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Systems of Weakly Coupled Hamilton–Jacobi Equations with Implicit Obstacles

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Abstract. In this paper we study systems of weakly coupled Hamilton–Jacobi equations with implicit obstacles that arise in optimal switching problems. Inspired by methods from the theory of viscosity solutions and weak KAM theory, we extend the notion of Aubry set for these systems. This enables us to prove a new result on existence and uniqueness of solutions for the Dirichlet problem, answering a question of F. Camilli, P. Loreti, and N. Yamada.

1 Introduction

On a Riemmanian manifold M the basic Hamiltonian in classical mechanics is of the form

$$H(x, p) = \frac{|p|^2}{2} + V(x),$$

where $x \in M$, $p \in T_x^*M$, and V is a scalar field (the potential). This Hamiltonian function is strictly convex in p, and it is *coercive*, *i.e.,* it tends to infinity with $|p|$, locally uniformly in x. Actually, in this case it is called superlinear, because $|p|^{-1}H(x, p)$ still tends to infinity with $|p|$.

Let L denote the corresponding Lagrangian, which can be obtained from H via the Legendre transformation. For two points $x, y \in M$, denote by $d_L(x, y)$ the infimum of all the action integrals $\int L(\gamma(t), \dot{\gamma}(t))dt$, where γ runs through all the continuously differentiable paths on M leading from x to y in any positive time. For each fixed $x_0 \in M$, the function $u(x) = d_L(x, x_0)$ is a viscosity solution of the Hamilton–Jacobi equation $H(x, Du(x)) = 0$ on $M \setminus \{x_0\}$. The study of the Hamilton–Jacobi equation $H(x, Du(x)) = 0$ is of fundamental importance in many situations.

In this work we study a generalization of these types of Hamilton–Jacobi equations on multi-layered domains of $\mathbb{R}^n$, *i.e.,* Cartesian products of some domain in $\mathbb{R}^n$ by a finite discrete set. We are concerned with systems of quasi-variational inequalities of the form

$$\max_{i,j}\left\{ H_i(x, Du_i), u_i(x) - u_j(x) - k(x, i, j)\right\} = 0, \tag{1.1}$$

where x belongs to a domain $\Omega \subset \mathbb{R}^n$, i, j run through a finite set of indices $\mathcal{I}$, k is a switching cost function described below in further detail, and the Hamiltonians

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$H_i = H_i(x, p)$ are continuous, convex in $p \in \mathbb{R}^n$, and satisfy a certain weak coercivity condition. The difference from the former, one-layered, case is that now the paths γ are allowed to jump occasionally between layers: the price to pay for each discontinuity is that a jump at the point x from the layer i to the layer j represents a positive increment $k(x, i, j)$ to the action.

These types of systems of Hamilton–Jacobi equations appear naturally in many applications. A typical example, which is discussed in [3], is the optimal management of a power plant with several modes of energy production and a cost associated with the switching of production modes. In that case the solution represents the cost of an optimal strategy for energy production. These so-called optimal switching problems have been the object of intensive study; see, for instance, [6, 11–13, 22, 29] and references therein.

Important results and a synthesis of the problem for the case where the evolution in each mode is governed by an ordinary differential equation are given in [5].

Our work is motivated by the results obtained in [4]. That article gives a complete solution of the obstacle problem, which bears a close relation with the former. In Remark 3, p. 1300, it is stated that a similar solution of the Dirichlet problem for the system (1.1) would be desirable. We will describe our formulation and answer this problem after a few preliminary remarks.

Given a real value $\overline{H} \in \mathbb{R}$, the stationary Hamilton–Jacobi equation looks for functions $u(x)$ on M such that

$$H(x, Du(x)) = \overline{H}\,. \tag{1.2}$$

A useful concept to deal with Hamilton–Jacobi equations is that of subsolution, *i.e.,* a function u such that $H(x, Du(x)) \le \overline{H}$. Just like subharmonic functions as compared to harmonic ones when dealing with the Laplacian, subsolutions form a set that is stable under more operations than that formed by solutions, and eventually solutions can be obtained in a manner similar to Perron's method (*cf.* [19]). An important consequence of the coercivity of H is that the subsolutions of such Hamilton–Jacobi equations are locally Lipschitz.

For many values of $\overline{H}$ no subsolutions will exist. For example, consider the Hamiltonian $H(x, p) = |p|^2$. The minimum value of $\overline{H}$ for which subsolutions exist is $\overline{H} = 0$, the subsolutions being in that case the constant functions (which are actually solutions). For $\overline{H} > 0$, equation (1.2) is called the eikonal and it describes, *e.g.,* the propagation of light rays in the manifold M.

The smallest value of $\overline{H}$ (if any) for which subsolutions exist is called in [17] the (Mañé) critical value. For compact manifolds, the existence of the critical value has been stated in [24, 25] but, due to the untimely death of R. Mañé, the proof appears in [9] (see also [7, 8, 16]). However, in a non-compact manifold, even strictly convex superlinear Hamiltonians may have no critical value. A trivial way to see that is just taking any convex, coercive Hamiltonian $H(x, p)$ and adding to it a continuous function $w(x)$ with fast enough growth to plus infinity: for each $k \in \mathbb{R}$, no matter how negative, there will exist x such that $H(x, p) + w(x) + k > 0$ for all p, so no subsolutions can ever exist. An example given at the end of Section 2 will show that deeper obstructions may occur.

In order to obtain general results for Hamilton–Jacobi equations on non-compact manifolds, it is essential to assume the existence of a critical value. This will be done here (as was also done in [21]), and, adding a constant to the Hamiltonian, there is no loss of generality in assuming that the critical value is not positive. As in [4], the subsolutions in our setting are the continuous functions which satisfy, in the viscosity sense, the inequality

$$\max_{i,j}\{H_i(x, Du_i), u_i(x) - u_j(x) - k(x, i, j)\} \leq 0\,. \tag{1.3}$$

Applications have required the study of Hamilton–Jacobi equations with more general Hamiltonians, with varied degrees of relaxation of their properties in the classical case. For the general results, our basic conditions on the Hamiltonian are the same as in [21]. Thus, we assume that it is continuous, convex (maybe not strictly so) in p, and that the critical value is not positive (an additional condition is included in the definition of subsolution to cope with the mode switching costs). Finally, the coercivity condition is also relaxed, but it still guarantees that subsolutions are locally Lipschitz.

The concept of the Aubry set has an essential role in our work. In the classical setting, it consists of those points a where any globally defined subsolution u of $H(x, Du(x)) \leq 0$ must satisfy $H(a, Du(a)) = 0$. Take $M = \mathbb{R}^n$ and $H(x, p) = |p|^2 + k$ (this is equivalent to the example discussed above, in the form $H(x, p) = |p|^2$ and $k = -\overline{H}$). The Aubry set is empty for $k < 0$ ($\overline{H} > 0$ in the previous formulation) and it is the whole $\mathbb{R}^n$ for $k = 0$ ($\overline{H} = 0$). In the other cases the equation is impossible, so the Aubry set is also $\mathbb{R}^n$. Another example is the oscillatory Hamiltonian $H(x, p) = |p|^2 - |\sin(2\pi x)|^2$ on the real line. In this case the Aubry set is the set of the integers, $\mathbb{Z}$.

The Aubry set is always a closed subset of the manifold M (maybe empty, maybe all of it), and its quotient under a certain equivalence relation becomes an additional boundary for the Cauchy problem, where the values of the solutions of a Hamilton–Jacobi equation can be pre-assigned, respecting a certain compatibility condition. In the important paper [17, Theorem 1.4], A. Fathi and A. Siconolfi give a very thorough description of the Aubry set. In this paper, we are able to characterize the Aubry set in terms of loops of arbitrarily low action, thus extending part (i)$\Leftrightarrow$(iv) of that theorem to our setting. In [20] this is done under less general conditions.

While our exposition does not require any familiarity with weak KAM theory or Aubry–Mather theory, it may serve as an introduction to some of its concepts and methods. A further exposition of these fascinating topics is beyond our scope; we refer the reader to Fathi's book [14].

Our approach to the study of (1.1) consists of considering it not as a system of equations but as a scalar Hamilton–Jacobi equation on a finitely layered space. This approach led to the variational formulation with paths with discontinuities, described before. We have included the switching costs in the definitions of subsolutions and solutions, extending in a natural manner the corresponding concepts from the theory of viscosity solutions of M. Crandall and P.-L. Lions [1, 23]. This enabled us to apply the techniques of [21] to the study of equation (1.1). Our final main result, Theorem 4.5, establishes the existence and uniqueness of solution for the Dirichlet problem. In order for the Dirichlet problem to become tractable, we assume two conditions which impose some regularity to both the boundary of the domain and the boundary behavior of the solutions. These conditions are also assumed in [4, 21]. We prove that, under these conditions, if a continuous function defined on the union of all layers of the boundary of the domain and on the Aubry set satisfies a certain compatibility condition, then it has a unique continuous extension that is a viscosity solution of (1.1). Since the mentioned compatibility condition is necessary for the existence of such an extension, we consider this to be a complete accomplishment of the project suggested by [4, Remark 3].

We hope that the way in which some technical issues are settled deserves additional independent interest from the reader. The equality $d_H = d_L$ has a partially different proof from that in [20] to account not only for the plurimodal setting, but also for the more general conditions. It requires Proposition 2.1, an existence result on differential inclusions that we were unable to find in the literature.

