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# RANKINGS AND RANKING FUNCTIONS

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**1. Introduction.** Suppose that $n$ competitors compete in $r$ races and in each race they are awarded placings $1, 2, 3, \ldots, n-1, n$. After the $r$ races each competitor has a result consisting of his $r$ placings. Let such a result be written $(\alpha_j)_{1 \leqq j \leqq r}$ where for convenience the positive integers $\alpha_j$ are arranged in ascending order. For example, if $n = 4$ and $r = 6$ a typical result is $(1, 2, 2, 3, 4, 4)$.

A final ranking of the $n$ competitors will be determined in all circumstances if a strict ordering is assigned to all possible results. In the next section it is shown that the number or results is $\binom{n+r-1}{r}$. Of course only some strict orderings of this set of results will be feasible in that there are criteria which a final ranking should satisfy. For example, $(1, 1, \ldots, 1)$ should clearly be ranked first in any feasible ranking.

In Section 2 we state axioms for a strict ordering of the set of $\binom{n+r-1}{r}$ results. A strict ordering which satisfies these axioms will be called a strict ranking. The problem posed is to characterise and if possible enumerate the strict rankings.

The theorems which follow depend on establishing a one to one correspondence between strict rankings and equivalence classes of stable ranking functions. A *ranking function* $F$ is a positive function defined on the first $n$ positive integers and satisfying the condition $F(j) > F(j+1)$, for $1 \leqq j \leqq n-1$. In each race the competitor finishing in the $j$th position is awarded a score $F(j)$. The sum of the scores over the $r$ races gives each competitor a final score and the competitors are ranked by the final scores. Two ranking functions are said to be *$n$:$r$ equivalent* if they give the same ranking of the set of $\binom{n+r-1}{r}$ results. (In general some equal rankings are possible.) Clearly $n$:$r$ equivalence is an equivalence relation in the usual sense.

Given any ranking function $F$, an $n$:$r$ equivalent function $G$ is defined by setting

$$G(j) = mF(j) + c, \quad 1 \leqq j \leqq n,$$

where $m$ and $c$ are positive constants. Thus without loss of generality we may assume that $F(n) = 0$ and $F(1) = 1$. The ranking functions are

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then characterised by an open subset $E(n)$ of $\mathbf{R}^{n-2}$ corresponding to the possible values of $(F(2), F(3), \ldots, F(n-1))$. We associate with $E(n)$ the usual topology of $\mathbf{R}^{n-2}$ restricted to $E(n)$.

A ranking function $F$ is said to be $n{:}r$ *stable* if there exists an open set $U \subset E(n)$ such that $F \in U$ and $U$ is contained in the $n{:}r$ equivalence class containing $F$. In other words, there exists a neighbourhood $U$ of $F$ such that all ranking functions in $U$ always rank the $\binom{n+r-1}{r}$ results in the same order as $F$. It is easily shown that the $n{:}r$ stable equivalence classes are represented by open convex sets in $E(n)$. Further examples and details concerning ranking functions are given in [**5**].

**2. Axioms for a strict ranking.** Before stating the axioms which define a strict ranking it is helpful to enumerate the results.

LEMMA 1. *If $n$ competitors compete in $r$ races the number of results is* $\binom{n+r-1}{r}$.

*Proof.* The number of results is the number of non-negative integer solutions to the equation $x_1 + x_2 + \ldots + x_n = r$. This is easily shown to be $\binom{n+r-1}{r}$. (See [**1**], p. 37.)

If $A = (\alpha_j)_{1 \leqq j \leqq r}$ and $B = (\beta_j)_{1 \leqq j \leqq r}$ are results we shall write $A > B$ to denote the ordering of $A$ above $B$. A *strict ranking* of the set of $\binom{n+r-1}{r}$ results will be defined to be a strict ordering which satisfies the following two axioms.

*Axiom I.* If two results $(\alpha_j)_{1 \leqq j \leqq r}$ and $(\beta_j)_{1 \leqq j \leqq r}$ satisfy the condition $\alpha_j \leqq \beta_j$, for $1 \leqq j \leqq r$, with at least one strict inequality, then the result $(\alpha_j)_{1 \leqq j \leqq r}$ is ranked above the result $(\beta_j)_{1 \leqq j \leqq r}$.

*Axiom* II. (A consistency axiom). There does not exist a result $R$ of length $pr$, $p \geqq 2$, such that $R = \bigcup_{i=1}^{p} A_i$ and $R = \bigcup_{i=1}^{p} B_i$ where $A_i$ and $B_i$ are results of length $r$ and $A_i > B_i$ for $1 \leqq i \leqq p$. (Here $\bigcup_{i=1}^{p} A_i$ and $\bigcup_{i=1}^{p} B_i$ are "unions" where repeated elements are permitted.)

