over $S^2$with three exceptional fibers, this already follows from [17].)

If $\hat{c}$ contains no fixed points, *i.e.*, if $\hat{M} = \tilde{M}$, then M is a Seifert bundle.

Suppose P is a fixed point of $\hat{c}$. If H is the fiber containing P then $\hat{c}(H) = H$ and there is exactly one other fixed point Q on H such that $\hat{c}: H \longrightarrow H$ is reflection on $P \cup Q$. Let $P_1, \ldots, P_m, Q_1, \ldots, Q_m$ be all the fixed points of $\hat{c}$, where P_i and Q_i lie on the $\hat{c}$-invariant fiber H_i $(c = 1, \ldots, m)$. Let N_i be an $\hat{c}$-invariant fibered solid torus neighborhood of H_i not containing any exceptional fibers (except possibly for H_i itself) and such that $N_i \cap N_j = \emptyset$ for $i \neq j$. We can assume that $N_i \subset \operatorname{Int} \hat{M}$. Represent N_i as $D^2 \times S^1$ with $D^2 = \left\{ z \in \mathbb{C} \mid |z| \leq 1 \right\}$ and $S^1 = \partial D^2$, and let $\hat{\alpha}: D^2 \longrightarrow D^2$ be $\hat{\alpha}(z) = -z$, $k: S^1 \longrightarrow S^1$ be $k(z) = \bar{z}$. Then by [8, p. 898], we can assume that $\hat{c}: N_i \longrightarrow N_i$ is the map $\hat{\alpha} \times k$. This is illustrated in Figure 2, where $p: N_i \longrightarrow p(N_i) = \mathbb{P}_i$ with $\mathbb{P}_i \cong \mathbb{P}$ and where the $\hat{c}$-invariant 3-balls B_{i1}, B_{i2} must be removed to get the covering $p: \tilde{M} \longrightarrow M$. A_{i1}, A_{i2} on ∂N_i map down to a special annulus A_i. The Klein bottle $p(\partial N_i)$ splits off a copy of $\mathbb{P}$ in M. It follows that M is a Seifert bundle mod $\mathbb{P}$. ∎

We now see that Thurston's Geometrization Conjecture holds for Seifert bundles mod $\mathbb{P}$. In fact, we have the following result.

COROLLARY 1. *Let M denote a compact, irreducible* 3*-manifold that does not contain a fake $P^2 \times I$ and whose fundamental group is infinite if M is orientable. If $\pi_1(M)$ contains a nontrivial, cyclic, normal subgroup, then Thurston's Geometrization Conjecture holds for M.*

PROOF. By Theorem A and Theorem 1, the manifold M is either Seifert fibered, or $P^2 \times I$, or a Seifert bundle mod $\mathbb{P}$ that is not Seifert fibered. In the first two cases, M is geometrically modeled on a Seifert geometry. In the latter case, $\hat{M}$ (as in the proof of Theorem 1) is Seifert fibered and the involution $\hat{c}: \hat{M} \longrightarrow \hat{M}$ preserves some Seifert fibration of $\hat{M}$, and so $\hat{c}$ preserves the geometric structure of $\hat{M}$ [12; p. 291]. It follows that the Geometrization Conjecture holds for M (see [14; §6]). ∎

REMARK. A special case of the above proof is when $M = \mathbb{P}$. There is a properly imbedded disk in $\mathbb{P}$ compressing the Klein bottle boundary component and splitting $\mathbb{P}$ into two copies of $P^2 \times I$, which of course has geometric structure. We did not mention this above, since when $M = \mathbb{P}$, $\hat{M}$ is a solid torus whose trivial fibration is $\hat{c}$-invariant, and the above proof holds. The point here is that there are *two* ways to show that the Geometrization Conjecture holds for $\mathbb{P}$. Similarly, to show the Geometrization Conjecture for M a Seifert bundle mod $\mathbb{P}$ one could split M by the Klein bottles (in the definition of Seifert bundle mod $\mathbb{P}$) into copies of $\mathbb{P}$ and Seifert bundles, where each piece clearly has a geometric structure.