This paper is organized as follows. In Section 2 we present the plurimodal setting. We state the conditions on the Hamiltonian and on the switching cost function, and we define the action and the subsolutions. The main result of this section is Theorem 2.7, which settles the Lagrangian formulation of the plurimodal Hamilton–Jacobi equation. In Section 3 we define the Aubry set and the solutions. The section closes with Theorem 3.6, the plurimodal version of (i)⇔(iv) of [17, Theorem 1.4]. Finally, in Section 4 we follow the techniques of [21] to study boundary value problems, concluding with the solution of the Dirichlet problem, Theorem 4.5.

2 Setting and Preliminairies

All functions are real valued unless explicitly stated otherwise.

We will denote by Ω an open connected subset of $\mathbb{R}^n$ and by $\mathcal{I}$ a discrete finite set $\{1, \dots, N_{\mathcal{I}}\}$, where $N_{\mathcal{I}}$ is some positive integer. The elements of $\mathcal{I}$ will be called *modes*, and the elements of Ω will be called *states*. Self-explanatory phrases like "state-mode space" may be used when convenient. Functions on the state-mode space will be called locally Lipschitz, differentiable, *etc.*, meaning that they are so with respect to the state-variable.

The Hamiltonian is a function $H\colon \Omega \times \mathcal{I} \times \mathbb{R}^n \to \mathbb{R}$ on which we impose the same conditions as in [21]. These conditions are as follows:

(A1) it is continuous ($\mathcal{I}$ has the discrete topology);
(A2) for each $(x, i) \in \Omega \times \mathcal{I}$ the map $p \mapsto H(x, i, p)$ is convex in $\mathbb{R}^n$;
(A3) H is locally coercive, *i.e.*, for each compact subset K of $\Omega \times \mathcal{I}$, there exists a positive constant R_K such that $H(x, i, p) > 0$ for every $(x, i) \in K$ and $|p| > R_K$.

An essential assumption (A4) will be formulated shortly.

Before we proceed, we will prove the following result concerning Hamiltonians that satisfy these conditions.

Proposition 2.1 *Let $\mu\colon \Omega \to \mathbb{R}^n$ be a continuous (co-)vector field. For every $(x_0, i) \in \Omega \times \mathcal{I}$ there exist $\delta > 0$ and a Lipschitz path $\gamma_0\colon [-\delta, \delta] \to \Omega$ such that $\gamma_0(0) = x_0$ and, for almost all s, $\dot\gamma_0(s) \in D_p^- H(\gamma_0(s), i, \mu(\gamma_0(s)))$.*

It is well known that convex functions on open convex subsets of $\mathbb{R}^n$ are locally Lipschitz. A slight adaptation of the usual argument shows that $H(x,i,p)$ is locally Lipschitz in p, with Lipschitz constant locally uniform in (x,i). This fact is used in the proof of Proposition 2.1, so we include its proof for the reader's convenience.

Lemma 2.2 *The Hamiltonian is locally Lipschitz in p, locally uniformly in (x,i).*

Proof Let K be a compact subset of $\Omega \times \mathcal{I}$, let B be a closed ball in $\mathbb{R}^n$, and let S be a sphere (*i.e.,* the boundary of a ball) with the same center as B and a larger radius. Let Λ be the maximum of

$$\left\{ \frac{|H(x,i,q)-H(x,i,p)|}{|q-p|} : p \in B,\ q \in S,\ (x,i) \in K \right\}.$$

Let p_1, p_2 be two different elements of B and suppose that $H(x,i,p_2) \geq H(x,i,p_1)$. Let q be the intersection with S of the half-line beginning at p_1 passing by p_2. Then, by the convexity of $H(x,i,\,\cdot\,)$,

$$\begin{aligned}\frac{|H(x,i,p_2)-H(x,i,p_1)|}{|p_2-p_1|} &= \frac{H(x,i,p_2)-H(x,i,p_1)}{|p_2-p_1|}\\ &\leq \frac{H(x,i,q)-H(x,i,p_1)}{|q-p_1|} \leq \Lambda. \qquad \blacksquare\end{aligned}$$

Proof of Proposition 2.1 Let ρ_ε be a smooth non-negative mollifier, and for each $\varepsilon > 0$ define $H_\varepsilon(x,i_0,p) := \int H(x,i_0,q)\rho_\varepsilon(p-q)\mathrm{d}q$. For each $\varepsilon > 0$, $\frac{\partial H_\varepsilon}{\partial p}(x,i_0,\mu(x))$ is a continuous (co-)vector field on a neighborhood of x_0. By Peano's theorem, there exists for some $\delta > 0$ a continuously differentiable path γ_ε such that $\gamma_\varepsilon(0) = x_0$ and $\dot\gamma_\varepsilon(s) = \frac{\partial H_\varepsilon}{\partial p}(\gamma_\varepsilon(s), i_0, \mu(\gamma_\varepsilon(s)))$. By Lemma 2.2, the same δ may be used for all (small enough) ε, and the paths γ_ε are Lipschitz uniformly on ε. Take this δ and for γ_0 take a uniform sublimit of the γ_ε as $\varepsilon \to 0$, whose existence is asserted by Ascoli–Arzela's theorem. A common Lipschitz constant of the γ_ε will be a Lipschitz constant of γ_0.

It only remains to prove that $\dot\gamma_0(s) \in D_p^- H(\gamma_0(s), i, \mu(\gamma_0(s)))$ for almost every $s \in [-\delta,\delta]$. Let $h \in \mathbb{R}^n$ be an arbitrary, but fixed, vector. As a function of s,

$$\beta_0(s;h) := H\Big(\gamma_0(s), i, \mu\big(\gamma_0(s)\big) + h\Big) - H\Big(\gamma_0(s), i, \mu\big(\gamma_0(s)\big)\Big) - \dot\gamma_0(s)h$$

is defined almost everywhere on $[-\delta,\delta]$. We need to prove that it is non-negative.

For $\varepsilon > 0$, define:

$$\beta_\varepsilon(s;h) := H_\varepsilon\Big(\gamma_\varepsilon(s), i, \mu\big(\gamma_\varepsilon(s)\big) + h\Big) - H_\varepsilon\Big(\gamma_\varepsilon(s), i, \mu\big(\gamma_\varepsilon(s)\big)\Big) - \dot\gamma_\varepsilon(s)h.$$

These functions are continuous and non-negative (everywhere) on $[-\delta,\delta]$ due to the convexity of $H_\varepsilon(x,i,\,\cdot\,)$. Since $\beta_0(s;h)$ is a weak sublimit of the $\beta_\varepsilon(s;h)$ as $\varepsilon \to 0$, we see that $\beta_0(s;h)$ must be a.e. non-negative.

Hence, for each vector $h \in \mathbb{R}^n$, there exists a null subset $E(h) \subset [-\delta,\delta]$ such that $\beta_0(s;h)$ is defined and non-negative for $s \notin E(h)$. Now let (h_k) be a dense sequence in

$\mathbb{R}^n$ and let $E \subset [-\delta, \delta]$ be the null set $\cup_k E(h_k)$. Now fix $s \in [-\delta, \delta] \setminus E$. The function $h \mapsto \beta_0(s; h)$ is defined and continuous on $\mathbb{R}^n$ and non-negative at each h_k, so it must be non-negative for all h. Thus, $\beta_0(s; h) \geq 0$ for all $s \in [-\delta, \delta] \setminus E$ and for all $h \in \mathbb{R}^n$, which means that $\dot{\gamma}_0(s) \in D_p^- H(\gamma_0(s), i, \mu(\gamma_0(s)))$ for s out of E. ∎

Following [5], the switching cost function is a continuous function k on $\Omega \times \mathcal{I} \times \mathcal{I}$ such that, for all $x \in \Omega$ and $i, j, j' \in \mathcal{I}$ with $i \neq j$,

- $k(x, i, i) = 0$,
- $k(x, i, j) > 0$, and
- $k(x, i, j) + k(x, j, j') \geq k(x, i, j')$.

Thus, each $k(x, \cdot, \cdot)$ may only fail to be a distance on $\mathcal{I}$ for lack of symmetry. Assuming the triangular inequality represents no loss of generality, as is observed in [22, p. 467].

We will denote by S_H^- the set of the continuous functions u on $\Omega \times \mathcal{I}$ that satisfy the following conditions:

(C1) For every state-mode (x, i) and every p in the superdifferential of $u(\cdot, i)$ at x, we have $H(x, i, p) \leq 0$.

(C2) Given $x \in \Omega$ and $i, j \in \mathcal{I}$, $u(x, i) - u(x, j) \leq k(x, i, j)$.

Condition (C1) states that on each mode i, the function $u_i = u(\cdot, i)$ is a viscosity subsolution of $H(x, Du_i(x), i) = 0$ ([1, 2, 18]). Condition (C2) represents the compatibility with the mode switching costs. The elements of S_H^- are called subsolutions, and they are the continuous functions that satisfy inequality (1.3) in the viscosity sense.

Condition (A4) can now be formulated:

(A4) The set S_H^- is not empty.

This condition is independent of the previous ones in a very essential way. This will be shown at the end of this section.

The conditions (A1) and (A2) on H allow the following simple criterion to check condition (C1).

