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Dupin Hypersurfaces in $\mathbb{R}^5$

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Abstract. We study Dupin hypersurfaces in $\mathbb{R}^5$ parametrized by lines of curvature, with four distinct principal curvatures. We characterize locally a generic family of such hypersurfaces in terms of the principal curvatures and four vector valued functions of one variable. We show that these vector valued functions are invariant by inversions and homotheties.

Introduction

Dupin surfaces were first studied by Dupin in 1822 and more recently by many authors [1–6, 9–14, 16, 17], who looked at several aspects of Dupin hypersurfaces. The class of Dupin hypersurfaces is invariant under Lie transformations [11]. Therefore, the classification of Dupin hypersurfaces is considered up to these transformations. The local classification of Dupin surfaces in $\mathbb{R}^3$ is well known. Pinkall [12] gave a complete classification up to Lie equivalence for Dupin hypersurfaces $M^3 \subset \mathbb{R}^4$, with three distinct principal curvatures. Niebergall [10] and more recently Cecil and Jensen [6] studied proper Dupin hypersurfaces with four distinct principal curvatures and constant Lie curvature (the cross-ratio of four principal curvatures).

In this paper we study generic Dupin hypersurfaces in $\mathbb{R}^5$, parametrized by lines of curvature, with four distinct principal curvatures. We obtain a local characterization of a generic family of such hypersurfaces (Theorem 3.1) in terms of the principal curvature functions and four vector valued functions of one variable. The characterization is based on the theory of higher-dimensional Laplace invariants introduced by Kamran–Tenenblat [7, 8].

We consider generic hypersurfaces in the sense that suitable generic conditions on the Laplace invariants are required. Moerover, assuming that the Dupin hypersurfaces are parametrized by lines of curvature, we eliminate those which are Lie equivalent to an isoparametric hypersurface in S^5. This follows from Pinkall's result [11], (see also [3]), which asserts that if M is an isoparametric hypersurface in S^n with more than two distinct principal curvatures, then M cannot be parametrized by lines of curvature. Further, it follows from [6] that if M is a Dupin hypersurface of R^5 parametrized by lines of curvature, with four distinct principal curvatures, then M is reducible in the sense of Pinkall or it has non constant Lie curvature.

We observe that if we consider a Dupin hypersurface $M^n \subset R^{n+1}$, parametrized by lines of curvature, with n distinct principal curvatures, satisfying the nongeneric assumption that all Laplace invariants vanish, we can show that M has constant Lie curvature. It follows from results of Pinkall [11] and Cecil–Jensen [6] that, when $n = 3$ or $n = 4$, M is reducible. This will appear in a forthcoming paper.

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In Section 1, we recall the main definitions and results on the Laplace invariants. In Section 2, we give some properties of Dupin hypersurfaces with distinct principal curvatures. In Section 3, Theorem 3.1 gives a local characterization of generic Dupin hypersurfaces in $\mathbb{R}^5$ with four distinct principal curvatures. In Section 4, we show that the vector valued functions, which appear in the characterization of Theorem 3.1 are invariant by inversions and homotheties.

The local characterization of Theorem 3.1 is given in terms of the principal curvatures λ_r and four functions of one variable $G_r(x_r)$, $1 \leq r \leq 4$, taking values in R^5. We observe that the principal curvatures λ_r change under inversions in spheres centered at the origin and under homotheties, while the functions G_r are invariant under such transformations. On the other hand, principal curvatures are preserved under isometries of R^5, but the functions G_r are not invariant under isometries. This is certainly expected, since parametrizations are not invariant under isometries. Therefore, the functions G_r are not invariant under the full group of Lie transformations of R^5.

We conclude this introduction by observing that, although the theory in this paper has been developped for $M^4 \subset R^5$, one can consider a similar theory for hypersurfaces $M^n \subset R^{n+1}$ in other dimensions n. For example, when $n = 3$, a similar classification result can be obtained with the generic condition that one of the Laplace invariants does not vanish. However, in this case one knows from Pinkall's result [13] that such hypersurfaces are reducible. For dimensions $n \geq 4$, the proofs get lenghthier as n increases. We observe that we do not have explicit examples in higher dimensions, which might help the reader to develop some understanding of the possible forms of the vector valued functions G_r. However, for some known examples, when $n = 3$, the functions G_r describe plane curves or points.

