BULL. AUSTRAL. MATH. SOC. 16A08
VOL. 15 (1976), 455-460.

Some functors arising from the consideration of torsion theories over noncommutative rings

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To each associative (but not necessarily commutative) ring R we assign the complete distributive lattice R-tors of (hereditary) torsion theories over R-mod . We consider two ways of making this process functorial - once contravariantly and once covariantly - by selecting appropriate subcategories of the category of associative rings. Combined with a functor due to Rota, this gives us functors from these subcategories to the category of commutative rings.

Let R be an associative (but not necessarily commutative) ring with unit element 1 . We denote by R-mod the category of all unitary left R-modules and by R-tors the complete distributive lattice of all (hereditary) torsion theories over R-mod . See [4] for details. If $\tau \in R$-tors and if M is a left R-module, then $T_\tau(M)$ will denote the τ-torsion submodule of M and $Q_\tau(M)$ will denote the localization of M at τ . The ring of quotients of R at τ will be denoted by R_τ . For each left R-module M , we have a homomorphism $\hat{\tau}_M : M \to Q_\tau(M)$ of left R-modules. Also, we have a ring homomorphism $\hat{\tau} : R \to R_\tau$. If this ring homomorphism is a left flat ring epimorphism (that is, if $\hat{\tau}$ is an epimorphism in the category of rings and if R_τ is flat as a *right* R-module) then the torsion theory τ is said to be *perfect*. Left flat

Received 3 August 1976.

https://doi.org/10.1017/S0004972700022899 Published online by Cambridge University Press

ring epimorphisms are studied in some detail in [9]. It is well-known that every left flat ring epimorphism is in fact of the form $\hat{\tau}$ for some perfect torsion theory τ. Moreover, $\tau \in R$-tors is perfect if and only if $Q_\tau(M) = R_\tau \otimes_R M$ for every left R-module M. If $\gamma : R \to S$ is a ring homomorphism then γ defines a function $\gamma_\# : R\text{-tors} \to S\text{-tors}$ which assigns to $\tau \in R$-tors that torsion theory $\sigma \in S$-tors characterized by the property that a left S-module N is σ-torsion if and only if it is τ-torsion as a left R-module. If S is flat as a right R-module, then γ also defines a function $\gamma^\# : S\text{-tors} \to R\text{-tors}$ which assigns to $\sigma \in S$-tors that torsion theory $\tau \in R$-tors characterized by the property that a left R-module M is τ-torsion if and only if $S \otimes_R M$ is σ-torsion. If $\gamma : R \to S$ is a left flat ring epimorphism, say $\gamma = \hat{\tau}$ for a perfect torsion theory $\tau \in R$-tors, then $\gamma_\#$ and $\gamma^\#$ induce a bijection between S-tors and $\{\tau' \in R\text{-tors} \mid \tau' \geq \tau\}$. See Proposition 17.14 of [4]. Thus we have a contravariant functor F from the category of rings and left flat ring epimorphisms to the category of complete lattices given by $F(R) = R$-tors and $F(\gamma) = \gamma^\#$.

We now seek another subcategory of the category of rings from which we can construct a *covariant* functor to the category of complete lattices which behaves on objects in the same way as F, namely which assigns to each ring R the complete lattice R-tors. For this we need some more definitions and remarks.

If $\gamma : R \to S$ is a ring homomorphism and if $\tau \in R$-tors, it is not necessarily true that a left S-module is $\gamma_\#(\tau)$-torsionfree if and only if it is τ-torsionfree as a left R-module. When this condition does hold, we say that the torsion theory τ is *compatible* with γ. The notion of compatibility between a torsion theory and a ring homomorphism has recently been studied by Louden [5], who also provides several examples of this phenomenon. Proposition 9.8 of [4] asserts that every torsion theory is compatible with a ring surjection; Proposition 17.15 of [4] asserts that if $\tau \in R$-tors is perfect and if $\tau' \geq \tau$ then τ' is compatible with $\hat{\tau}$. A torsion theory $\tau \in R$-tors is *stable* if and only if the class of all τ-torsion left R-modules is closed under taking essential extensions.