Lemma 2.3 *Let u be a locally Liscphitz function on $\Omega \times \mathcal{I}$. Then u satisfies* (C1) *if and only if for all $i \in \mathcal{I}$, $H(\cdot, i, Du(\cdot, i)) \leq 0$ almost everywhere on Ω.*

Proof To simplify the notation, fix any mode $i \in \mathcal{I}$ and let u denote $u(\cdot, i)$. By the Rademacher theorem u is differentiable almost everywhere, and the "only if" part is obvious. Suppose now that $H(x, i, Du(x)) \leq 0$ a.e., let ρ_ε be smooth non-negative mollifiers ($\varepsilon > 0$). Then $Du_\varepsilon(x) = \int \rho_\varepsilon(x - y) Du(y) \mathrm{d}y$. By (A2) and Jensen's inequality, $H(x, i,_\varepsilon (x)) \leq \int \rho_\varepsilon(x - y) H(x, i, Du(y)) \mathrm{d}y$. By (A1), for every $\delta > 0$ we will have $H(x, i, Du(y)) < \delta$ for y on the support of $\rho_\varepsilon(x - y)$ and small enough $\varepsilon > 0$, so $H(x, i, Du_\varepsilon(x)) < \delta$. Sending $\varepsilon \to 0$, the usual argument of stability of viscosity subsolutions under locally uniform convergence ([2, 18]) completes the proof. ∎

Condition (A3) provides a local Lipschitz constant for all subsolutions. This fact is standard material,, but we find it convenient to state it explicitly.

Proposition 2.4 *Let $i \in \mathfrak{I}$ and let B be an open ball whose closure $\overline{B}$ is contained in Ω. Let $K = \overline{B} \times \mathfrak{I}$ and let R_K be a constant as assigned to K by condition* (A3). *Then, for every $u \in S_H^-$, $u(\,\cdot\,, i)$ is R_K-Lipschitz on $\overline{B}$.*

The proof is similar to that of [2, Lemme 2.5, pp. 33-34] (although the statement there is not exactly the same).

A very important consequence of Proposition 2.4 is that, by the Ascoli–Arzela theorem, any locally uniformly bounded subset of S_H^- is relatively compact in $C(\Omega \times \mathfrak{I})$ for the topology of locally uniform convergence, because it is equicontinuous on each compact.

The set of subsolutions has a considerable structure, summarized in the proposition below. The second part of the proof is abbreviated as it also follows from closely standard material. See, for example, [1, 2, 18].

Proposition 2.5 *The set S_H^- is closed under the lattice operations* max *and* min. *It is actually a convex complete lattice.*

Proof Let $u, v \in S_H^-$. It is easy to check that both $\max(u, v)$ and $\min(u, v)$ satisfy (C2) and $\max(u, v)$ satisfies (C1). To prove that $\min(u, v)$ satisfies (C1), we note that it is locally Lipschitz too, and we will apply Lemma 2.3. Once more, to simplify the notation, fix any mode $i \in \mathfrak{I}$ and let u, v denote $u(\,\cdot\,, i)$, $v(\,\cdot\,, i)$, respectively. If $u(x) < v(x)$ then $\min(u, v)$ is differentiable at x if and only if u is, and in that case its differential is $Du(x)$ (ikewise if $v(x) < u(x)$). If $u(x) = v(x)$ and both u and v are differentiable at x, it can be checked that $\min(u, v)$ is differentiable at x if and only if $Du(x) = Dv(x)$ and in this case these differentials equal that of $\min(u, v)$ as well. Since u, v, and $\min(u, v)$ are differentiable a.e. on Ω, we have $H(\,\cdot\,, i, \min(u, v)'(\,\cdot\,, i)) \leq 0$ a.e. on Ω, and Lemma 2.3 applies.

The convexity of S_H^- is a consequence of condition (A2), and its completeness is a consequence of the remark following Proposition 2.4 and the stability of conditions (C1) and (C2) under locally uniform convergence. ∎

2.1 The Lagrangian

The Lagrangian L of H is the $\mathbb{R} \cup \{+\infty\}$-valued Legendre transform

$$L(x, i, \xi) = \sup\{\xi \cdot p - H(x, i, p) : p \in \mathbb{R}^n\} .$$

We note two useful facts about the Lagrangian in the following lemma.

Lemma 2.6 *For every $(x_0, i_0) \in \Omega \times \mathfrak{I}$ there exist $\rho > 0$, $a > 0$, and $b > 0$ such that*

(i) *$L(x, i_0, \xi) \leq b$ when $|x - x_0| \leq \rho$ and $|\xi| \leq a$, and*
(ii) *$|\xi|^{-1} L(x, i_0, \xi) \to +\infty$ uniformly on $|x - x_0| \leq \rho$ when $|\xi| \to +\infty$.*

Proof (i) We claim that for some $p_0 \in \mathbb{R}^n$, $H(x_0, i_0, p_0) \leq 0$. Indeed, by local coerciveness, given a closed ball K centered at x_0, there exists $R_K > 0$ such that $H(x, i_0, p) > 0$ for $x \in K$ and $|p| \geq R_K$. If p_0 as above did not exist, there would exist a possibly smaller closed ball K' centered at x_0 such that the minimum of the continuous function $H(x, i_0, p)$ for $x \in K'$ and $|p| \leq R_K$ were positive. That would

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mean that $H(x, i_0, p)$ was positive for all $x \in K'$ and $p \in \mathbb{R}^n$. This is impossible because S_H^- is not empty, and its elements have non-empty superdifferentials almost everywhere.

Now let $p_0 \in \mathbb{R}^n$ be such that $H(x_0, i_0, p_0) \le 0$, let K be a closed ball centered at x_0, and let $R_K > 0$ be such that $H(x, i_0, p) > 0$ for $x \in K$ and $|p - p_0| \ge R_K$. For some smaller closed ball K' centered at x_0, the number $h_0 := \max\{H(x, i_0, p_0) : x \in K'\}$ is smaller than the positive number $h_1 := \min\{H(x, i_0, p) : x \in K,\ |p - p_0| = R_K\}$. Let ρ be the radius of K' and let a be any positive number smaller than $(h_1 - h_0)/R_K$. For each $|\xi| \le a$ and $x \in K'$ the concave function $\varphi(p) = p \cdot \xi - H(x, i_0, p) - p_0 \cdot \xi$ satisfies $\varphi(p_0) \le h_0$ and for all $|p - p_0| = R_K$, $\varphi(p) > h_0$. By concavity, φ must attain its maximum on the R_K ball of p_0, and that maximum equals $L(x, i_0, \xi) - p_0 \cdot \xi$. This shows that $L(x, i_0, \xi)$ is finite for $x \in K'$ and $|p - p_0| \le a$, and a crude estimate for b is, *e.g.*, $(R_K + |p_0|)a + \max\{H(x, i_0, p) : x \in K',\ |p - p_0| \le R_K\}$.

(ii) Let $C > 0$ be any large constant and $h_\rho := \max\{H(x, i_0, p) : |x - x_0| \le \rho,\ |p| = C + 1\}$. Then for all $|\xi| \ge h_\rho$ and $|x - x_0| \le \rho$,

$$
\begin{aligned}
|\xi|^{-1} L(x, i_0, \xi) &\ge \max\{p \cdot |\xi|^{-1}\xi - |\xi|^{-1} H(x, i_0, p) : |x - x_0| \le \rho,\ |p| = C + 1\} \\
&\ge C. \qquad \blacksquare
\end{aligned}
$$

Note that it may happen that, for large enough $|\xi|$, $L(x, i, \xi)$ takes the value $+\infty$, because we do not require $H(x, i, p)$ to be superlinear in p.

2.2 Distance-like Functions

Given two state-modes $A = (x, i)$ and $B = (y, j)$ and a positive real $t > 0$, we will denote by $\Gamma_t(A, B)$ the set of all paths $\gamma = (\gamma_\Omega, \gamma_{\mathcal{I}})\colon [0, t] \to \Omega \times \mathcal{I}$ such that γ_Ω is Lipschitz, $\gamma(0) = B$, $\gamma(t) = A$ and, for some finite partition $0 = t_0 < t_1 < \cdots < t_m = t$ of $[0, t]$, on each subinterval $[t_a, t_{a+1})$ the mode component $\gamma_{\mathcal{I}}$ is constant. In these conditions, we will define:

$$
\mathcal{S}(\gamma) = \sum_{a=0}^{m-1} \int_{t_a}^{t_{a+1}} L\big(\gamma(s), \dot{\gamma}_\Omega(s)\big)\, ds \ + \ k\big(\gamma_\Omega(t_{a+1}), \gamma_{\mathcal{I}}(t_{a+1}), \gamma_{\mathcal{I}}(t_a)\big).
$$

Now we define two distance-like functions between state-modes. In the unimodal case, the function d_L is called Mañé potential in Aubry–Mather theory, and the symbol d_H for the other function is used in [21]. For each pair of state-modes A, B:

$$
\begin{aligned}
d_H(A, B) &= \sup\{u(A) - u(B) : u \in S_H^-\} \\
d_L(A, B) &= \inf\{\mathcal{S}(\gamma) : t > 0, \gamma \in \Gamma_t(A, B)\}.
\end{aligned}
$$

Some remarks are in order:

(1) The function d_H is well defined because subsolutions exist by condition (A4), Ω is connected and k takes finite values, so by Proposition 2.4 the set of differences defining $d_H(A, B)$ is bounded. In particular, for every state-mode A, $d_H(A, A) = 0$.

(2) Both d_H and d_L satisfy the triangular inequality, but may fail to be distances, because they may not be positive or symmetric.
(3) The Lagrangian L must be locally bounded below: for instance, $L(A,\xi) \geq -H(A,0)$. Thus, by Lemma 2.6(i) and the triangular inequality, d_L is locally Lipschitz in both state variables.
(4) The same is true for d_H, as a consequence of condition (A3) and Proposition 2.4.