1 The Higher-Dimensional Laplace Invariants

In this section we briefly review the theory of higher-dimensional Laplace invariants [7,8] which will be necessary in the following sections. We consider linear systems of second-order partial differential equations of the form

$$Y_{,kl} + a_{kl}^k Y_{,k} + a_{kl}^l Y_{,l} + c_{kl} Y = 0, \quad 1 \leq k \neq l \leq n, \tag{1}$$

where Y is a scalar function of the independent variables $x_1, x_2, \ldots, x_n$, and the coefficients a and c are smooth functions of $x_1, x_2, \ldots, x_n$ which are symmetric in the pair of lower indices and satisfy certain compatibility conditions, and $Y_{,l}$ denotes the derivative of Y with respect to x_l.

The general form of system (1) is preserved under *admissible transformations*

$$Y = \varphi(x_1, x_2, \ldots, x_n)\bar{Y}, \tag{2}$$

$$x_i = f_i(\bar{x}_i), \quad 1 \leq i \leq n, \tag{3}$$

where φ is a smooth and non-vanishing function and the f_i's are smooth functions whose derivatives do not vanish. It is easily verified that under an admissible transformation, the coefficients a and c transform according to

$$\begin{aligned} \bar{a}^l_{lk} &= f_k'\big[a^l_{lk} + (\log\varphi)_{,k}\big],\\ \bar{c}_{lk} &= f_l' f_k'\Big[c_{lk} + a^l_{lk}(\log\varphi)_{,l} + a^k_{lk}(\log\varphi)_{,k} + \frac{\varphi_{,kl}}{\varphi}\Big], \quad 1 \le k \ne l \le n, \end{aligned} \tag{4}$$

and system (1) transforms into

$$\bar{Y}_{,kl} + \bar{a}^k_{kl}\bar{Y}_{,k} + \bar{a}^l_{kl}\bar{Y}_{,l} + \bar{c}_{kl}\bar{Y} = 0, \quad 1 \le k \ne l \le n.$$

The *higher-dimensional Laplace invariants* of (1), introduced in [7], are the $n(n-1)^2$ functions given by

$$m_{ij} = a^i_{ij,i} + a^i_{ij}a^j_{ij} - c_{ij}, \quad m_{ijk} = a^k_{kj} - a^i_{ij}, \quad k \ne i, j, \tag{5}$$

for all ordered pairs (i, j), $1 \le i \ne j \le n$.

Lemma 1.1 *The higher-dimensional Laplace invariants of a compatible system* (1) *satisfy the following relations: for* $1 \le i, j, k, l \le n$, *with* i, j, k, l *distinct,*

$$\begin{aligned} m_{ijk} + m_{kji} &= 0,\\ m_{ijk,k} - m_{ijk}m_{jki} - m_{kj} &= 0,\\ m_{ij,k} + m_{ijk}m_{ik} + m_{ikj}m_{ij} &= 0,\\ m_{ijk} - m_{ijl} - m_{ljk} &= 0,\\ m_{lik,j} + m_{ijl}m_{kil} + m_{ljk}m_{kij} &= 0. \end{aligned} \tag{6}$$

The functions m_{ij}, m_{ijk} are invariant under pure rescalings (2). The expression of a system (1) in terms of its higher-dimensional Laplace invariants is given in the following result proved in [8].

Theorem 1.2 *Given any collection of* $n(n-1)^2$, $n \ge 3$ *smooth functions of* $x_1, x_2, \dots, x_n$, m_{ij}, m_{ijk}, $1 \le i, j, k \le n$, *with* i, j, k *distinct, satisfying the constraints* (6), *there exists a linear system* (1) *whose higher-dimensional Laplace invariants are the given functions* m_{ij}, m_{ijk}. *Any such system is defined up to rescaling* (2). *A representative is given by*

$$\begin{aligned} Y_{,ij} + AY_{,j} - m_{ij}Y &= 0,\\ Y_{,ik} + (A + m_{jik})Y_{,k} - m_{ik}Y &= 0,\\ Y_{,jk} + m_{ikj}Y_{,j} + m_{ijk}Y_{,k} &= 0,\\ Y_{,lk} + m_{ikl}Y_{,l} + m_{ilk}Y_{,k} &= 0, \end{aligned} \tag{7}$$

where (i, j) *is a fixed (ordered) pair,* $1 \le i, j, k, l \le n$ *are distinct and* A *is a function which satisfies*

$$A_{,j} = m_{ji} - m_{ij} \quad A_{,k} = -m_{jki,i}. \tag{8}$$

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2 Properties of Dupin Hypersurfaces with Distinct Principal Curvatures

