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A SHARP L_2 INEQUALITY OF OSTROWSKI TYPE

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Abstract

A sharp L_2 inequality of Ostrowski type is established, which provides a generalization of some previous results and gives some other interesting results as special cases. Applications in numerical integration are also given.

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1. Introduction

In [1] and [2], we may find the following two interesting sharp bounds for the errors in the corrected trapezoid rule and corrected midpoint rule.

THEOREM 1.1. *Let $f : [a, b] \to \mathbb{R}$ be such that f' is absolutely continuous on $[a, b]$ and $f'' \in L_2[a, b]$. Then*

$$\left| \int_a^b f(t)\, dt - \frac{b-a}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(b) - f'(a)] \right| \le \frac{(b-a)^{(5/2)}}{12\sqrt{5}} \sqrt{\sigma(f'')}, \tag{1.1}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\left(\int_a^b f(t)\, dt\right)^2 \tag{1.2}$$

and $\|f\|_2 := [\int_a^b f^2(t)\, dt]^{(1/2)}$. Inequality (1.1) is sharp in the sense that the constant $(1/(12\sqrt{5}))$ cannot be replaced by a smaller one.

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THEOREM 1.2. *Under the assumptions of Theorem 1.1,*

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^{(5/2)}}{12\sqrt{5}}\sqrt{\sigma(f'')}. \tag{1.3}$$

Inequality (1.3) is sharp in the sense that the constant $1/12\sqrt{5}$ *cannot be replaced by a smaller one.*

In this work, we will derive a new sharp inequality of Ostrowski type for functions whose first derivatives are absolutely continuous and whose second derivatives belong to $L_2(a,b)$. This will not only provide a generalization of inequalities (1.1) and (1.3), but will also give some other interesting sharp inequalities as special cases. Moreover, we show that the corrected Simpson rule (see [3–5]) gives better results than the Simpson rule and, in particular, the corrected averaged midpoint-trapezoid quadrature rule is optimal. Applications in numerical integration are also given.

2. The results

THEOREM 2.1. *Let the assumptions of Theorem 1.1 hold. Then for any* $\theta\in[0,1]$ *and* $x\in[a,b]$*,*

$$\begin{aligned}&\left|\int_a^b f(t)\,dt-(b-a)\left[(1-\theta)f(x)+\theta\frac{f(a)+f(b)}{2}\right]\right.\\&+(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x)\\&\left.-\left[\frac{1-\theta}{2}\left(x-\frac{a+b}{2}\right)^2+\frac{1-3\theta}{24}(b-a)^2\right][f'(b)-f'(a)]\right|\\&\quad\leq\left[\frac{\theta(1-\theta)}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4+\frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x-\frac{a+b}{2}\right)^2\right.\\&\qquad\left.+\frac{15\theta^2-15\theta+4}{2880}(b-a)^5\right]^{(1/2)}\sqrt{\sigma(f'')}.\end{aligned} \tag{2.1}$$

The inequality (2.1) is sharp in the sense that the coefficient constant 1 *of the right-hand side cannot be replaced by a smaller one.*

PROOF. Let us define the function

$$K(x,t):=\begin{cases}\dfrac{(t-a)^2}{2}-\dfrac{\theta(b-a)}{2}(t-a), & t\in[a,x],\\[2mm] \dfrac{(t-b)^2}{2}+\dfrac{\theta(b-a)}{2}(t-b), & t\in(x,b].\end{cases}$$

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Integrating by parts, we obtain

$$\int_a^b K(x,t)f''(t)\,dt = \int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] + (1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x). \tag{2.2}$$

We also have

$$\int_a^b K(x,t)\,dt = \frac{1-\theta}{2}(b-a)\left(x-\frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^3 \tag{2.3}$$

and

$$\int_a^b f''(t)\,dt = f'(b) - f'(a). \tag{2.4}$$

From (2.2)–(2.4), it follows that

$$\begin{aligned}\int_a^b &\left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right]\left[f''(t) - \frac{1}{b-a}\int_a^b f''(s)\,ds\right]dt \\ &= \int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] \\ &\quad + (1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x) \\ &\quad - \left[\frac{1-\theta}{2}\left(x-\frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^2\right][f'(b)-f'(a)].\end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned}&\left|\int_a^b \left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right]\left[f''(t) - \frac{1}{b-a}\int_a^b f''(s)\,ds\right]dt\right| \\ &\quad \le \left\|K(x,\cdot) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right\|_2 \left\|f'' - \frac{1}{b-a}\int_a^b f''(s)\,ds\right\|_2.\end{aligned} \tag{2.6}$$

We also have

$$\begin{aligned}&\left\|K(x,\cdot) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right\|_2^2 \\ &\quad = \frac{\theta(1-\theta)}{4}(b-a)\left(x-\frac{a+b}{2}\right)^4 + \frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x-\frac{a+b}{2}\right)^2 \\ &\qquad + \frac{15\theta^2-15\theta+4}{4}(b-a)^5\end{aligned} \tag{2.7}$$