The following theorem asserts that these two functions are actually the same. This has been proved under slightly stronger conditions in [20] and elsewhere.

Theorem 2.7 *For all state-modes A and B, $d_H(A,B) = d_L(A,B)$.*

Proof Let A and B be state modes. To prove that $d_H(A,B) \leq d_L(A,B)$ it is enough to show that given $u \in S_H^-$, $t > 0$, and $\gamma \in \Gamma_t(A,B)$, we have $u(A) - u(B) \leq \mathcal{S}(\gamma)$. Let t_a, $0 \leq a \leq m$ be the partition mentioned in the definition of $\Gamma_t(A,B)$. Let $x, y \in \Omega$ be two states connected by a continuously differentiable path η on Ω and let $i \in \mathcal{I}$ be a mode. Due to the continuity of H and to its convexity in p, for each small enough $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the mollification u_ε of $u(\,\cdot\,,i)$ is a subsolution of $H + \delta$ on a neighborhood of the image of the path (η, i). Then

$$\begin{aligned} u_\varepsilon(x) - u_\varepsilon(y) = \int u_\varepsilon'(\eta(s))\dot\eta(s)\mathrm{d}s &\leq \int (L(\eta(s),\dot\eta(s)) + H(\eta(s), u_\varepsilon'(\eta(s)))\mathrm{d}s \\ &\leq \int (L(\eta(s),\dot\eta(s)) - \delta)\mathrm{d}s\,, \end{aligned}$$

and letting ε tend to zero, $\delta = \delta(\varepsilon)$ can also tend to zero, and we obtain

$$u(x) - u(y) \leq \int L(\eta(s),\dot\eta(s))\mathrm{d}s\,. \tag{2.1}$$

Now we have:

$$\begin{aligned} u(A) - u(B) = \sum_{a=0}^{m-1} (u(\gamma(t_{a+1})) - u(\gamma(t_a))) = \\ \sum_{a=0}^{m-1} (u(\gamma_\Omega(t_{a+1}), \gamma_{\mathcal{I}}(t_a)) - u(\gamma_\Omega(t_a), \gamma_{\mathcal{I}}(t_a)))+ \\ \sum_{a=0}^{m-1} (u(\gamma_\Omega(t_{a+1}), \gamma_{\mathcal{I}}(t_{a+1})) - u(\gamma_\Omega(t_{a+1}), \gamma_{\mathcal{I}}(t_a)))\,. \end{aligned}$$

Applying inequality (2.1) to the next-to-last sum and the condition (C2) in the definition of S_H^- to the last sum, it is proved that $u(A) - u(B) \leq \mathcal{S}(\gamma)$.

We now prove that $d_H(A,B) \geq d_L(A,B)$, by showing that for each state-mode B the function $u(A) = d_L(A,B)$ belongs to S_H^-. Firstly, we will prove that u satisfies condition (C2). Let $x \in \Omega$, $i, j \in \mathcal{I}$. It suffices to prove that for every $\varepsilon > 0$, $u(x,i) - u(x,j) - \varepsilon < k(x,i,j)$. For some $t > 0$ there must exist $\gamma \in \Gamma_t((x,j),B)$ such that $\mathcal{S}(\gamma) < u(x, j) + \varepsilon$. For each $\delta > 0$, let γ_δ be the extention of γ to $[0, t + \delta]$ such that $\gamma_\delta(s) = \gamma(t)$ for $t < s < t + \delta$ and $\gamma_\delta(t + \delta) = (x, i)$. Then $\gamma_\delta \in \Gamma_{t+\delta}((x, i), B)$ and $\mathcal{S}(\gamma_\delta) \to \mathcal{S}(\gamma) + k(x, i, j)$ when $\delta \to 0$, which shows that $u(x, i) < u(x, j) + \varepsilon + k(x, i, j)$.

To prove that u satisfies condition (C1) we first note that by remark (3) after the definitions of d_H and d_L, u is locally Lipschitz and so, by Rademacher's Theorem, for every $i \in \mathcal{I}$ $u(\,\cdot\,, i)$ is differentiable almost everywhere in Ω.

Now let $(x_0, i) \in \Omega \times \mathcal{I}$ and suppose that $u(\,\cdot\,, i)$ has differential p_0 at x_0. By Lemma 2.3, it suffices to prove that $H(x_0, i, p_0) \leq 0$. Suppose that $H(x_0, i, p_0) > c > 0$, and we will arrive at a contradiction. Define on Ω the constant field $\mu(x) = p_0$. By Proposition 2.1 there exist a $\delta > 0$ and a Lipschitz path $\gamma_0 : [-\delta, \delta] \to \Omega$ such that $\dot\gamma_0(s) \in D_p^-(H(\gamma_0(s), i, p_0))$ a.e. on $[-\delta, \delta]$, *i.e.*, $p_0\dot\gamma_0(s) - H(\gamma_0(s), i, p_0) = L(\gamma_0(s), i, \dot\gamma_0(s))$ for almost every $s \in [-\delta, \delta]$. For $0 < t < \delta$ small enough, $H(\gamma_0(s), i, p_0) > c > 0$ for all $s \in [0, t]$ and then we have

$$\begin{aligned}
&d_L\big(\big(\gamma_0(t), i\big), (x_0, i)\big)\\
&\quad\geq u\big(\gamma_0(t), i\big) - u\big(\gamma_0(0), i\big) = p_0 \cdot \big(\gamma_0(t) - \gamma_0(0)\big) + o\big(|\gamma_0(t) - \gamma_0(0)|\big)\\
&\quad= \int_0^t p_0 \cdot \dot\gamma_0(s) \mathrm{d}s + o\big(|\gamma_0(t) - \gamma_0(0)|\big)\\
&\quad= \int_0^t \Big(H\big(\gamma_0(s), i, p_0\big) + L\big(\gamma_0(s), i, \dot\gamma_0(s)\big)\Big) \,\mathrm{d}s + o\big(|\gamma_0(t) - \gamma_0(0)|\big)\\
&\quad\geq ct + \int_0^t L\big(\gamma_0(s), i, \dot\gamma_0(s)\big) \,\mathrm{d}s + o\big(|\gamma_0(t) - \gamma_0(0)|\big)\\
&\quad\geq ct + d_L\big(\big(\gamma_0(t), i\big), (x_0, i)\big) + o\big(|\gamma_0(t) - \gamma_0(0)|\big),
\end{aligned}$$

(the first inequality is a consequence of the triangular inequality for d_L) so, cancelling out $d_L((\gamma_0(t), i), (x_0, i))$, dividing over by t, making $t \to 0$, and observing that γ_0 is Lipschitz, we have $c \leq 0$, a contradiction. ∎

We make the two following remarks about this theorem and its proof:

(1) One consequence of the theorem is that $d_L(A, A) = 0$ for every state-mode A and thus there can be no loops (paths beginning and ending at the same state-mode) of negative action. This fact will be important in the next section.

(2) In the first part of the proof it is shown that if u is a subsolution and γ is a Lipschitz path with initial and final times t_1 and t_2 respectively, then $u(\gamma(t_2)) - u(\gamma(t_1)) \leq \mathcal{S}(\gamma)$. This fact will be used right ahead.

We conclude this section with an example of a Hamiltonian H such that for every $k \in \mathbb{R}$, $H + k$ satisfies conditions (A1) through (A3), but it does not satisfy condition (A4). In other words, H does not have a critical value.

Let Ω be the open square $\{(x, y) \in \mathbb{R}^2 : |x|, |y| < 1\}$ and let $\mathcal{I}$ be a singular set. Since this is an unimodal example, the mode variable will be suppressed to simplify the notation. Let $\alpha = \alpha(x, y)$ be a continuous (or even smooth) function on Ω, which is never smaller than 1 and constant equal to 1 outside of the square $[-\frac{1}{2}, \frac{1}{2}] \times$

$[-1,0]$ and equal to $1/(y+1)$ on the square $[-\frac{1}{3},\frac{1}{3}]\times(-1,-\frac{1}{3}]$. Let $\mathbf{e_1}$ denote the vector $(1,0)$, $q=(x,y)$ ("position"), $p=(p_x,p_y)$ ("moment"), and $v=(v_x,v_y)$ ("velocity"). The Hamiltonian and the corresponding Lagrangian are

$$H(q,p)=\frac{|p|^2}{2}+\alpha(q)p_x+\frac{\alpha(q)^2}{4},$$
$$L(q,v)=\frac{1}{2}|v-\alpha(q)\mathbf{e_1}|^2-\frac{\alpha(q)^2}{4}.$$

It is easily checked that for every $k\in\mathbb{R}$, $H+k$ satisfies conditions (A1) through (A3), and its Lagrangian is $L-k$.

First let $k=0$. For $-1<b<-1/3$ let γ_b be a rectangular path starting at $(-\frac{1}{2},0)$, travelling downwards at speed one until $(-\frac{1}{2},b)$, then right at speed $\alpha(x,b)$ until $(\frac{1}{2},b)$ and then upwards at speed one until $(\frac{1}{2},0)$. The Lagrangian L is negative on the horizontal track, so the action here must not only be negative, but also smaller than the action of the part where $|x|\le 1/3$, which is $-1/(6(b+1))$. The sum of the actions of the vertical tracks of γ_b is $-3b/2$. Hence, the action of γ_b tends to $-\infty$ as $b\to -1$.