We consider the Euclidean space $\mathbb{R}^n$ endowed with the Euclidian metric $\langle\ ,\ \rangle$. Let Ω be an open subset of $\mathbb{R}^n$ and $x = (x_1, x_2, \dots, x_n) \in \Omega$.

Definition 2.1 An immersion $X\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ is a parametrized *Dupin hypersurface* if each principal curvature is constant along its corresponding lines of curvature. If the multiplicity of the principal curvatures is constant then the Dupin submanifold is said to be *proper*.

Remark 2.2 Let $X\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ be a proper Dupin hypersurface parametrized by lines of curvature, with distinct principal curvatures λ_i, $1 \le i \le n$ and $N\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$ a unit normal vector field of X. Then

$$\begin{aligned} &\langle X_{,i}, X_{,j}\rangle = \delta_{ij} g_{ii} \quad 1 \le i, j \le n, \\ &N_{,i} = -\lambda_i X_{,i}, \quad \lambda_{i,i} = 0. \end{aligned} \tag{9}$$

Moreover,

$$X_{,ij} - \Gamma^i_{ij} X_{,i} - \Gamma^j_{ij} X_{,j} = 0, \quad 1 \le i \ne j \le n, \tag{10}$$

$$\Gamma^i_{ij} = \frac{\lambda_{i,j}}{\lambda_j - \lambda_i}, \quad 1 \le i \ne j \le n, \tag{11}$$

where Γ^k_{ij} are the Christoffel symbols.

We now consider the higher-dimensional Laplace invariants of the system of equations (10), defined by (5).

$$\begin{aligned} m_{ij} &= -\Gamma^i_{ij,i} + \Gamma^i_{ij}\Gamma^j_{ij}, \\ m_{ijk} &= \Gamma^i_{ij} - \Gamma^k_{kj}, \quad k \ne i, j,\ 1 \le k \le n. \end{aligned} \tag{12}$$

As a consequence of (11) and Lemma 1.1, we obtain for $1 \le i, j, k, l \le n$, i, j, k, l distinct,

$$\begin{aligned} &m_{ij} = 0, \quad m_{ijk} + m_{kji} = 0, \quad m_{ijk,k} - m_{ijk} m_{jki} = 0, \\ &m_{ijk} - m_{ijl} - m_{ljk} = 0, \quad m_{lik,j} + m_{ijl} m_{kil} + m_{ljk} m_{kij} = 0. \end{aligned} \tag{13}$$

We now consider the effect on the principal curvatures and on the higher-dimensional Laplace invariants of a Dupin hypersurface under an inversion or a homothety. Let $X\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1} - \{0\}$, $n \ge 2$, be a proper Dupin hypersurface parametrized by lines of curvature with distinct principal curvatures λ_i, $1 \le i \le n$. Consider an inversion

$$\begin{aligned} &I^{n+1}\colon \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\} \\ &X \to I^{n+1}(X) = \frac{X}{\langle X, X\rangle}. \end{aligned} \tag{14}$$

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Denoting $I^{n+1}(X) = \tilde{X}$, we have that

$$\tilde{N} = -2\frac{\langle X, N\rangle}{\langle X, X\rangle}X + N$$

is a unit vector field normal to $\tilde{X}$. Then $\tilde{X}$ is a Dupin hypersurface parametrized by lines of curvature, with distinct principal curvatures given by

$$\tilde{\lambda}_i = \langle X, X\rangle\lambda_i + 2\langle X, N\rangle \ , 1 \leq i \leq n.$$

Moreover, $\tilde{X}$ satisfies the system

$$\tilde{X}_{,kl} - \tilde{\Gamma}^k_{kl}\tilde{X}_{,k} - \tilde{\Gamma}^l_{kl}\tilde{X}_{,l} = 0, \quad 1 \leq k \neq l \leq n, \ k \neq i, j,$$

and since $\tilde{X}$ is a rescaling of X, we conclude that for $n \geq 3$, an inversion does not change the higher-dimensional Laplace invariants *i.e.,* $\tilde{m}_{ij} = m_{ij} = 0$ and $\tilde{m}_{ijk} = m_{ijk}$.

