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# MAXIMUM SIZE OF SUBSETS OF PAIRWISE NONCOMMUTING ELEMENTS IN FINITE METACYCLIC *p*-GROUPS

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**Abstract**

Let $G$ be a finite group. A subset $X$ of $G$ is a set of pairwise noncommuting elements if any two distinct elements of $X$ do not commute. In this paper we determine the maximum size of these subsets in any finite nonabelian metacyclic $p$-group for an odd prime $p$.

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## 1. Introduction

Let $G$ be a finite nonabelian group and let $X$ be a subset of pairwise noncommuting elements of $G$ such that $|X| \geq |Y|$ for any other set of pairwise noncommuting elements $Y$ in $G$. Then the subset $X$ is said to have the maximum size, and this size is denoted by $\omega(G)$. Also $\omega(G)$ is the maximum clique size in the noncommuting graph of a finite group $G$. Let $Z(G)$ be the centre of $G$. The noncommuting graph of a group $G$ is defined as a graph whose $G \backslash Z(G)$ is the set of vertices and two vertices are joined if and only if they do not commute. Various attempts have been made to find $\omega(G)$ for some groups $G$; see, for example, [1, 2, 5, 7, 10, 11]. Moreover, there is a connection between sets of pairwise noncommuting elements in a finite group $G$ and coverings of $G$. Following [4, Section 116], we say that a finite group $G$ is covered by proper subgroups $A_1, \ldots, A_n$ if $G = A_1 \cup \cdots \cup A_n$. As a matter of fact, if $G$ is covered by $n$ proper abelian subgroups, then $\omega(G) \leq n$ since two elements that do not commute cannot be in the same abelian subgroup. Answering a question of Erdős and Straus [6], Mason [9] has shown that any finite group $G$ can be covered with at most $\lfloor |G|/2 \rfloor + 1$ abelian subgroups.

In this paper we prove the following main theorem.

Theorem 1.1. *Let $G$ be a finite nonabelian metacyclic $p$-group with $p > 2$. Then $\omega(G) = |G'|(1 + p)/p$.*

To prove this theorem for a finite nonabelian metacyclic $p$-group $G$ with $p > 2$, our main strategy is to find an upper bound and a lower bound for $\omega(G)$, both of which are equal to $|G'|(1 + p)/p$. For an upper bound we give an abelian covering for $G$ and for a lower bound we find a set of pairwise noncommuting elements in $G$. We note that maximal subgroups of $G$ play an important role in our proofs.

Throughout this paper the following notation is used. All groups are assumed to be finite. The letter $p$ denotes a prime number. In a $p$-group $G$, we define $\mho_i(G) = \langle x^{p^i} \mid x \in G\rangle$. The minimal number of generators of $G$ is denoted by $d(G)$. We write $[a, b]$ for $a^{-1}b^{-1}ab$. Also, a minimal nonabelian group is a nonabelian group such that all its proper subgroups are abelian.

## 2. Some basic results

In this section we give some basic results for metacyclic $p$-groups that are needed for the main results of the paper.

Let $G$ be a finite metacyclic $p$-group. We know that there exists a normal cyclic subgroup $\langle a\rangle$ of $G$ such that $G/\langle a\rangle$ is cyclic. Therefore we may choose an element $b \in G$ and a number $k \geq 1$ such that $G = \langle b, a\rangle$ and $b^{-1}ab = a^k$ and so any element of $G$ has the form $b^j a^i$ for $j, i \geq 0$.

For the rest of the paper we fix the above notation.

Lemma 2.1. *Let $G$ be a nonabelian metacyclic $p$-group. Then:*

(i) $k \equiv 1 \pmod p$;
(ii) $[a^i, b^j] = [a, b]^{i(1+k+\cdots+k^{j-1})}$ *for* $i, j \geq 1$;
(iii) $G' = \langle [a, b]\rangle$;
(iv) *any two arbitrary elements* $x = b^j a^i$ *and* $y = b^s a^r$ *in $G$ commute if and only if* $(1 + k + \cdots + k^{s-1})i \equiv (1 + k + \cdots + k^{j-1})r \pmod{|G'|}$, *where* $i, j, r, s \geq 0$ *and we take* $1 + k + \cdots + k^{m-1} = 0$ *when* $m = 0$;
(v) $(ba^i)^n = b^n a^{i(1+k+\cdots+k^{n-1})}$ *for* $i, n \geq 1$;
(vi) $\Phi(G) = \langle b^p, a^p\rangle$.