Now let k be any real value. The action of each path γ_b equals its action in the previous case minus k times the total travelling time, which is no greater than three. This shows that there cannot be a subsolution u of $H+k$ for any value of $k\in\mathbb{R}$, no matter how negative, because the difference $u(\frac{1}{2},0)-u(-\frac{1}{2},0)$ should be smaller than the action of any γ_b. However, unlike the example in the introduction, this time, for every $k<0$ $H(q,p)+k$ has a negative value at each position q: take $p=-\alpha(q)\mathbf{e_1}$.

This type of Lagrangian goes back to Mañé's works; see [15] and the references therein.

3 The Aubry Set

We will denote by S_H the set of those $u\in S_H^-$ that satisfy one of the following conditions at each state-mode (x,i):

(C1') For every $p\in D^-u(\cdot,i)(x)$, $H(x,i,p)\ge 0$.
(C2') For some mode $j\ne i$, $u(x,i)-u(x,j)=k(x,i,j)$.

Conditions (C1) and (C1') together mean that on each mode i, $u_i=u(\cdot,i)$ is a viscosity solution of $H(x,i,Du_i(x))=0$. Here condition (C2') replaces the weaker condition (C2). The mode switching costs must now have the exact prescribed values. The elements of S_H are called solutions, and they are the continuous functions that satisfy equality (1.1) in the viscosity sense. Note that both sets S_H^- and S_H are preserved by the addition of constants. Given a suitable subset U of the state-mode space, we will use $S_H^-(U)$ to denote $S_{H|_U}^-$ and likewise for $S_H(U)$. The following proposition extends a well-known property of unimodal Hamiltonians to the present setting, under a minor adaptation of its standard proof.

Proposition 3.1 *For each state-mode B, $d_H(\cdot,B)\in S_H(\Omega\times\mathcal{I}\setminus\{B\})\cap S_H^-$.*

Proof Let $B=(y,j)$ be a state-mode. For notational convenience, we will call u to $d_H(\cdot,B)$. Suppose that there exists a state-mode $A=(x,i)$ different from B, where u yields strict inequalities in condition (C2) and for some $p \in D^- u(\,\cdot\,,i)(x)$, $H(x,i,p) < 0$. Then there exists a smooth function ϕ defined on a neighborhood of x such that $\phi'(x) = p$ and $u(\,\cdot\,,i) - \phi(\cdot)$ attains zero as a strict local minimum at x (see, *e.g.*, [2, pp. 18–19] or [18, Prop. 26, pp. 70–72]). Let $\rho > 0$ be small enough so that, for $|z-x| \le \rho$ and $i' \ne i$, $H(z,i,\phi'(z)) < 0$ and $u(z,i) - u(z,i') < k(z,i,i')$. Let $\varepsilon > 0$ be smaller than the positive maximum of $u(z,i) - \phi(z)$ on $|z-x| = \rho$ and the positive maximum of $k(z,i,i') - u(z,i) + u(z,i')$ on $|z-x| \le \rho$ and $i' \ne i$. Then define $v(z,i) = \max(\phi(z) + \varepsilon, u(z,i))$ for $|z-x| \le \rho$ and $v = u$ on the remaining state-modes. Then $v \in S_H^-$ and $v(A) > u(A)$, while $v(B) = u(B) = 0$. Hence, $d_H(A,B) = u(A) - u(B) < v(A) - v(B)$, a contradiction. ∎

The Aubry set is the set $\mathcal{A}$ of those state-modes B such that for all (equivalently, for some) $T > 0$ the infimum of the actions of all paths in $\cup_{t \ge T} \Gamma_t(B,B)$ is zero.

Proposition 3.2 *If $B = (y,j) \in \mathcal{A}$, then $d_H(\,\cdot\,,B) \in S_H$.*

Proof If $H(B,\,\cdot\,) \ge 0$, then the conclusion follows. Otherwise let $p \in \mathbb{R}^n$ be such that $H(B,p) < 0$. Assume that $B \in \mathcal{A}$ and let (γ_n) be a sequence of paths in $\cup_{t \ge 1} \Gamma_t(B,B)$ such that $\mathcal{S}(\gamma_n) \to 0$. For small enough $\rho > 0$, $H(\,\cdot\,,j,p)$ has a negative minimum $-c < 0$ on the closed ρ-ball centered at y. If some γ_n stays in $\{(x,j) : |x-y| \le \rho\}$, then its action is no less than $\int_0^t p\cdot\dot\gamma(s) - H(\gamma(s),p)\mathrm{d}s \ge ct \ge c$, so, taking a subsequence if necessary, one may assume that no path γ_n stays in that set. This means that every path γ_n must pass through some state-mode E_n in the set $\mathbf{K}$ that is the union of the sets $\{(x,j) : |x-y| = \rho\}$ and $\{(x,j') : |x-y| \le \rho,\ j' \ne j\}$. Using, respectively, the triangle inequality for d_H, Theorem 2.7, the definition of d_L, and the fact that γ_n passes through E_n, we have

$$0 \le d_H(B,E_n) + d_H(E_n,B) = d_L(B,E_n) + d_L(E_n,B) \le \mathcal{S}(\gamma_n),$$

hence $d_H(B,E_n) + d_H(E_n,B)$ tends to zero. Let $E \in \mathbf{K}$ be a sublimit of the state-modes E_n, which must exist because $\mathbf{K}$ is compact. Then $E \ne B$ and by continuity of d_H, one has $d_H(E,B) + d_H(B,E) = 0$. The argument employed in [21, p. 2169], can be directly adapted to this situation to show that $d_H(\,\cdot\,,B)$ is a solution (and so is $d_H(\,\cdot\,,E)$, because the roles of E and B may be switched). Applying the triangle inequality twice and the previous equality once, we have

$$d_H(\,\cdot\,,B) \le d_H(\,\cdot\,,E) + d_H(E,B) \le d_H(\,\cdot\,,B) + d_H(B,E) + d_H(E,B) = d_H(\,\cdot\,,B),$$

so $d_H(\,\cdot\,,B) = d_H(\,\cdot\,,E) + d_H(E,B)$ and, by Proposition 3.1, $d_H(\,\cdot\,,B)$ is a solution away from B, and $d_H(\,\cdot\,,E)$ is a solution away from E, but a solution plus a constant is still a solution. Given any state-mode A, repeated usage of the triangle inequality for d_H gives:

$$d_H(A,E) \le d_H(A,B) + d_H(B,E) = d_H(A,B) - d_H(E,B), \text{ so}$$

$$d_H(A,E) + d_H(E,B) \le d_H(A,B) \le d_H(A,E) + d_H(E,B),$$

showing that $d_H(A,B) = d_H(A,E) + d_H(E,B)$. Since $E \ne B$, Proposition 3.1 states that $d_H(\,\cdot\,,E)$ is a solution on a small enough neighborhood of B, and solutions are closed under addition of constants. Thus, $d_H(\,\cdot\,,B)$ is a solution near B (and everywhere else, by Proposition 3.1 once more), so the proof is complete. ∎

We distinguish a set of state-modes $B \in \Omega \times \mathcal{I}$ that satisfy a special locality condition:

(LC) For every neighborhood U of B there exists a neighborhood V of B such that for every $A \in V$ there exists a sequence of paths $\gamma_n \in \Gamma_{t_n}(A, B)$ in U such that $\mathcal{S}(\gamma_n) \to d_L(A, B)$.

Proposition 3.3 *If $B = (x_0, i_0)$ does not satisfy* (LC)*, then $B \in \mathcal{A}$.*

The proof uses the following Lemma.

Lemma 3.4 *Let $(x_0, i_0) \in \Omega \times \mathcal{I}$ and $\rho > 0$ be as in Lemma* 2.6*. For each natural n let $\gamma_n \in \Gamma_{t_n}(x_n, i, x_0, i_0)$ be a path whose state component $\gamma_{n,\Omega}$ remains in the closed ρ-ball about x_0 $\overline{B_\rho(x_0)}$, with $|x_n - x_0| = \rho$. If $t_n \to 0$, then $\mathcal{S}(\gamma_n) \to +\infty$.*

Proof Let $r_n = \frac{\rho}{2t_n}$, $F_n = \{s \in [0, \delta_n] : |\dot\gamma_{n,\Omega}(s)| > r_n\}$ and $S_n = [0, t_n] \setminus F_n$. We have

$$\int_{S_n} L\big(\gamma_n(s), \dot\gamma_{n,\Omega}(s)\big)\, \mathrm{d}s \ge -\int_{S_n} H\big(\gamma_n(s), 0\big)\, \mathrm{d}s,$$

which are bounded below because $H(\,\cdot\,, 0)$ is bounded below on $\overline{B_\rho(x_0)} \times \mathcal{I}$. Now, for $r > 0$ let $C(r)$ be the minimum of $\{|\xi|^{-1}L(x, i_0, \xi) : |\xi| \ge r,\ |x - x_0| \le \rho\}$. By Lemma 2.6(ii), $C(r) \to +\infty$ as $r \to +\infty$. Then

$$\int_{F_n} L(\gamma_n(s), \dot\gamma_{n,\Omega}(s))\mathrm{d}s \ge \int_{F_n} C(r_n)|\dot\gamma_{n,\Omega}(s)|\mathrm{d}s \ge C(r_n)\frac{\rho}{2} \to +\infty \ \text{ as } n \to \infty.$$

The last inequality is due to the fact that the distance ran on the state space during S_n cannot exceed $\rho/2$. Now note that for each n, the action $\mathcal{S}(\gamma_n)$ equals the sum of these two integrals plus positive mode switching costs. ■

Proof of Proposition 3.3 Suppose that $B = (x_0, i_0)$ does not satisfy condition (LC). Then there exists $\rho > 0$ such that:

(1) for every $j \in \mathcal{I}$, Lemma 2.6 holds on $\overline{B_\rho(x_0)} \times \{j\}$;
(2) for every $x, x' \in \overline{B_\rho(x_0)}$ and $i, i' \in \mathcal{I}$, both $|d_L(x, i, x', i)|$ and $|d_L(x', i, x, i)|$ are smaller than both $k(x, i, i')/3$ and $k(x, i', i)/3$;
(3) there exists a sequence $x_n \to x_0$ and a sequence of paths $\gamma_n \in \Gamma_{t_n}(x_n, x_0)$ that do not stay in $\overline{B_\rho(x_0)} \times \{i_0\}$ and whose actions satisfy $d_H(x_n, i_0, x_0, i_0) \le \mathcal{S}(\gamma_n) < d_H(x_n, i_0, x_0, i_0) + 1/n$.