*Example* 1. Let $n = 4$ and $r = 2$ and consider the strict ordering

$$(1, 1)\ (1, 2)\ (1, 3)\ (2, 2)\ (2, 3)\ (1, 4)\ (2, 4)\ (3, 3)\ (3, 4)\ (4, 4).$$

This is not a strict ranking because Axiom II is not satisfied when $R = (1, 2, 2, 3, 3, 4)$ and the decomposition of $R$ is taken to be $(1, 3) > (2, 2)$, $(2, 3) > (1, 4)$, $(2, 4) > (3, 3)$.

LEMMA 2. *Let a strict ranking be given and suppose that two results* $(\alpha_k)_{1\leqq k\leqq r}$ *and* $(\beta_k)_{1\leqq k\leqq r}$ *have a subset of common elements* $(\gamma_j)_{1\leqq j\leqq s}$. *Let* $(\alpha_k')_{1\leqq k\leqq r}$ *and* $(\beta_k')_{1\leqq k\leqq r}$ *be results produced by replacing the elements* $(\gamma_j)_{1\leqq j\leqq s}$ *by the elements* $(\gamma_j')_{1\leqq j\leqq s}$ *in each of the results* $(\alpha_k)_{1\leqq k\leqq r}$ *and* $(\beta_k)_{1\leqq k\leqq r}$ *respectively. Then the ranking of* $(\alpha_k)_{1\leqq k\leqq r}$ *relative* to $(\beta_k)_{1\leqq k\leqq r}$ *is the same as the ranking of* $(\alpha_k')_{1\leqq k\leqq r}$ *relative to* $(\beta_k')_{1\leqq k\leqq r}$. *(We refer to this latter statement as a substitution condition.)*

*Proof.* Let us assume that

$$(\alpha_k)_{1\leqq k\leqq r} > (\beta_k)_{1\leqq k\leqq r}$$

and

$$(\beta_k')_{1\leqq k\leqq r} > (\alpha_k')_{1\leqq k\leqq r}.$$

Then if we set

$$\begin{aligned} R &= (\alpha_k)_{1\leqq k\leqq r} \cup (\beta_k')_{1\leqq k\leqq r} \\ &= (\alpha_k')_{1\leqq k\leqq r} \cup (\beta_k)_{1\leqq k\leqq r} \end{aligned}$$

we see that Axiom II is violated and we have a contradiction.

LEMMA 3. *If* $s < r$ *a strict ranking of all results of length* $r$ *induces a uniquely defined strict ranking of all results of length* $s$.

*Proof.* The proof follows from Lemma 2.

We note that Example 1 gives a strict order which satisfies Axiom I and the substitution condition of Lemma 2. Since this strict order does not satisfy Axiom II this demonstrates that Axiom I and the substitution condition together do not imply Axiom II. In the earlier paper [**4**] the substitution condition itself took the role of an axiom but here it is superceded by Axiom II.

**3. The one to one correspondence.** The following lemmas will be used to demonstrate a one to one correspondence between strict rankings and $n{:}r$ equivalence classes of stable ranking functions. Suppose $x = (x_1, x_2, \ldots, x_n)$ and $V_i(x)$, $1 \leqq i \leqq p$, are linear functionals on Euclidean $n$-space $\mathbf{R}^n$.

LEMMA 4. *The system of inequalities* $V_i(x) > 0$, $1 \leqq i \leqq p$, *has a solution if and only if the zero-functional is not in the convex hull of* $V_1, V_2, \ldots, V_p$.

*Proof.* See [**2**, p. 115].

Let $n$ and $r$ be fixed and let $\mathscr{S}$ be a strict ranking. For each pair of results $A = (\alpha_k)_{1\leqq k\leqq r}$ and $B = (\beta_k)_{1\leqq k\leqq r}$, where $A > B$, define

$$V_{AB}(x) = \sum_{k=1}^{r} x_{\alpha_k} - \sum_{k=1}^{r} x_{\beta_k}.$$

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Let $L = L(\mathscr{S})$ be the set of all linear functionals defined in this way.