Proof. (i) Obviously $G' \leq \langle a\rangle$ and $\langle a^{k-1}\rangle \leq G'$. Now if $(p, k - 1) = 1$, then $G' = \langle a\rangle$, a contradiction.
(ii) This follows from $b^{-1}ab = a^k$.
(iii) We have $G' = \langle [x, y] \mid x, y \in G\rangle$, which completes the proof by using (ii).
(iv) This is a consequence of (ii).
(v) We use induction on $n$.
(vi) On setting $H = \langle a^p, b^p\rangle$, we see that $G' \leq H$ by (i) and (iii) and so $|G/H| \leq p^2$. Now we can complete the proof since $H \leq \Phi(G)$ and $d(G) = 2$. □

Following [3, Section 26], we state the definition of powerful $p$-groups for $p > 2$ and some of their properties which will be used in the following. Let $p > 2$. A finite $p$-group $G$ is said to be powerful if $\mho_1(G) = \Phi(G)$.

Lemma 2.2. *Let $G$ be a metacyclic $p$-group with $p > 2$. Then:*

(i) *see $G$ and all its subgroups are powerful;*
(ii) *see $G = \langle a \rangle \langle b \rangle$;*
(iii) *see $\mho_i(G) = \{x^{p^i} | x \in G\}$ for $i \geq 1$.*

Proof. (i) [3, Section 26, Exercise 1].
(ii) [3, Corollary 26.12].
(iii) [3, Proposition 26.10]. □

## 3. Maximal subgroups

In this section we proceed to find all maximal subgroups of a noncyclic metacyclic $p$-group $G$ for an odd prime $p$. Obviously the number of maximal subgroups of $G$ is $1 + p$ since $d(G) = 2$.

First we state the following lemma due to Berkovich [4].

Lemma 3.1 [4, Lemma 124.27]. *Let $G$ be a nontrivial metacyclic $p$-group and suppose that $A < B \leq G$. Then $A' < B'$ unless $B$ is abelian.*

Corollary 3.2. *Let $G$ be a metacyclic $p$-group and $|G'| = p$. Then $G$ is minimal nonabelian.*

Proof. By Lemma 3.1, for any maximal subgroup $M$ of $G$ we have $M' < G'$, as desired. □

Theorem 3.3. *Let $G$ be a noncyclic metacyclic $p$-group with $p > 2$. Then:*

(i) *$K_1 = \langle b, a^p \rangle$, $K_2 = \langle b^p, a \rangle$ and $H_i = \langle ba^i, a^p \rangle$, for $1 \leq i \leq p-1$, are all distinct maximal subgroups of $G$;*
(ii) *$|K_j'| = |H_i'| = |G'|/p$, for $1 \leq j \leq 2$ and $1 \leq i \leq p-1$, when $G$ is not abelian.*

Proof. (i) First we see that $\Phi(G) < K_j < G$, for $1 \leq j \leq 2$, by Lemma 2.1(vi). Also, by considering $(ba^i)^p$, we see that $b^p \in H_i$, for $1 \leq i \leq p-1$, by Lemma 2.1(v), (i). Therefore $\Phi(G) < H_i < G$ for $1 \leq i \leq p-1$. Moreover, it is easy to check that these maximal subgroups are distinct.

(ii) Obviously $\langle [a^p, ba^i] \rangle$ and $\langle [a^p, b] \rangle$ are subgroups of $H_i'$ and $K_1'$ respectively and are of order $|G'|/p$ by Lemma 2.1(ii), (iii), for $1 \leq i \leq p-1$. Also, by Lemma 2.1(i), (ii), (iii), $\langle [a, b^p] \rangle \leq K_2'$ is of order $|G'|/p$ since $p$ divides $1 + \cdots + k^{p-1}$ and $p^2$ does not divide $1 + \cdots + k^{p-1}$. Moreover, by Lemma 3.1, we see that $H_i' < G'$ and $K_j' < G'$, for $1 \leq i \leq p-1$ and $1 \leq j \leq 2$, which completes the proof. □

Corollary 3.4. *If $G$ is a nonabelian metacyclic $p$-group with $p > 2$ and $G$ possesses an abelian maximal subgroup, then $|G'| = p$.*

Proof. This follows from Theorem 3.3(ii). □

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## 4. Covering and pairwise noncommuting elements