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and

$$\left\| f'' - \frac{1}{b-a}\int_a^b f''(s)\,ds \right\|_2^2 = \|f''\|_2^2 - \frac{(f'(b)-f'(a))^2}{b-a}. \tag{2.8}$$

From (2.5)–(2.8), we can easily get (2.1), since by (1.2),

$$\sqrt{\sigma(f'')} = \left[\|f''\|_2^2 - \frac{(f'(b)-f'(a))^2}{b-a} \right]^{(1/2)}.$$

In order to prove that the inequality (2.1) is sharp, we define the function

$$f(t) = \begin{cases} \dfrac{t^4}{24} - \dfrac{\theta t^3}{12}, & t \in [0, x], \\ \dfrac{(t-1)^4}{24} + \dfrac{\theta(t-1)^3}{12} + \left[\dfrac{1-\theta}{2}\left(x - \dfrac{1}{2}\right)^2 + \dfrac{1-3\theta}{24}\right] & \\ \quad \times\left(t - \dfrac{1}{2}\right) - \dfrac{1-\theta}{3}\left(x - \dfrac{1}{2}\right)^3, & t \in (x, 1], \end{cases} \tag{2.9}$$

where $x \in [0, 1]$. It follows that

$$f'(t) = \begin{cases} \dfrac{t^3}{6} - \dfrac{\theta t^2}{4}, & t \in [0, x], \\ \dfrac{(t-1)^3}{6} + \dfrac{\theta(t-1)^2}{4} + \dfrac{1-\theta}{2}\left(x - \dfrac{1}{2}\right)^2 + \dfrac{1-3\theta}{24}, & t \in (x, 1] \end{cases} \tag{2.10}$$

and

$$f''(t) = \begin{cases} \dfrac{t^2}{2} - \dfrac{\theta}{2}t, & t \in [0, x], \\ \dfrac{(t-1)^2}{2} + \dfrac{\theta}{2}(t-1), & t \in (x, 1]. \end{cases} \tag{2.11}$$

Clearly, the function given in (2.10) is absolutely continuous since it is a continuous piecewise polynomial function.

We now suppose that (2.1) holds with a constant $C > 0$ as

$$\begin{aligned} &\left| \int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] \right. \\ &\quad + (1-\theta)(b-a)\left(x - \frac{a+b}{2}\right)f'(x) \\ &\quad \left. - \left[\frac{1-\theta}{2}\left(x - \frac{a+b}{2}\right)^2 + \frac{1-3\theta}{24}(b-a)^2\right][f'(b)-f'(a)] \right| \\ &\qquad \le C\left[\frac{\theta(1-\theta)}{4}(b-a)\left(x - \frac{a+b}{2}\right)^4 + \frac{3\theta^2-5\theta+2}{24}(b-a)^3\left(x - \frac{a+b}{2}\right)^2 \right. \\ &\qquad\quad \left. + \frac{15\theta^2-15\theta+4}{2880}(b-a)^5\right]^{(1/2)}\sqrt{\sigma(f'')}. \end{aligned} \tag{2.12}$$

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Choosing $a = 0$, $b = 1$, and f defined in (2.9) with (2.10), (2.11), we get

$$\int_0^1 f(t)\,dt = \frac{1-\theta}{8}\Big(x-\frac{1}{2}\Big)^4 - \frac{1-\theta}{6}\Big(x-\frac{1}{2}\Big)^3 + \frac{1-\theta}{16}\Big(x-\frac{1}{2}\Big)^2 + \frac{11-35\theta}{1920},$$

$$f(0) = 0, \quad f(1) = \frac{1-\theta}{4}\Big(x-\frac{1}{2}\Big)^2 - \frac{1-\theta}{3}\Big(x-\frac{1}{2}\Big)^3 + \frac{1-3\theta}{48},$$

$$f(x) = \frac{1}{24}\Big(x-\frac{1}{2}\Big)^4 + \frac{1-\theta}{12}\Big(x-\frac{1}{2}\Big)^3 + \frac{1-2\theta}{16}\Big(x-\frac{1}{2}\Big)^2 + \frac{1-3\theta}{48}\Big(x-\frac{1}{2}\Big) + \frac{1-4\theta}{384},$$

$$f'(0) = 0, \quad f'(1) = \frac{1-\theta}{2}\Big(x-\frac{1}{2}\Big)^2 + \frac{1-3\theta}{24},$$

$$f'(x) = \frac{1}{6}\Big(x-\frac{1}{2}\Big)^3 + \frac{1-\theta}{4}\Big(x-\frac{1}{2}\Big)^2 + \frac{1-2\theta}{8}\Big(x-\frac{1}{2}\Big) + \frac{1-3\theta}{48}$$

and

$$\int_0^1 (f''(t))^2\,dt = \frac{1-\theta}{4}\Big(x-\frac{1}{2}\Big)^4 + \frac{2\theta^2-3\theta+1}{8}\Big(x-\frac{1}{2}\Big)^2 + \frac{20\theta^2-15\theta+3}{960}$$

such that the left-hand side becomes

$$\text{L.H.S. (2.12)} = \frac{\theta(1-\theta)}{4}\Big(x-\frac{1}{2}\Big)^4 + \frac{3\theta^2-5\theta+2}{24}\Big(x-\frac{1}{2}\Big)^2 + \frac{15\theta^2-15\theta+4}{2880}. \tag{2.13}$$

We also find that the right-hand side is

$$\text{R.H.S. (2.12)} = C\Big[\frac{\theta(1-\theta)}{4}\Big(x-\frac{1}{2}\Big)^4 + \frac{3\theta^2-5\theta+2}{24}\Big(x-\frac{1}{2}\Big)^2 + \frac{15\theta^2-15\theta+4}{2880}\Big]. \tag{2.14}$$

From (2.12)–(2.14), we find that $C \geq 1$, proving that the coefficient constant 1 is the best possible in (2.1).