By continuity of d_H, $\mathcal{S}(\gamma_n)$ tends to zero. Condition (2) on ρ implies that the state components of γ_n can not stay in $\overline{B_\rho(x_0)}$. If they do, since each path of constant mode i from x to x' cannot have action lower than $-|d_L(x', i, x, i)|$, the action $\mathcal{S}(\gamma_n)$ must exceed one third of the positive minimum switching cost for states on $\overline{B_\rho(x_0)}$, a contradiction.

Each γ_n may be continued to a loop closing at B in such a way that the actions of such loops still tend to zero (*e.g.,* just move from x_n straight to x_0 at a constant speed smaller than the a in Lemma 2.6(i)). The proof is complete if we show that the sequence t_n cannot tend to zero. Suppose then that it does.

For each n, let δ_n be the first instant when γ_n hits the boundary $\partial B_\rho(y) \times \mathcal{I}$. By Lemma 3.4, $\int_0^{\delta_n} L(\gamma_n(s), \dot{\gamma}_n(s))\mathrm{d}s$ tends to $+\infty$. In that case the actions of the γ_n on $[\delta_n, t_n]$ tend to $-\infty$, thus becoming lower than $-d_L(\gamma_n(\delta_n), B)$ for large enough n. For such an n, one might continue the restriction of γ_n to $[\delta_n, t_n]$ to a loop closing at $\gamma_n(\delta_n)$ with negative action, contradicting the remark at the end of Section 2. ∎

Proposition 3.5 *If $d_H(\,\cdot\,, B) \in S_H$, then $B \in \mathcal{A}$.*

Proof By Proposition 3.1, the hypothesis simply means that $d_H(\,\cdot\,, B)$ satisfies one of the conditions (C1') or (C2') at $B = (x_0, i_0)$.

Suppose first that $d_H(\,\cdot\,, B)$ satisfies condition (C1') at B. If $H(B, \,\cdot\,) \geq 0$, then the constant path $\gamma(s) = B$ has zero action for every time length and the conclusion holds. So, once more, let $p \in \mathbb{R}^n$ such that $H(B, p) < 0$.

By Proposition 3.3, if B does not satisfy condition (LC), then $B \in \mathcal{A}$. If it does, let U be a neighborhood of B where $H(\,\cdot\,, p)$ has a negative upper bound $-c < 0$. For any path γ in U (say, of duration t),

$$\mathcal{S}(\gamma) \geq \int_0^t p \cdot \dot{\gamma}_\Omega(s) - H\big(\gamma(s), p\big)\,\mathrm{d}s \geq p \cdot (x - x_0) + ct \geq p \cdot (x - x_0)\,.$$

Let V be a neighborhood of B as prescribed by condition (LC). Then we argue as in the first part of the proof of [20, Proposition A.3, p. 264]. Given any $A \in V$, $d_H(A, B)$ can be approximated by actions $\mathcal{S}(\gamma)$ of paths γ in U, so, by the above inequality, p belongs to the subdifferential of $d_H(\,\cdot\,, i_0, B)$ at x_0. This contradicts the hypothesis, since $H(B, p) < 0$.

Suppose now that $d_H(\,\cdot\,, B)$ satisfies condition (C2') at B. Then for some mode $i \neq i_0$, $d_H(x_0, i, x_0, i_0) = -k(x_0, i, i_0)$. Let $A = (x_0, i)$. There exists a sequence of paths $\gamma_n \in \Gamma_{t_n}(A, B)$ such that $\mathcal{S}(\gamma_n) \to -k(x_0, i, i_0)$. For each n we may remain at A for time $1/n$, if necessary, and then switch mode back to B with cost $k(x_0, i, i_0)$. This creates a sequence of loops at B whose actions tend to zero. The proof will be complete if we show that t_n cannot tend to zero.

Suppose that $t_n \to 0$. Choose $\rho > 0$ so to fulfill condition (2) in the proof of Proposition 3.3, preventing the state components of the loops from remaining in $\overline{B_\rho(x_0)}$. An argument similar to the end of that proof leads once more to a contradiction. ∎

Propositions 3.2 and 3.5 are thus summarized by the following theorem.

Theorem 3.6 *Let B be a state-mode. Then $B \in \mathcal{A}$ if and only if $d_H(\,\cdot\,, B) \in S_H$.*

This is part (i)⇔(iv) of [17, Theorem 1.4] for the present setting.

We make the following observations.

(1) The Aubry set $\mathcal{A}$ is closed in $\Omega \times \mathcal{I}$. This can be seen either directly from its definition or by noting that solutions are stable under locally uniform convergence.

(2) Given a suitable subset $U \subset \Omega \times \mathcal{I}$, the Aubry set of the restriction $H|_U$ is contained in the the Aubry set of H. This follows from the definition. In particular, for each mode $i \in \mathcal{I}$, let $\mathcal{A}_i$ be the Aubry set of the unimodal Hamiltonian $H(\,\cdot\,, i, \,\cdot\,)$. Then $\mathcal{A}_i \times \{i\} \subset \mathcal{A}$.

The following example shows that the Aubry set may have elements not arising from the subsets mentioned in observation (2) above. It consists of translations of the eikonal equation on two half-lines. The switching cost tends to zero at the origin and, until the point 1, it turns a specific subsolution into a solution. Further ahead, it grows quadratically, and the two layers become independent there.

Example 3.7 Let $\Omega = (0, +\infty)$, $\mathcal{I} = \{1, 2\}$, and

$$H(x, i, p) = \begin{cases} |p - 1| - 1, & i = 1 \\ |p + 2| - 1, & i = 2 \end{cases} \quad k(x, 1, 2) = k(x, 2, 1) = \begin{cases} x, & 0 < x \le 1 \\ x^2, & x > 1. \end{cases}$$

Conditions (A1)–(A3) are easily verified, and $u(x, i) = (1 - i)x$ is a subsolution (actually, a solution), so (A4) holds too. The Lagrangian is

$$L(x, i, \xi) = \begin{cases} 1 + \xi, & i = 1 \\ 1 - 2\xi, & i = 2 \end{cases} \text{ if } |\xi| \le 1 \text{ and } +\infty \text{ if } |\xi| > 1.$$

Both sets $\mathcal{A}_1$, $\mathcal{A}_2$ as defined in observation (2) above are empty, but $\mathcal{A} = (0, 1] \times \{1, 2\}$. On $\mathcal{A}$, every subsolution u must equal $(1 - i)x + C$, where $C = \lim_{x \to 0} u(x, i)$, $i = 1, 2$.

Note that this example, well within our setting, is slightly more general than the systems considered in [4]. No continuous function u can satisfy both inequalities $H(x, 1, Du(x)) \le 0$ and $H(x, 2, Du(x)) \le 0$ (even in the viscosity sense), so this Hamiltonian does not satisfy [4, condition (7), p. 1293], which assumed the existence of smooth functions ψ and $f_i \ge 0$ ($i \in \mathcal{I}$) such that $H(x, i, D\psi) \le -f_i(x)$. In order to highlight a new feature in our approach we generalize that condition. Let $\psi \in S_H^-$ be of class C^1 (in the state variable) and let $\mathcal{B}$ denote the set of state-modes where ψ satisfies any of the conditions (C1') or (C2'). With the straightforward adaptations to the plurimodal case, this function ψ can be used in the same way as the function ψ in the proof of [20, Proposition A.3] to prove that $\mathcal{A} \subset \mathcal{B}$. In particular, the disjoint union of the zero sets of the functions f_i (denoted by $\mathcal{B}_i$ in [4]) contains the Aubry set as we define it here.

4 Boundary Values

We recall that regularity properties, differentials, subdifferentials, *etc.*, of functions on the state-mode space refer to the state variable.

Let U be an open subset of $\Omega \times \mathcal{I}$ and let $\overline{U}_\Omega$ be the projection onto Ω of its closure $\overline{U}$. We define the extended boundary of U in the following way: $\partial_{\mathcal{I}} U := \overline{U}_\Omega \times \mathcal{I} \setminus U$.

The result below is the essence of the proof of [21, Theorem 2.3]. We extend their argument to present setting, the plurimodal case.

Theorem 4.1 *Let U be the interior of a compact subset of $\Omega \times \mathcal{I} \setminus \mathcal{A}$ and let $v \in S_H$. Then for every $A \in U$, $v(A) = \min\{v(B) + d_H(A, B) : B \in \partial_{\mathcal{I}} U\}$.*

If U is empty, then the theorem is trivially true, so we will assume that it is not. The proof requires the following lemma.