Similarly, let $X\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 2$ be a proper Dupin hypersurface parametrized by lines of curvature with distinct principal curvatures λ_i, $1 \leq i \leq n$. Consider the homothety

$$\begin{aligned} D\colon \mathbb{R}^{n+1} &\to \mathbb{R}^{n+1} \\ X \to D(X) &= aX, \quad a \in \mathbb{R}, \ a \neq 0. \end{aligned} \tag{15}$$

Denoting $\bar{X} = D(X)$, we have that $\bar{N} = N$ is a unit vector field normal to $\bar{X}$. Then $\bar{X}$ is a Dupin hypersurface parametrized by lines of curvature, with distinct principal curvatures given by $\bar{\lambda}_i = \lambda_i/a$, $1 \leq i \leq n$. Moreover, since $\bar{X}$ satisfies the system

$$\bar{X}_{,kl} - \bar{\Gamma}^k_{kl}\bar{X}_{,k} - \bar{\Gamma}^l_{kl}\bar{X}_{,l} = 0, \quad 1 \leq k \neq l \leq n, \ k \neq i, j,$$

and $\bar{X}$ is a rescaling of X, for $n \geq 3$ the higher-dimensional Laplace invariants do not change, *i.e.,* $\bar{m}_{ij} = m_{ij} = 0$ and $\bar{m}_{ijk} = m_{ijk}$.

The following result provides some properties which are satisfied by the principal curvatures of a Dupin hypersurface in $\mathbb{R}^{n+1}$ parametrized by lines of curvature. Its proof is a straightforward computation which follows from (13).

Lemma 2.3 *Let $\lambda_r\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be smooth functions distinct at each point, such that $\lambda_{r,r} = 0$. Consider functions m_{ijk} defined by* (11) *and* (12)*. Then for i, j fixed, $1 \leq i \neq j \leq n$, the following properties hold*

$$\Big[\frac{\lambda_k - \lambda_j}{\lambda_j - \lambda_i}m_{jki}\Big]_{,i} = -m_{jki,i}, \tag{16}$$

$$\Big[\frac{\lambda_k - \lambda_j}{\lambda_j - \lambda_i}m_{jki}\Big]_{,j} = 0, \tag{17}$$

$$\Big[\frac{\lambda_k - \lambda_j}{\lambda_j - \lambda_i}m_{jki}\Big]_{,l} = \Big[\frac{\lambda_l - \lambda_j}{\lambda_j - \lambda_i}m_{jli}\Big]_{,k}, \tag{18}$$

where $1 \leq k \neq l \leq n$ are distinct from i and j.

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The next result is a straightforward application of Theorem 1.2.

Lemma 2.4 *Let $X\colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a proper Dupin hypersurface parametrized by lines of curvature, with n distinct principal curvatures λ_r, $1 \leq r \leq n$. For i, j, k fixed, $1 \leq i \neq j \neq k \leq n$, the transformation*

$$X = V\bar{X}, \quad \text{where } V = \frac{e^{\int \frac{\lambda_k - \lambda_j}{\lambda_j - \lambda_i} m_{jki}\, dx_k}}{\lambda_j - \lambda_i}, \tag{19}$$

transforms system (10) *into*

$$\begin{aligned} \bar{X}_{,ij} + A\bar{X}_{,j} &= 0, \\ \bar{X}_{,ir} + (A + m_{jir})\bar{X}_{,r} &= 0, \\ \bar{X}_{,jr} + m_{irj}\bar{X}_{,j} + m_{ijr}\bar{X}_{,r} &= 0, \\ \bar{X}_{,rl} + m_{ilr}\bar{X}_{,r} + m_{irl}\bar{X}_{,l} &= 0, \end{aligned} \tag{20}$$