Let $G$ be a finite nonabelian metacyclic $p$-group with $p > 2$. In this section we show that $\omega(G) = |G'|(1 + p)/p$. Our strategy is to find an upper bound and a lower bound for $\omega(G)$ both of which are equal to $|G'|(1 + p)/p$. To find an upper bound for $\omega(G)$, we cover $G$ by its abelian subgroups; in fact any maximal subgroup of $G$ is covered by abelian subgroups. To find a lower bound, we give a set of pairwise noncommuting elements in $G$, again by finding a set of pairwise noncommuting elements in each maximal subgroup of $G$. We note that if $|G'| = p$, then by [4, Lemma 116.1(a)], $\omega(G) = 1 + p$ since $G$ is minimal nonabelian.

Lemma 4.1. *Let $G$ be a powerful $p$-group with $p > 2$ and $G = M_1 \cup \cdots \cup M_t \cup \Phi(G)$, where $M_i$ are subgroups of $G$. Then $G = M_1 \cup \cdots \cup M_t$.*

Proof. Assume that $1 \neq x \in \Phi(G)$. It is enough to show that $x \in M_i$ for some $1 \leq i \leq t$. Since $G$ is finite, we may write $1 = \mho_s(G) \leq \cdots \leq \mho_2(G) \leq \mho_1(G) = \Phi(G)$ for some $s > 1$. Therefore there exists $1 \leq k \leq s - 1$ such that $x \in \mho_k(G) \setminus \mho_{k+1}(G)$. By [3, Proposition 26.10], $x = g^{p^k}$ for some $g \in G$. We see that $g \notin \mho_1(G) = \Phi(G)$. Therefore $g \in M_i$ for some $1 \leq i \leq t$ and so $x \in M_i$. □

Theorem 4.2. *Let $G$ be a nonabelian metacyclic $p$-group with $p > 2$ and $|G'| = p^n$. Then any maximal subgroup of $G$ is covered by $\Phi(G)$ and $p^{n-1}$ abelian subgroups of $G$.*

Proof. We use induction on $n$. For $n = 1$ this is obvious by Corollary 3.2. Now assume that $n \geq 2$ and that the result holds for any nonabelian metacyclic $p$-group with the derived subgroup of order $p^{n-1}$. Let $H$ be a maximal subgroup of $G$. Then by Corollary 3.4, $H$ is not abelian and so $H$ has $1 + p$ maximal subgroups, of which $\Phi(G)$ is one. Therefore $H = M_1 \cup \cdots \cup M_p \cup \Phi(G)$, where the elements of the union are all maximal subgroups of $H$. Now by Theorem 3.3(ii), $|H'| = p^{n-1}$ and so by the induction hypothesis $M_i = \bigcup_{j=1}^{p^{n-2}} A_{ij} \cup \Phi(H)$, where $A_{ij}$ is abelian for $1 \leq i \leq p$ and $1 \leq j \leq p^{n-2}$. Hence we can complete the proof by the fact that $\Phi(H) \leq \Phi(G)$. □

Corollary 4.3. *If $G$ is a nonabelian metacyclic $p$-group with $p > 2$, then $G$ is covered by $|G'|(1 + p)/p$ abelian subgroups. Therefore $\omega(G) \leq |G'|(1 + p)/p$.*

Proof. It is clear that $G = H_1 \cup \cdots \cup H_{1+p}$, where $H_1, \ldots, H_{1+p}$ are all maximal subgroups of $G$. Therefore by Theorem 4.2, $G$ is covered by $|G'|(1 + p)/p$ abelian subgroups and $\Phi(G)$. Now the result follows immediately from Lemma 2.2(i) and Lemma 4.1. □

Now we proceed to find the lower bound for $\omega(G)$.

Lemma 4.4. *Let $G$ be a nonabelian metacyclic $p$-group with $p > 2$ and $H$, $K$ be two distinct maximal subgroups of $G$. Then:*

(i) *for any $x \in H \setminus \Phi(G)$ and any $y \in K \setminus \Phi(G)$, $xy \neq yx$;*
(ii) *$H \cap K = \Phi(G)$.*