3. **The Torus Theorem for nonorientable 3-manifolds.**

The Torus Theorem for compact nonorientable 3-manifolds M is proved by applying the Torus Theorem to the 2-fold orientable cover $\tilde{M}$ of M. If $\tilde{M}$ is irreducible and if $\pi_1(\tilde{M})$ contains $\mathbb{Z} \oplus \mathbb{Z}$, but $\tilde{M}$ does not contain an incompressible torus, then $\tilde{M}$ is a "small" Seifert fiber space, *i.e.*, $\tilde{M}$ does not contain a vertical torus. In this case the orbit surface S of $\tilde{M}$ is either S^2, D^2 or P^2 and we have one of the following cases, where n is the number of exceptional fibers of $\tilde{M}$:

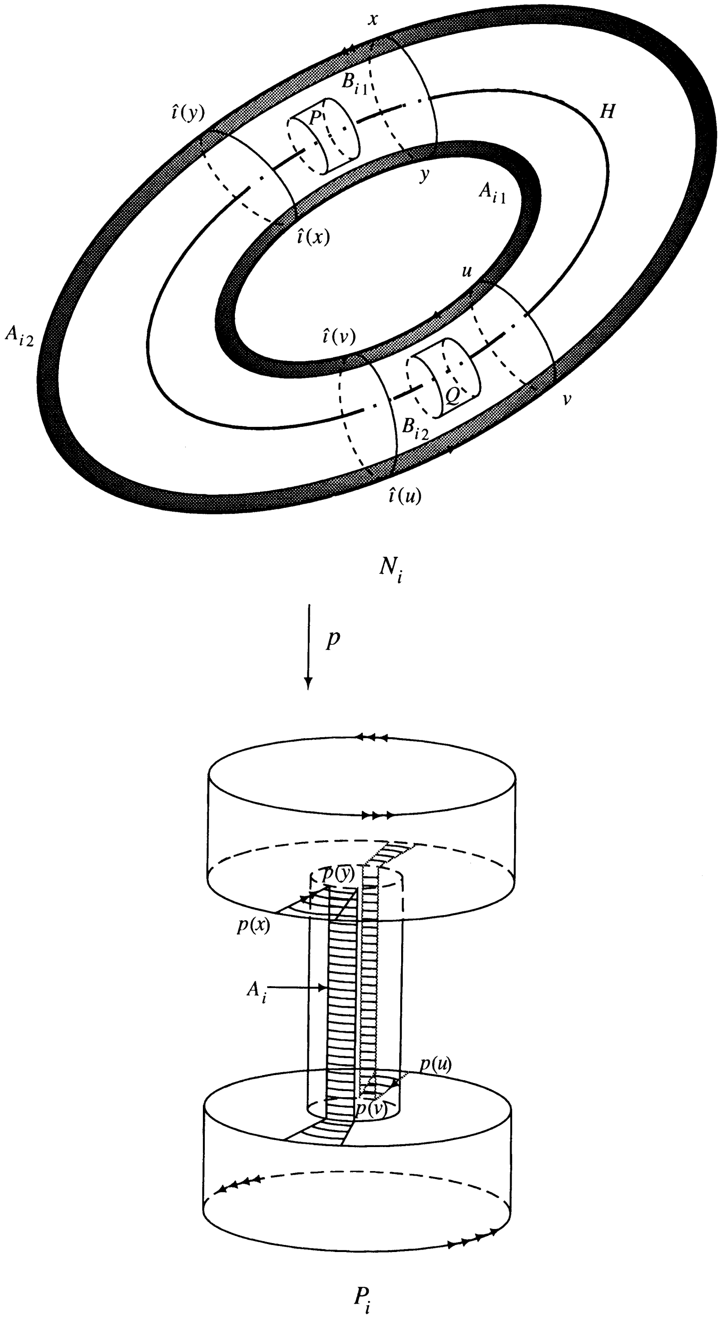

FIGURE 2

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(i) $S = S^2, n \leq 3$.
(ii) $S = D^2, n \leq 1$.
(iii) $S = P^2, n \leq 1$.

In case (i) $n = 3$, since otherwise $\tilde{M}$ is a lens space. Also case (ii) cannot happen since here $\tilde{M}$ is a solid torus. In case (iii) $\pi_1(M)$ is either $\mathbb{Z}_2 * \mathbb{Z}_2$ or finite [13; (6.2)], hence does not contain $\mathbb{Z} \oplus \mathbb{Z}$. Thus the only small Seifert fiber spaces that do not contain an incompressible torus but whose fundamental groups contain $\mathbb{Z} \oplus \mathbb{Z}$ are those in the next lemma.