where l and r are such that $1 \leq r \neq l \neq i \neq j \leq n$ and

$$A = -\int m_{jki,i}\, dx_k. \tag{21}$$

Moreover,

$$A_{,j} = 0, \quad A_{,r} = -m_{jri,i}, \tag{22}$$

Remark 2.5 For subsequent use, we will compute the derivatives of the function V given by (19). It follows from Lemma 2.3 that,

$$\begin{aligned} V_{,i} = \left(A + \Gamma^j_{ji}\right) V, \quad V_{,j} = \Gamma^i_{ij} V, \\ V_{,k} = \Gamma^i_{ik} V, \quad V_{,l} = \Gamma^i_{il} V, \end{aligned} \tag{23}$$

where A is given by (21) and l is distinct from i, j, k.

3 A Characterization of Dupin Hypersurfaces in $\mathbb{R}^5$

In this section, we prove our main result which provides a local characterization of generic Dupin hypersurfaces in $\mathbb{R}^5$, with four distinct principal curvatures. We start by introducing some functions that will be useful in what follows. Considering the higher-dimensional Laplace invariants, satisfying (13), for $1 \leq i \neq j \neq k \neq l \leq 4$ *fixed* we introduce the following functions admitting that $m_{jik} \neq 0$, $m_{jil} \neq 0$ and $m_{kil} \neq 0$,

$$T_{ijkl} = m_{jil} + \left[\log\left(\frac{m_{jik}}{m_{kil}}\right)\right]_{,i}, \tag{24}$$

$$U_{ijkl} = m_{kil} + \left[\log\left(\frac{m_{jik}}{m_{jil}}\right)\right]_{,i} \tag{25}$$

$$P^4_j = m_{jik}T_{ijkl}, \quad P^4_k = m_{jik}U_{ijkl}, \quad P^4_l = m_{jil}U_{ijkl}. \tag{26}$$

Theorem 3.1 *Let* $X\colon \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$, *be a proper Dupin hypersurface, parametrized by lines of curvature, with four distinct principal curvatures* λ_r. *For* i, j, k, l *distinct fixed indices, suppose* $m_{jkl} \neq 0$, $m_{ljk} \neq 0$ *and* $T_{ijkl} \neq 0$ *then*

$$X = V\left[B_j^4 - B_k^4 + B_l^4\right], \tag{27}$$

where

$$V = \frac{e^{\int \frac{\lambda_k-\lambda_j}{\lambda_j-\lambda_i} m_{jki}\,dx_k}}{\lambda_j-\lambda_i}, \quad B_s^4 = \frac{1}{Q_s^4}\left[\int \frac{Q_s^4 G_i(x_i)}{P_s^4}\,dx_i + G_s(x_s)\right], \quad s \neq i, \tag{28}$$

P_s^4 *are defined by* (26), $G_r(x_r), 1 \leq r \leq 4$, *are vector valued functions of* $\mathbb{R}^5$, $A = -\int m_{jki,i}\,dx_k$, *and*

$$Q_s^4 = \begin{cases} e^{\int A\,dx_i} & \text{if } s = j,\\ e^{\int (A+m_{jis})\,dx_i} & \text{if } s = k, l.\end{cases} \tag{29}$$

Moreover, considering

$$\alpha^i = \left(A + \frac{\lambda_{j,i}}{\lambda_i-\lambda_j}\right)M + M_{,i}, \quad \alpha^s = \frac{\lambda_{i,s}}{\lambda_s-\lambda_i}M + M_{,s}, \quad s \neq i, \tag{30}$$

where $M = B_j^4 - B_k^4 + B_l^4$, *the functions* $G_r(x_r)$ *satisfy the following properties in* Ω, *for* $1 \leq r \neq t \leq 4$:

(a) $\alpha^r \neq 0$,

(b) $\langle \alpha^r, \alpha^t\rangle = 0, r \neq t$,

(c) $\lambda_r = \dfrac{\langle \alpha^r_{,r}, \alpha^i \times \alpha^j \times \alpha^k \times \alpha^l\rangle}{V|\alpha^r|^2\,|\alpha^i|\,|\alpha^j|\,|\alpha^k|\,|\alpha^l|}$.