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Proof. (i) By Theorem 3.3(i), distinct maximal subgroups of $G$ are $K_1 = \langle b, a^p\rangle$, $K_2 = \langle b^p, a\rangle$ and $H_i = \langle ba^i, a^p\rangle$ for $1 \le i \le p-1$. Therefore we may assume, for example, that $H = H_i$ and $K = H_j$ for $1 \le i < j \le p-1$. Now if $x \in H \setminus \Phi(G)$ and $y \in K \setminus \Phi(G)$, then by Lemmas 2.1(vi) and 2.2(i), (ii), $x = (ba^i)^n a^{pm}$ and $y = (ba^j)^r a^{ps}$, where $n \ne 0$, $r \ne 0$, $(n, p) = (r, p) = 1$ and $m, s \ge 0$. So by way of contradiction, if $xy = yx$, then

$$\begin{aligned}(i(1 + k + \cdots + k^{n-1}) + pm)&(1 + k + \cdots + k^{r-1})\\ &\equiv (j(1 + k + \cdots + k^{r-1}) + ps)(1 + k + \cdots + k^{n-1}) \pmod{|G'|},\end{aligned}$$

by Lemma 2.1(v), (iv). This implies that $inr \equiv jrn \pmod{p}$, by using Lemma 2.1(i), a contradiction. The proof of other cases for $H$ and $K$ is the same as above.

(ii) This is evident. □

Theorem 4.5. *Let $G$ be a nonabelian metacyclic $p$-group $(p > 2)$ with $|G'| = p^n$, where $n \ge 2$ and let $H$ be a maximal subgroup of $G$. Then there exist $p^{n-1}$ pairwise noncommuting elements in $H \setminus \Phi(G)$.*

Proof. We use induction on $n$. For $n = 2$, by Theorem 3.3(ii), we see that $|H'| = p$ and so $H$ is minimal nonabelian by Corollary 3.2. Therefore $\omega(H) = 1 + p$ by [4, Lemma 116.1(a)]. Hence there exist $p$ pairwise noncommuting elements in $H \setminus \Phi(G)$ since $\Phi(G)$ is abelian. Now suppose that $n \ge 3$ and that the result holds for any nonabelian metacyclic $p$-group with the derived subgroup of order $p^{n-1}$. Let $M_1, \ldots, M_{1+p}$ be all distinct maximal subgroups of $H$; obviously we may assume that $M_{1+p} = \Phi(G)$. By Theorem 3.3(ii), $|H'| = p^{n-1}$ and so $H$ is nonabelian. Now by using the induction hypothesis for $H$, we see that there exists a subset $A_i$ of pairwise noncommuting elements in $M_i \setminus \Phi(H)$ such that $|A_i| = p^{n-2}$ for $1 \le i \le p$. On setting $A = A_1 \cup \cdots \cup A_p$, we see that $A \subseteq H \setminus \Phi(G)$ and $|A| = p^{n-1}$ by Lemma 4.4(ii). Moreover, by Lemma 4.4(i), elements of $A_i$ and $A_j$ do not commute for $1 \le i < j \le p$, as desired. □

Corollary 4.6. *If $G$ is a nonabelian metacyclic $p$-group with $p > 2$, then $|G'|(1 + p)/p \le \omega(G)$.*

Proof. If $|G'| = p$, then by Corollary 3.2, $G$ is minimal nonabelian and so $\omega(G) = 1 + p$ by [4, Lemma 116.1(a)]. Now let $|G'| = p^n$ and $n \ge 2$. Assume that $H_1, \ldots, H_{1+p}$ are all maximal subgroups of $G$. Then by Theorem 4.5, there exists a subset $A_i$ of pairwise noncommuting elements in $H_i \setminus \Phi(G)$ such that $|A_i| = p^{n-1}$, for $1 \le i \le p + 1$. On setting $A = A_1 \cup \cdots \cup A_{1+p}$, we see that $|A| = (1 + p)p^{n-1}$ by Lemma 4.4(ii) and $A$ is a set of pairwise noncommuting elements in $G$ by Lemma 4.4(i), as desired. □

Proof of Theorem 1.1. This is an immediate consequence of Corollaries 4.3 and 4.6.

Remark. We note that Theorem 1.1 does not hold for nonabelian metacyclic 2-groups. To verify this we use GAP [8]. The notation group$(m, n)$ is used for the $n$th group of order $m$ as quoted in the 'Small Groups' library of GAP. For example, if $G =$

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group(16, 7), then $|G'| = 4$ and $\omega(G) = 5$. If $G$ = group(32, 18), then $|G'| = 8$ and $\omega(G) = 9$. If $G$ = group(64, 46), then $|G'| = 8$ and $\omega(G) = 11$.

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