Lemma 4.2 *There exists a smooth function ψ on a neighborhood of $\overline{U}$ such that, for all (x,i), (x,i') in its domain, $H(x,i,\psi'(x,i)) < 0$ and $\psi(x,i) - \psi(x,i') < k(x,i,i')$ if $i \neq i'$.*

Proof First we proceed as in the proof of Proposition 3.1. For notational convenience, given a state-mode A, we will call u_A to $d_H(\,\cdot\,,A)$. Since $\overline{U}$ does not intersect $\mathcal{A}$, for each $A = (x,i) \in \overline{U}$ there exists a smooth function ϕ_A on a neighborhood of x such that $u_A - \phi_A$ has a zero strict local minimum at A and a radius $\rho_A > 0$ such that, in the closed ball $|z - x| \leq \rho_A$, $H(z,i,\phi_A'(z))$ has a negative maximum and u_A yields strict switching cost inequalities. Let $\varepsilon > 0$ be smaller than the positive maximum of $u_A(z,i) - \phi_A(z)$ on $|z-x| = \rho_A$ and the positive maximum of $k(z,i,i') - u_A(z,i) + u_A(z,i')$ on $|z-x| \leq \rho_A$ and $i' \neq i$. Define the function ψ_A by $\psi_A(z,i) = \max(\phi_A(z) + \varepsilon, u(z,i))$ for $|z-x| \leq \rho_A$ and by $\psi_A = d_H(\,\cdot\,,A)$ elsewhere. Then $\psi_A \in S_H^-$ and, on some neighborhood U_A of A, it is smooth, its differentials have negative Hamiltonian, and it yields strict switching cost inequalities.

The rest is similar to the proof of [21, Theorem 1.5], so we only provide a sketch here. Since $\overline{U}$ is compact, a finite number of such neighborhoods U_A must cover it. On the union of these neighborhoods, the average of the corresponding functions ψ_A is differentiable a.e., its differentials have negative Hamiltonians, and it yields strict switching cost inequalities. A sharp enough mollification of this function is the desired function ψ. ∎

Proof of Theorem 4.1 For $A \in \Omega \times \mathcal{I}$, define $\tilde{v}(A) = \min\{v(B) + d_H(A,B) : B \in \partial_{\mathcal{I}} U\}$. The minimum exists by the compactness of $\partial_{\mathcal{I}} U$, and $\tilde{v}$ is continuous, because d_H is locally Lipschitz in both (state) variables. By Proposition 2.5, $\tilde{v} \in S_H^-$. It follows from the definition of d_H that $\tilde{v} \geq v$ and $\tilde{v} = v$ on $\partial_{\mathcal{I}} U$ because $d_H(B,B) = 0$. It remains to prove that for all $A \in U$, $\tilde{v}(A) \leq v(A)$.

Suppose that for some $A \in U$ we have $\tilde{v}(A) > v(A)$. Then $\tilde{v} - v$ attains a positive maximum $\beta > 0$ over $\overline{U}$ at some point in $A_0 = (x_0, i_0) \in U$.

Let ψ be the function from Lemma 4.2 and let ν be the maximum of $|\psi - v|$ over $\overline{U}$. Let $\lambda > 0$ be a positive real smaller than $\frac{\beta}{\beta+2\nu} < 1$ and define $v_\lambda = (1-\lambda)\tilde{v} + \lambda\psi$ on the domain of ψ. Being a convex combination of $\tilde{v}$ and ψ, v_λ is a strict subsolution on the domain of ψ, *i.e.,* it is a subsolution on that set and it does not satisfy (C1') or (C2') anywhere. For $B \in \overline{U} \setminus U$, $v_\lambda(B) - v(B) = \lambda(\psi(B) - v(B))$, since $B \in \partial_{\mathcal{I}} U$. Hence, for $B \in \overline{U} \setminus U$, $v_\lambda(B) - v(B) \leq \lambda\nu$.

We have $v_\lambda(A_0) - v(A_0) \geq (1-\lambda)\beta - \lambda\nu > \lambda\nu$, so $v_\lambda - v$ has a positive maximum over $\overline{U}$. Let T be the set of the points where that maximum is attained. By the inequalities above, $T \subset U$. We claim that v cannot satisfy condition (C2') at any point of T.

Suppose that v satisfies condition (C2') at $A_1 = (x_1, i_1) \in T$. Then for some $i_2 \neq i_1$, $v(x_1,i_1) - v(x_1,i_2) = k(x_1,i_1,i_2)$. If $A_2 := (x_1,i_2) \in \overline{U}$, then $v_\lambda(x_1,i_1) - v_\lambda(x_1,i_2) < k(x_1,i_1,i_2)$, so $v_\lambda(A_1) - v(A_1) < v_\lambda(A_2) - v(A_2)$, which is a contradiction. Thus, we must have $A_2 \in \partial_{\mathcal{I}} U \setminus \overline{U}$. But then $\tilde{v}(A_2) = v(A_2)$ and $\tilde{v}(A_1) - \tilde{v}(A_2) \leq$

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$k(x_1, i_1, i_2)$, so

$$v_\lambda(A_1) - v(A_1) = (1-\lambda)\big(\tilde{v}(A_1) - v(A_1)\big) + \lambda\big(\psi(A_1) - v(A_1)\big)$$
$$\leq \lambda\big(\psi(A_1) - v(A_1)\big) \leq \lambda\nu$$

could not even reach $(1-\lambda)\beta - \lambda\nu \leq v_\lambda(A_0) - v(A_0)$, again a contradiction.

By the continuity of the switching cost functions, v cannot satisfy condition (C2') on a neighborhood W of T in U. We are now ready for the final step, the well-known argument of doubling variables [1,2,18,23].

Let $A, B \in \overline{U}$, where $A = (x, i)$, $B = (y, j)$ and let $\varepsilon > 0$. Define

$$\Psi_\varepsilon(A, B) = v_\lambda(A) - v(B) - \varepsilon^{-2}(|x-y|^2 + 1 - \delta_{ij}),$$

where δ is the usual Kronecker symbol. For each $\varepsilon > 0$, Ψ_ε is continuous on $\overline{U} \times \overline{U}$, and for $A_1 \in T$, $\Psi_\varepsilon(A_1, A_1) = v_\lambda(A_1) - v(A_1)$ is the positive maximum of $v_\lambda - v$ over $\overline{U}$, which we shall now denote by β_λ. Thus, for each $\varepsilon > 0$ there exists $(A_\varepsilon, B_\varepsilon) \in \overline{U} \times \overline{U}$, which maximizes Ψ_ε and $\Psi_\varepsilon(A_\varepsilon, B_\varepsilon) \geq \beta_\lambda$. Let $A_\varepsilon = (x_\varepsilon, i_\varepsilon)$ and $B_\varepsilon = (y_\varepsilon, j_\varepsilon)$. We have

$$0 \leq |x_\varepsilon - y_\varepsilon|^2 + 1 - \delta_{i_\varepsilon j_\varepsilon} \leq \varepsilon^2\big(\beta_\lambda - v_\lambda(A_\varepsilon) + v(B_\varepsilon)\big) .$$

This inequality leads to the following conclusions:

(1) Since both v_λ and v are bounded on $\overline{U}$, for small enough ε we have $i_\varepsilon = j_\varepsilon$, and $|x_\varepsilon - y_\varepsilon| \to 0$ as $\varepsilon \to 0$.
(2) By the compactness of $\overline{U}$, there exist in $\overline{U} \times \overline{U}$ sublimits of $(A_\varepsilon, B_\varepsilon)$ as $\varepsilon \to 0$.
(3) By the uniform continuity of $v_\lambda - v$, such sublimits must be of the form $(A, A) \in T \times T$.

Now take $\varepsilon > 0$ small enough so that $i_\varepsilon = j_\varepsilon$ and $(A_\varepsilon, B_\varepsilon) \in W \times W$. Let $p_\varepsilon = 2\varepsilon^{-2}(x_\varepsilon - y_\varepsilon)$. Fixing B_ε, we have that p_ε belongs to the superdifferential of $v_\lambda(\,\cdot\,, i_\varepsilon)$ at x_ε. Note that v_λ is Lipschitz on $\overline{U}$, so the vectors p_ε form a bounded set.

By the convexity condition (A2), we have that $H(A_\varepsilon, p_\varepsilon) \leq -\lambda c < 0$, where $-c < 0$ is the negative minimum of $\{H(A, \psi'(A)) : A \in \overline{U}\}$. On the other hand, fixing A_ε, we have that p_ε belongs to the subdifferential of $v(\,\cdot\,, i_\varepsilon)$ at y_ε. Since $v \in S_H$ does not satisfy condition (C2') at $B_\varepsilon \in W$, it must satisfy condition (C1'), so $H(B_\varepsilon, p_\varepsilon) \geq 0$. Taking any sublimit of $(A_\varepsilon, B_\varepsilon, p_\varepsilon)$ as $\varepsilon \to 0$, we obtain a contradiction. Hence β_λ cannot be positive, and the proof is complete. ∎

As in [21, Proposition 6.1, p. 2176], the technique in the above proof establishes which continuous functions on $\mathcal{A}$ have an extension in S_H and describes how to construct such an extension that is maximal.

Proposition 4.3 *A continuous function g on $\mathcal{A}$ has an extension in S_H if and only if for all $B, B' \in \mathcal{A}$,*

$$g(B') - g(B) \leq d_H(B', B). \tag{4.1}$$

Moreover, define on $\Omega \times \mathcal{I}$ $\tilde{g}(A) = \inf\{g(B) + d_H(A, B) : B \in \mathcal{A}\}$. Then $\tilde{g} \in S_H$; it extends g, and for all extension $u \in S_H$ of g, $u \leq \tilde{g}$.