Conversely, let $\lambda_r\colon \Omega \subset \mathbb{R}^4 \to \mathbb{R}, r = 1, \ldots, 4$ *be real functions, distinct at each point, such that* $\lambda_{r,r} = 0$. *Assume that the functions* m_{rts} *defined by*

$$m_{rts} = \frac{\lambda_{r,t}}{\lambda_t-\lambda_r} - \frac{\lambda_{s,t}}{\lambda_t-\lambda_s}, \quad 1 \leq r \neq t \neq s \leq 4, \tag{31}$$

satisfy (13), *and for* i, j, k, l *distinct fixed indices,* $m_{jkl} \neq 0$, $m_{ljk} \neq 0$, $T_{ijkl} \neq 0$. *Then for any vector valued functions* $G_r(x_r)$ *satisfying properties* (a), (b), (c) *where* α^r *is defined by* (30), *the function* $X\colon \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$ *given by* (27) *describes a Dupin hypersurface, parametrized by lines of curvature, whose principal curvatures are the functions* λ_r.

Before proving this theorem, we observe that $T_{ijkl} = 0$ if and only if $U_{ijkl} = 0$. In fact, using the relation $m_{kil} + m_{mjik} = m_{jil}$ (see (13)) twice, it follows from (24) and (25) that $m_{ijk}m_{kil}T_{ijkl} = U_{ijkl}$. Hence, the hypothesis of Theorem 3.1 implies that $P_s^4 \neq 0$, for $s \neq i$.

For the proof of Theorem 3.1 we will need three lemmas.

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Lemma 3.2 *Let X be a Dupin hypersurface as in Theorem 3.1, then*

$$X = \frac{V}{m_{jik}}\left[W^k - W^j\right], \tag{32}$$

where $W^k(x_i, x_j, x_l)$ and $W^j(x_i, x_k, x_l)$ satisfy the following systems of equations,

$$\begin{aligned} W^k_{,ij} + \Big(A - \frac{m_{jik,i}}{m_{jik}}\Big) W^k_{,j} + m_{kji} m_{jik} W^k &= 0, \\ W^k_{,il} + \Big(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\Big) W^k_{,l} + m_{ilk} m_{kil} W^k &= 0, \\ W^k_{,jl} + \frac{m_{jlk} m_{kil}}{m_{jik}} W^k_{,j} + \frac{m_{ljk} m_{jik}}{m_{kil}} W^k_{,l} &= 0. \end{aligned} \tag{33}$$

$$\begin{aligned} W^j_{,ik} + \Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\Big) W^j_{,k} + m_{ikj} m_{jik} W^j &= 0, \\ W^j_{,il} + \Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}}\Big) W^j_{,l} + m_{ilj} m_{jil} W^j &= 0, \\ W^j_{,kl} + \frac{m_{jlk} m_{jil}}{m_{jik}} W^j_{,k} + \frac{m_{jkl} m_{jik}}{m_{jil}} W^j_{,l} &= 0. \end{aligned} \tag{34}$$

Proof From Remark 2.2, we have,

$$X_{,sr} - \Gamma^s_{sr} X_{,s} - \Gamma^r_{sr} X_{,r} = 0, \quad 1 \le s \ne r \le 4. \tag{35}$$

For fixed distinct indices i, j, k, we consider the transformation

$$X = V\bar{X}, \tag{36}$$

as in Lemma 2.4, where V is given by (19). Then system (35) reduces to

$$\begin{aligned} \bar{X}_{,ij} + A\bar{X}_{,j} &= 0, \\ \bar{X}_{,ir} + (A + m_{jir})\bar{X}_{,r} &= 0, \\ \bar{X}_{,jr} + m_{irj}\bar{X}_{,j} + m_{ijr}\bar{X}_{,r} &= 0, \\ \bar{X}_{,kl} + m_{ilk}\bar{X}_{,k} + m_{ikl}\bar{X}_{,l} &= 0, \end{aligned} \tag{37}$$

where $r = k, l$ and $k \ne l$,

$$A_{,j} = 0, \quad A_{,r} = -m_{jri,i}. \tag{38}$$

It follows from the third and second equations of (13) and (38), that

$$(A + m_{jir})_{,r} = 0, \quad r = k, l. \tag{39}$$

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Using (38) and (39) in the first two equations of (37), we have that