Stable torsion theories are very important in the theory of noncommutative localization; they were first introduced and characterized in [2]. The Goldie torsion theory is a common example of a stable torsion theory. A ring R is *left stable* if and only if every member of R-tors is stable. Such rings have been studied in [1, 6, 7]; commutative noetherian rings are left stable and left stability is a Morita invariant.

In [4] we showed that the lattice R-tors is always distributive. Raynaud has extended this result to show that R-tors in fact always satisfies the join-infinite distributive identity, namely that for every $\tau \in R$-tors and every $U \subseteq R$-tors , we have

$$\tau \wedge (\vee U) = \vee\{\tau \wedge \tau' \mid \tau' \in U\} .$$

Thus the lattice R-tors is brouwerian [8]. In a private conversation, D. Strauss has pointed out that if R is left stable then the dual of this, namely the meet-infinite distributive identity also holds. Thus we have:

PROPOSITION 1. *If R is a left stable ring then for every $\tau \in R$-tors and every $U \subseteq R$-tors we have* $\tau \vee (\wedge U) = \wedge\{\tau \vee \tau' \mid \tau' \in U\}$.

Proof. For each $\tau' \in U$ we have $\tau' \geq \wedge U$ and so $\tau \vee \tau' \geq \tau \vee (\wedge U)$. If $\tau'' = \wedge\{\tau \vee \tau' \mid \tau' \in U\}$ we then have that $\tau'' \geq \tau \vee (\wedge U)$. Now assume that this inequality is strict. Then there exists a nonzero left R-module M which is τ''-torsion but not $[\tau \vee (\wedge U)]$-torsion. Indeed, replacing M by $M/T_{\tau\vee(\wedge U)}(M)$, we can assume that M is $[\tau \vee (\wedge U)]$-torsionfree and hence that it is both τ-torsionfree and $(\wedge U)$-torsionfree, as well as being τ''-torsion. Since τ'' is stable, the injective hull $E(M)$ of M also has these properties. In particular, if $\tau' \in U$ then $E(M)$ cannot be τ'-torsionfree. By [4, Proposition 11.2], there then exists a submodule N of $E(M)$ such that $E(M) = N \oplus T_{\tau'}(E(M))$. This implies that N is both τ-torsionfree and τ'-torsionfree and so is $(\tau \vee \tau')$-torsionfree, a contradiction unless $N = 0$. Thus $E(M)$ must be τ'-torsion for every $\tau' \in U$. This implies that $E(M)$ is $(\wedge U)$-torsion, which is also a contradiction. □

Another interesting aspect of stability is the following.

PROPOSITION 2. *Let $\tau \in R$-tors be perfect and let $\tau' \in R$-tors be stable. Then* $\hat{\tau}_{\#}(\tau \vee \tau') = \hat{\tau}_{\#}(\tau')$.

https://doi.org/10.1017/S0004972700022899 Published online by Cambridge University Press