We will need the following lemma. Unlike S_H^-, the set of solutions S_H may not be convex or closed under maxima, but two important properties remain.

Lemma 4.4 *Given an open subset U of $\Omega \times \mathcal{I}$, $S_H(U)$ is stable under the operation* min *and locally uniform convergence.*

Proof Let $u, v \in S_H(U)$ and $A \in U$. If $\min(u, v)(A) = u(A)$ and u satisfies condition (C2') at A, then so does $\min(u, v)$, and likewise for v. Otherwise both u and v satisfy condition (C1') at A.

Let v be a locally uniform limit of a sequence (v_n) in S_H. If v does not satisfy condition (C2') at $A \in \Omega \times \mathcal{I}$, then neither do the functions (v_n) for large enough n on a some neighborhood of A (independent of n).

To prove that in the remaining cases condition (C1') is satisfied at A, one may now follow the usual stability argument introduced in [10, Theorem 1.2, pp. 5–6], which can also be found in, *e.g.,* [1, 2, 18, 23]. ∎

Proof of Proposition 4.3 Condition (4.1) is necessary by the definition of d_H. Suppose now that it holds. It follows from Lemma 4.4 that $\tilde{g} \in S_H$. If $A \in \mathcal{A}$, then choosing $B = A$ we get $\tilde{g}(A) \geq g(A)$ and by (4.1), for all $B \in \mathcal{A}$, $g(B) + d_H(A, B) \geq g(A)$, so $\tilde{g} = g$ on $\mathcal{A}$. The maximality follows once more from the definition of d_H. ∎

In response to [4, Remark 3, p. 1300], we come now to the solution of the Dirichlet problem. Given a continuous function g on $\partial\Omega \times \mathcal{I} \cup \mathcal{A}$, find a continuous extension v of g to $\overline{\Omega} \times \mathcal{I}$ whose restriction to $\Omega \times \mathcal{I}$ belongs to S_H. The function g is called the boundary data of the Dirichlet problem.

Following the notation of [21], formulate following the conditions.

(A3') The Hamiltonian H is uniformly coercive, *i.e.,* there exists a fixed positive real $R > 0$ such that for all $A \in \Omega \times \mathcal{I}$, if $|p| > R$, then $H(A, p) > 0$.

(A6) The domain Ω is bounded, and it is a Lipschitz manifold with boundary.

By Proposition 2.4, condition (A3') implies that d_H is uniformly locally Lipschitz in both state variables.

Let us expand on condition (A6). It means that given $z \in \partial\Omega$, there exists $\varepsilon > 0$ and an index $1 \leq k \leq n$ in the following conditions. For $x \in \mathbb{R}^n$, call x' to the element of $\mathbb{R}^{n-1}$ obtained by removing the k-coordinate from x. Let U be the open cylinder $B_\varepsilon(z') \times (z_k - \varepsilon, z_k + \varepsilon)$ (where $B_\varepsilon(z')$ is the open ε-ball about z' in $\mathbb{R}^{n-1}$). Then there exists a Lipschitz map $\Phi\colon B_\varepsilon(z') \to \mathbb{R}$ such that $\Omega \cap U$ is one of the sets $\{x_k > \Phi(x')\}$ or $\{x_k < \Phi(x')\}$.

Proposition 8.1 [21, p. 2181] carries through, *mutatis mutandis*, to the plurimodal case: if both conditions (A3') and (A6) hold, then d_H extends continuously to $\overline{\Omega} \times \mathcal{I} \times \overline{\Omega} \times \mathcal{I}$.

In view of the above remarks, we believe that the following theorem is quite a complete answer to [4, Remark 3, p. 1300]. While only the uniform continuity of d_H is used in the proof, which follows along the lines of that of [21, Theorem 3.3], we choose to include condition (A6) in the statement. This condition is quite essential for the present formulation of the Dirichlet problem, as is shown in [21, Example 5.6, pp. 2174–2175]. The functions $d_H(\,\cdot\,, y)$ for the eikonal Hamiltonian $H(x, p) = |p| - 1$ on a disk with a radial slit have two different sublimits at each point of the slit. As for condition (A3'), it is assumed throughout in [4], where it is simply called coercivity, but the explicit reference in the proof of [4, Lemma 3.1, p. 1293], identifies it with the hypothesis (H3), [2, p. 32].

Theorem 4.5 *Assume that condition* (A6) *holds and that d_H is uniformly continuous. Let g be a continuous function on $\partial\Omega\times\mathcal{I}\cup\mathcal{A}$. Then the Dirichlet problem with boundary data g has a unique solution v if and only if for all $B, B' \in \partial\Omega\times\mathcal{I}\cup\mathcal{A}$,*

$$g(B') - g(B) \le d_H(B', B). \tag{4.2}$$

In that case, for all $A \in \overline{\Omega}\times\mathcal{I}$, $v(A) = \min\{g(B) + d_H(A,B) : B \in \partial\Omega\times\mathcal{I}\cup\mathcal{A}\}$.

Proof We denote by d_H its continuous extension to $\overline{\Omega}\times\mathcal{I}\times\overline{\Omega}\times\mathcal{I}$. By continuity, condition (4.2) is obviously necessary for the existence of a solution.

Suppose now that condition (4.2) holds. For $A \in \Omega\times\mathcal{I}$ define

$$\tilde{g}(A) = \inf\{g(B) + d_H(A,B) : B \in \partial\Omega\times\mathcal{I}\cup\mathcal{A}\}.$$

The infimum is actually a minimum, because $\partial\Omega\times\mathcal{I}\cup\mathcal{A}$ is compact, and g and d_H are continuous on their domains. Hence $\tilde{g}$ is well defined.

We claim that if $B \in \partial\Omega\times\mathcal{I}$, then $g(B) + d_H(\cdot\,,B) \in S_H$. Let (B_n) be a sequence in $\Omega\times\mathcal{I}$ tending to B. Then $g(B) + d_H(\cdot\,,B_n)$ converges locally uniformly to $g(B) + d_H(\cdot\,,B)$, so it belongs to S_H^-. Let U be the interior of any compact subset of $\Omega\times\mathcal{I}$. For large enough n, $B_n \notin U$, so $g(B) + d_H(\cdot\,,B_n) \in S_H(U)$ by Proposition 3.1. The claim now follows from Lemma 4.4.

To prove that $\tilde{g}$ is continuous and that its restriction to $\Omega\times\mathcal{I}$ is in S_H, let (B_n) be a dense sequence in $\partial\Omega\times\mathcal{I}\cup\mathcal{A}$. Let $N \in \mathbb{N}$ and $A \in \overline{\Omega}\times\mathcal{I}$, and define

$$g_N(A) = \min\{g(B_n) + d_H(A,B_n) : 1 \le n \le N\}.$$

By Lemma 4.4, each function g_N has the two properties we want to establish for $\tilde{g}$. These functions converge pointwisely to $\tilde{g}$. Moreover, they are uniformly bounded because g is bounded, $\overline{\Omega}$ is compact and connected, and $\mathcal{I}$ is finite, and they are uniformly locally Lipschitz by condition (A3') and a previous remark. Thus, by the Ascoli–Arzela theorem they converge locally uniformly to $\tilde{g}$. This proves both properties for $\tilde{g}$ (using once more Lemma 4.4).

To prove that $\tilde{g}$ extends g, let $B, B' \in \partial\Omega\times\mathcal{I}\cup\mathcal{A}$. By condition (4.2), $g(B') + d_H(B,B') \ge g(B) = g(B) + d_H(B,B)$, which shows that $\tilde{g}(B') = g(B)$.

To prove the uniqueness of the solution, let v be any solution of the Dirichlet problem with boundary data g. By continuity and the original definition of d_H, we have that $v \le \tilde{g}$. To prove the reverse inequality, let U_n be a sequence of open sets of compact closure such that for all n, $\overline{U_n} \subset U_{n+1} \subset \Omega\times\mathcal{I}\setminus\mathcal{A}$, and whose union is $\Omega\times\mathcal{I}\setminus\mathcal{A}$. Let $A \in \Omega\times\mathcal{I}$. For all large enough n, $A \in U_n$ and by Theorem 4.1 there exists $B_n \in \partial U_n$ such that $v(A) = v(B_n) + d_H(A,B_n)$. Let B be a sublimit of (B_n), which must belong to $\partial\Omega\times\mathcal{I}\cup\mathcal{A}$. Taking the same sublimit in the previous equality,

$$v(A) = v(B) + d_H(A,B) = g(B) + d_H(A,B) \ge \tilde{g}(A). \qquad \blacksquare$$

Example 4.6 In Example 3.7, conditions (A3') and (A6) hold. Any solution of the Dirichlet problem is uniquely determined by one value prescribed at any point of $[0,1] \times \{1,2\}$ because given A, B in that set, $d_H(A,B) = u(A) - u(B)$, where $u(x,i) = (1-i)x$.

The binary relation $d_H(A,B) = -d_H(B,A)$ is an equivalence relation (the transitivity follows from the triangular inequality for d_H) and d_H-compatible functions in the sense of condition (4.2) are actually defined on the space of equivalence classes. In the particular case of the example above, the Aubry set and the boundary form a single equivalence class. In general this is false, even in the scalar (unimodal) case and under much less general conditions than the ones assumed here (see, *e.g.,* the oscillatory Hamiltonian shown in the introduction), and J. Mather [27] has recently shown that the quotient Aubry set may be isometric to an interval. Discussions of the concept of a quotient Aubry set may be found in [26, 28].

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