$$\bar{X}_{,i} + A\bar{X} = W^j(x_i, x_k, x_l), \tag{40}$$

$$\bar{X}_{,i} + (A + m_{jik})\bar{X} = W^k(x_i, x_j, x_l), \tag{41}$$

$$\bar{X}_{,i} + (A + m_{jil})\bar{X} = W^l(x_i, x_j, x_k). \tag{42}$$

where W^j, W^k and W^l are functions that do not depend on x_j, x_k and x_l, respectively. Since $m_{jik} \neq 0$, from (40) and (41) we have

$$\bar{X} = \frac{1}{m_{jik}}[W^k - W^j]. \tag{43}$$

Therefore, it follows from (36) that X is given by (32).

We will now obtain the differential equations that W^k and W^j must satisfy, by using (37), (40)–(42).

The substitution of $\bar{X}$ and $\bar{X}_{,i}$ into (40), (41) and (42), gives

$$\left(A - \frac{m_{jik,i}}{m_{jik}}\right)[W^k - W^j] + [W^k - W^j]_{,i} = m_{jik}W^j, \tag{44}$$

$$\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)[W^k - W^j] + [W^k - W^j]_{,i} = m_{jik}W^k, \tag{45}$$

$$\left(A + m_{jil} - \frac{m_{jik,i}}{m_{jik}}\right)[W^k - W^j] + [W^k - W^j]_{,i} = m_{jik}W^l. \tag{46}$$

Substituting $\bar{X}_{,j}$ and $\bar{X}_{,ij}$ into the first equation of (37), it follows from (13), that

$$W^k_{,ij} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right)W^k_{,j} - m_{ijk}\left\{\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)[W^k - W^j] + [W^k - W^j]_{,i}\right\} = 0.$$

Using (45) in this equation, we conclude that W^k satisfies the first equation of (33).

Similarly, substituting $\bar{X}_{,k}$ and $\bar{X}_{,ik}$ in the second equation of (37) for $r = k$, and using (13) we obtain that

$$W^j_{,ik} + \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)W^j_{,k} + m_{ikj}\left\{\left(A - \frac{m_{jik,i}}{m_{jik}}\right)[W^k - W^j] + [W^k - W^j]_{,i}\right\} = 0.$$

It follows from (44) that this equation reduces to the first equation of (34).

Now substituting $\bar{X}_{,l}$ and $\bar{X}_{,il}$ in the second equation of (37) with $r = l$, and using (13) we obtain a relation equivalent to (46). The substitution of $\bar{X}_{,j}$, $\bar{X}_{,k}$ and $\bar{X}_{,jk}$ in the third equation of (37) with $r = k$, gives an identity.

Using $\bar{X}_{,j}$, $\bar{X}_{,l}$ and $\bar{X}_{,jl}$ in the third equation of (37) with $r = l$, and using (13), we have that

$$W^k_{,jl} + \frac{m_{jlk}m_{kil}}{m_{jik}}W^k_{,j} + m_{ljk}\left\{-[W^k - W^j]_{,l} + \frac{m_{klj}m_{jil}}{m_{jik}}[W^k - W^j]\right\} = 0. \tag{47}$$

From (40), (41) and (42) we get,

$$W^k - W^j = \frac{m_{jik}}{m_{kil}}[W^l - W^k]. \tag{48}$$

Differentiating (48) with respect to x_l, and using (13), we obtain

$$[W^k - W^j]_{,l} = \frac{m_{klj}m_{jil}}{m_{jik}}[W^k - W^j] - \frac{m_{jik}}{m_{kil}}W^k_{,l}, \tag{49}$$

which substituted into (47), provides the third equation of (33).

Similarly, substituting $\bar{X}_{,k}$, $\bar{X}_{,l}$ and $\bar{X}_{,kl}$ in the last equation of (37), and using (13), we have that

$$W^j_{,kl} + \frac{m_{jlk}m_{jil}}{m_{jik}}W^j_{,k} + m_{lkj}W^k_{,l} - m_{lkj}W^j_{,l} + \frac{m_{lkj}m_{jlk}m_{kil}}{m_{jik}}[W^k - W^j] = 0. \tag{50}$$

From (49) we get,

$$W^k_{,l} = \frac{m_{kil}}{m_{jil}}W^j_{,l} + \frac{m_{kil}m_{klj}}{m_{jik}}[W^k - W^j].$$

Using this relation into (50), we obtain the third equation of (34).