COROLLARY 2.2. *Let the assumptions of Theorem 2.1 hold. Then, for any $\theta \in [0, 1]$,*

$$\Big|\int_a^b f(t)\,dt - (b-a)\Big[(1-\theta)f\Big(\frac{a+b}{2}\Big) + \theta\frac{f(a)+f(b)}{2}\Big] - \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\Big| \leq \frac{(b-a)^{(5/2)}}{24\sqrt{5}}(15\theta^2-15\theta+4)^{(1/2)}\sqrt{\sigma(f'')}. \tag{2.15}$$

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PROOF. We set $x = (a+b)/2$ in (2.1) to get (2.15). □

REMARK 1. If we take $\theta = 1$ and $\theta = 0$ in (2.15), then the sharp corrected trapezoid inequality (1.1) and the sharp corrected midpoint inequality (1.3) are recaptured. Thus Theorem 2.1 may be regarded as a generalization of Theorems 1.1 and 1.2.

REMARK 2. If we take $\theta = 1/3$, we get a sharp Simpson-type inequality

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^{(5/2)}}{12\sqrt{30}}\sqrt{\sigma(f'')}. \tag{2.16}$$

If we take $\theta = 7/15$, we get a sharp corrected Simpson-type inequality

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{30}\left[7f(a) + 16f\left(\frac{a+b}{2}\right) + 7f(b)\right] + \frac{(b-a)^2}{60}[f'(b) - f'(a)]\right| \le \frac{(b-a)^{(5/2)}}{60\sqrt{3}}\sqrt{\sigma(f'')}. \tag{2.17}$$

From (2.16) and (2.17), we see that the corrected Simpson rule gives better results than the Simpson rule.

REMARK 3. If we take $\theta = 1/2$, we get a sharp corrected averaged midpoint-trapezoid-type inequality as

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] + \frac{(b-a)^2}{48}[f'(b) - f'(a)]\right| \le \frac{(b-a)^{(5/2)}}{48\sqrt{5}}\sqrt{\sigma(f'')}. \tag{2.18}$$

It is interesting to note that the smallest bound for (2.1) is obtained at $x = (a+b)/2$ and $\theta = 1/2$. Thus the corrected averaged midpoint-trapezoid rule is optimal in the current situation.

3. Applications in numerical integration

We restrict further considerations to the corrected averaged midpoint-trapezoid quadrature rule. We also emphasize that similar considerations can be given for all quadrature rules considered in the previous section.

THEOREM 3.1. *Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = (b-a)/n$ and let the assumptions of Theorem 2.1 hold. Then*

$$\left|\int_a^b f(t)\,dt - \frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i) + 2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right] + \frac{(b-a)^2}{48n^2}[f'(b) - f'(a)]\right| \le \frac{(b-a)^{(5/2)}}{48\sqrt{5}n^2}\sqrt{\sigma(f'')}. \tag{3.1}$$

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PROOF. From (2.18) we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{4}\left[f(x_i) + 2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right] + \frac{h^2}{48}[f'(x_{i+1}) - f'(x_i)]\right|$$
$$\leq \frac{h^{(5/2)}}{48\sqrt{5}}\left[\int_{x_i}^{x_{i+i}} (f''(t))^2\,dt - \frac{1}{h}(f'(x_{i+1}) - f'(x_i))^2\right]^{(1/2)}. \tag{3.2}$$

By summing (3.2) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\left|\int_a^b f(t)\,dt - \frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i) + 2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right] + \frac{h^2}{48}[f'(b) - f'(a)]\right|$$
$$\leq \frac{h^{(5/2)}}{48\sqrt{5}}\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+i}} (f''(t))^2\,dt - \frac{1}{h}(f'(x_{i+1}) - f'(x_i))^2\right]^{(1/2)}. \tag{3.3}$$

By using the Cauchy inequality twice, it is not difficult to obtain

$$\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} (f''(t))^2\,dt - \frac{1}{h}(f'(x_{i+1}) - f'(x_i))^2\right]^{(1/2)}$$
$$\leq \sqrt{n}\left[\|f''\|_2^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}(f'(x_{i+1}) - f'(x_i))^2\right]^{(1/2)}$$
$$\leq \sqrt{n}\left[\|f''\|_2^2 - \frac{(f'(b) - f'(a))^2}{b-a}\right]^{(1/2)} = \sqrt{n\sigma(f'')}. \tag{3.4}$$

Consequently, the inequality (3.1) follows from (3.3) and (3.4). □

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