LEMMA 5. *There exists a ranking function which ranks all results in the same order as the strict ranking $\mathscr{S}$ if and only if the zero-functional is not in the convex hull of $L = L(\mathscr{S})$.*

*Proof.* Let $F$ be a ranking function and set $F(j) = x_j$, $1 \leqq j \leqq n$. Then the ranking function $F$ ranks all results in the same order as $\mathscr{S}$ if and only if each of the inequalities $V_{AB}(x) > 0$ is satisfied for all $V_{AB} \in L$. Hence by Lemma 4 there exists a ranking function which ranks all results in the same order as $\mathscr{S}$ if and only if the zero-functional is not in the convex hull of $L$. To check the sufficiency of this condition note that if the zero-functional is not in the convex hull of $L$ then there exists a solution to the system $V_{AB}(x) > 0$, $V_{AB} \in L$. But the inequalities $x_j - x_{j+1} > 0$, $1 \leqq j \leqq n - 1$, are included in this system. Hence the solution can indeed be used to define a ranking function by choosing $m$ and $c$ so that $F(j) = mx_j + c$ and $1 = F(1) = mx_1 + c$, $0 = F(n) = mx_n + c$.

THEOREM 1. *For all positive integers $n$ and $r$ there is a one to one correspondence between strict rankings and $n{:}r$ equivalence classes of stable ranking functions.*

*Proof.* Let a strict ranking $\mathscr{S}$ be given. We shall assume the zero-functional belongs to the convex hull of $L = L(\mathscr{S})$ and show that this leads to a contradiction.

Let us suppose there exists $K \subseteq L$ and constants $\lambda_{AB} > 0$ such that

$$\sum_{V_{AB} \in K} \lambda_{AB} V_{AB}(x) = 0.$$

This equation can then be rewritten as a homogeneous system of $n$ equations in the $\lambda_{AB}$ by setting the coefficient of $x_j$ to be zero for $1 \leqq j \leqq n$. With the $\lambda_{AB}$ taken to be the variables this system has integer coefficients. By using Gaussian elimination we can obtain the general solution to this homogeneous system. Because the coefficients are integers the general solution will contain rational coefficients and arbitrary constants.

The conditions that $\lambda_{AB} > 0$, for $V_{AB} \in K$, define a system of inequalities which the arbitrary constants must satisfy. The set of possible values is open and by the assumption nonempty. Hence the arbitrary constants can be chosen to be rational and the $\lambda_{AB}$ can be taken to be positive rational numbers. Further, since the system is homogeneous the $\lambda_{AB}$ can be multiplied by a suitable constant to give a solution of positive integers.

To summarise we have shown that if the zero-functional belongs to the convex hull of $L$ there exist $K \subseteq L$ and positive integers $\lambda_{AB}$ such that

$$\sum_{V_{AB} \in K} \lambda_{AB} V_{AB} = 0.$$

If we now define

$$R = \bigcup_{V_{AB} \in K} (\lambda_{AB} \text{ copies of } A)$$

and

$$T = \bigcup_{V_{AB} \in K} (\lambda_{AB} \text{ copies of } B)$$

then this equation and the definition of $V_{AB}$ imply that $R = T$. Using the set $R$ and the rankings $A > B$, $V_{AB} \in K$, we contradict Axiom II. This completes the proof that the zero-functional does not belong to the convex hull of $L$. Hence by Lemma 5 there exists a ranking function which ranks all results in the same order as $\mathscr{S}$. The inequalities $V_{AB}(x) > 0$ define an open set in $E(n)$ and hence the corresponding $n{:}r$ equivalence class is $n{:}r$ stable.

Now let an $n{:}r$ equivalence class of stable ranking functions be given. We must show that it defines a strict ranking. Firstly it certainly defines a strict order of results since if two results were ranked equal this equality would be destroyed by an arbitrarily small perturbation of the ranking functions which would not then be stable. Secondly it is easy to show that a strict order defined by a ranking function satisfies Axioms I and II and hence defines a strict ranking. This establishes the one to one correspondence.

*Example* 2. For $n = 3$ it is shown in [**5**] that the number of $3{:}r$ equivalence classes of stable ranking functions is $\sum_{j=1}^{r} \phi(j)$ where $\phi$ is Euler's $\phi$ function. By Theorem 1 this is also the number of strict rankings. From [**3**, p. 266] we obtain the asymptotic estimate

$$\sum_{j=1}^{r} \phi(j) = \frac{3r^2}{\pi^2} + O(r \log r).$$

## References


**1.** R. A. Brualdi, *Introductory combinatorics* (North Holland, New York, 1977).

**2.** K. Fan, *Systems of linear inequalities*, in: *Linear inequalities and related systems* (Princeton Univ. Press, Princeton, N.J., 1956).

**3.** G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers* (Oxford Press, 1938).

**4.** W. J. Walker, *Algebraic and combinatorial results for ranking competitors in a sequence of races*, Discrete Math. *14* (1976), 297–304.

**5.** ——— *The enumeration and bifurcations of ranking functions*, Bull. Austral. Math. Soc. *18* (1978), 287–292.