LEMMA 3. *Let $\tilde{M}$ be a Seifert fiber space with orbit surface S^2 and with three exceptional fibers. Then $\tilde{M}$ does not admit an orientation reversing and fiber preserving involution i with at most isolated fixed points.*

PROOF. Suppose $c: \tilde{M} \longrightarrow \tilde{M}$ is an orientation reversing, fiber preserving involution. Then c permutes the exceptional fibers and therefore there is an exceptional fiber $\bar{H}$ which is c-invariant. Let V be an invariant fibered solid torus neighborhood of $\bar{H}$. If c has no fixed points on $\bar{H}$ we can assume that c has no fixed points on V; hence $c|V$ is a covering translation and $c: V \longrightarrow V$ is orientation reversing. But then V would cover a fibered solid Klein bottle and that fibering lifts to a trivial fibering of V, a contradiction. Therefore $\bar{H}$ contains a fixed point of c and we have the situation of Figure 2 with $c = \hat{\alpha} \times k$. For canonical generators $a = 1 \times S^1 \subset D^2 \times S^1 = V$ and $b = \partial D^2 \times S^1$ of $\pi_1(\partial V)$ we have $c_*(a) = a^{-1}$, $c_*(b) = b$. Since $\bar{H}$ is exceptional, a regular fiber $H \sim a^\mu b^\nu$ with $|\mu| > 1$. But then $c_*(H) = H^{\pm 1}$ since c is fiber preserving and $c_*(H) = a^{-\mu} b^\nu$. This implies that $\mu = 0$ or ± 1, a contradiction. ∎

THEOREM 2 (TORUS THEOREM). *If M is an irreducible nonorientable 3-manifold and $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(M)$ then M contains an incompressible torus or Klein bottle.*

PROOF: CASE (1). M is compact and P^2-irreducible. The squares of the two generators of $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$ lift to loops in the 2-fold orientable cover $\tilde{M}$ of M and generate a subgroup $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(\tilde{M})$. By the Torus Theorem (for the orientable irreducible case) $\tilde{M}$ either contains an incompressible torus or $\tilde{M}$ is a small Seifert fiber space. In the first case, by [11, Corollary (3.14)], $\tilde{M}$ contains an c-equivariant incompressible torus, where c is the covering translation. Therefore M contains an incompressible torus or Klein bottle.

The second case cannot happen by Lemma 3 and the discussion preceding Lemma 3.

CASE (2). M is compact irreducible and contains projective planes.

(2a) Every P^2 in M is parallel to ∂M.

Let $\hat{M}$ be obtained from $\tilde{M}$ by capping off the 2-spheres of $\partial \tilde{M}$ with 3-balls. By the proof of Lemma 2, $\hat{M}$ is irreducible and $\pi_1(\hat{M})$ contains a $\mathbb{Z} \oplus \mathbb{Z}$. The covering map c on $\tilde{M}$ extends to an involution $\hat{c}$ with isolated fixed points. Now the argument of Case 1 applies: By Luft's result $\hat{M}$ contains an $\hat{c}$-equivariant incompressible torus T that is disjoint from the fixed points. Hence $T \subset \tilde{M}$ projects to an incompressible torus or Klein bottle in M.

(2b) M contains non-boundary parallel P^2's.

Let φ be a maximal collection of non-parallel and non-boundary parallel projective planes in M. Then $M \setminus \varphi = M_1 \cup \cdots \cup M_n$ and $\tilde{M} \setminus p^{-1}(\varphi) = \tilde{M}_1 \cup \cdots \cup \tilde{M}_n$ (where $p: \tilde{M} \longrightarrow M$ is the covering map), hence $\pi_1(\tilde{M}) = \pi_1(\tilde{M}_1) * \cdots * \pi_1(\tilde{M}_n) * \mathbb{Z} * \cdots * \mathbb{Z}$ (where the $\mathbb{Z}$-factors come from 2-spheres in the boundary of a $\tilde{M}_i$ which are identified in $\tilde{M}$). Therefore $\mathbb{Z} \oplus \mathbb{Z} \subset \pi_1(\tilde{M}_k)$ for some k. Now M_k and $\tilde{M}_k$ are as in case (a). So there is an incompressible torus or Klein bottle in M_k and hence in M.

CASE (3). M is not compact.

This case is reduced to the compact case by the argument in the proof of Corollary (9.6) in [7]. ■

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