Proof. By [4, Proposition 17.14] there exists a torsion theory $\tau'' \geq \tau$ satisfying the condition that $\hat{\tau}_{\#}(\tau'') = \hat{\tau}_{\#}(\tau')$. Indeed, τ'' is characterized by the property that a left R-module M is τ''-torsion if and only if $Q_{\tau}(M)$ is $\hat{\tau}_{\#}(\tau')$-torsion as a left R_{τ}-module. We therefore want to show that $\tau'' = \tau \vee \tau'$. Let M be a $(\tau \vee \tau')$-torsionfree left R-module. Then M is τ-torsionfree and so the homomorphism $\hat{\tau}_M : M \to Q_{\tau}(M)$ is an essential extension, implying that $Q_{\tau}(M)$ is τ'-torsionfree as a left R-module. Therefore $Q_{\tau}(M)$ is $\hat{\tau}_{\#}(\tau')$-torsionfree as a left R_{τ}-module and so M is τ''-torsionfree since τ'' is compatible with $\hat{\tau}$. This proves that $\tau \vee \tau' \geq \tau''$. Assume that this inequality is strict. Then by [4, Proposition 17.14] we have that $\hat{\tau}_{\#}(\tau \vee \tau') > \hat{\tau}_{\#}(\tau'') = \hat{\tau}_{\#}(\tau')$ and so there exists a left R_{τ}-module N which is $\hat{\tau}_{\#}(\tau \vee \tau')$-torsion but not $\hat{\tau}_{\#}(\tau')$-torsion. Indeed, by factoring out the $\hat{\tau}_{\#}(\tau')$-torsion submodule of N we can assume that N is $\hat{\tau}_{\#}(\tau')$-torsionfree. Since N is a left R_{τ}-module and since τ is perfect, we know that N is τ-torsionfree as a left R-module. Since N is $\hat{\tau}_{\#}(\tau \vee \tau')$-torsion, it is $(\tau \vee \tau')$-torsion as a left R-module and so cannot be τ'-torsionfree as a left R-module. Let M be the τ'-torsion submodule of N. Since M is also τ-torsionfree, the homomorphism $\hat{\tau}_M$ is an essential extension and so $Q_{\tau}(M)$ is also τ'-torsion by the stability of τ'. Therefore $Q_{\tau}(M)$ is $\hat{\tau}_{\#}(\tau')$-torsion as a left R_{τ}-module. But $Q_{\tau}(M)$ is an R_{τ}-submodule of the $\hat{\tau}_{\#}(\tau')$-torsionfree left R-module N, which implies that we must have $M = 0$, a contradiction. This proves that $\tau \vee \tau' = \tau''$. □

As a consequence of this result we obtain the following proposition.

PROPOSITION 3. *If* R *is a left stable ring and if* $\tau \in R$-tors *is perfect then* $\hat{\tau}_{\#} : R\text{-tors} \to R_{\tau}\text{-tors}$ *is a morphism of complete lattices.*

Proof. It is an immediate consequence of the definitions that $\hat{\tau}_{\#}$ is a morphism of complete meet semilattices. If U is a subset of R-tors and if N is a left R_{τ}-module, then, by Proposition 2, we know that N

is $\hat{\tau}_{\#}(\vee U)$-torsionfree if and only if N is $\hat{\tau}_{\#}[(\vee U) \vee \tau]$-torsionfree. Since $(\vee U) \vee \tau$ is compatible with $\hat{\tau}$, this holds if and only if N is $[(\vee U) \vee \tau]$-torsionfree when considered as a left R-module; that is, it holds if and only if N is $(\tau' \vee \tau)$-torsionfree for every $\tau' \in U$. Since $\tau' \vee \tau$ is compatible with $\hat{\tau}$, this holds if and only if N is $\hat{\tau}_{\#}(\tau' \vee \tau)$-torsionfree as a left R_τ-module for every $\tau' \in U$ and again by Proposition 2 this holds if and only if N is $\hat{\tau}_{\#}(\tau')$-torsionfree for every $\tau' \in U$ or, in other words, if and only if N is torsionfree with respect to $\vee\{\hat{\tau}_{\#}(\tau') \mid \tau' \in U\}$. □

Thus we now have a covariant functor G from the category of left stable rings and left flat ring epimorphisms to the category of complete lattices and morphisms of complete lattices given by $G(R) = R\text{-tors}$ and $G(\gamma) = \gamma_{\#}$.

In conclusion, let us note one possible use of the functors F and G. Rota has defined a "valuation" functor $V(-)$ from the category of distributive lattices and lattice homomorphisms to the category of commutative rings and ring homomorphisms. See [3] for details. If we combine this functor with the functor F [respectively the functor G] defined above we obtain a contravariant [respectively covariant] functor from the category of [left stable] rings and left flat ring epimorphisms to the category of commutative rings and ring homomorphisms.

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