Differentiating the first equation of (33) with respect to x_l and using the third equation of (33), we get

$$\begin{aligned} W^k_{,ijl} &+ \left[\left(A - \frac{m_{jik,i}}{m_{jik}} \right)_{,l} - \left(A - \frac{m_{jik,i}}{m_{jik}} \right) \frac{m_{jlk}m_{kil}}{m_{jik}} \right] W^k_{,j} \\ &+ \left[m_{kji}m_{ji} - \left(A - \frac{m_{jik,i}}{m_{jik}} \right) \frac{m_{ljk}m_{jik}}{m_{kil}} \right] W^k_{,l} + (m_{kji}m_{jik})_{,l}W^k = 0. \end{aligned} \tag{51}$$

Differentiating the third equation of (33) with respect to x_i, it follows from the first equation of (33), that

$$\begin{aligned} W^k_{,jli} &+ \left[\left(\frac{m_{jlk}m_{kil}}{m_{jik}} \right)_{,i} - \left(A - \frac{m_{jik,i}}{m_{jik}} \right) \frac{m_{jlk}m_{kil}}{m_{jik}} \right] W^k_{,j} + \left(\frac{m_{ljk}m_{jik}}{m_{kil}} \right)_{,i} W^k_{,l} \\ &+ \frac{m_{ljk}m_{jik}}{m_{kil}}W^k_{,li} + m_{jlk}m_{kil}m_{ijk}W^k = 0. \end{aligned} \tag{52}$$

Since $m_{ljk} \neq 0$ and $W^k_{,jli} = W^k_{,ijl}$, from (51), (52), (38) and (13), we obtain the second equation of (33).

Similar computations provide the second equation of (34). In fact, differentiating the first equation of (34) with respect to x_l, and using the third equation of (34), we get

$$\begin{aligned} W^j_{,ikl} &+ \left[\left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} \right)_{,l} - \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} \right) \frac{m_{jlk}m_{jil}}{m_{jik}} \right] W^j_{,k} \\ &+ \left[m_{ikj}m_{jik} - \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} \right) \frac{m_{jkl}m_{jik}}{m_{jil}} \right] W^j_{,l} + (m_{ikj}m_{jik})_{,l}W^j = 0. \end{aligned} \tag{53}$$

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Differentiating the third equation of (34) with respect to x_i and using the first equation of (34), we obtain

$$W^j_{,kli} + \left[\left(\frac{m_{jlk}m_{jil}}{m_{jik}}\right)_{,i} - \left(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\right)\frac{m_{jlk}m_{jil}}{m_{jik}}\right]W^j_{,k} + \left(\frac{m_{jkl}m_{jik}}{m_{jil}}\right)_{,i}W^j_{,l} + \frac{m_{jkl}m_{jik}}{m_{jil}}W^j_{,li} + m_{klj}m_{jil}m_{ikj}W^j = 0. \tag{54}$$

Since $m_{jkl} \neq 0$ and $W^j_{,ikl} = W^j_{,kli}$, from (53), (54), (38), and (13), we obtain the second equation of (34). Which concludes the proof of Lemma 3.2. ∎

The solutions of the systems of equations (33) and (34) are given in the following two lemmas.

Lemma 3.3 *The solution of* (33) *is given by*

$$W^k = \frac{m_{jik}}{Q^4_j}\left[\int \frac{Q^4_j G_i(x_i)}{P^4_j}dx_i + G_j(x_j)\right] - \frac{m_{kil}}{Q^4_l}\left[\int \frac{Q^4_l G_i(x_i)}{P^4_l}dx_i + G_l(x_l)\right]. \tag{55}$$

Proof Using equations of (13) and (38), we have that

$$\left(A - \frac{m_{jik,i}}{m_{jik}}\right)_{,j} = m_{jik}m_{kji}, \tag{56}$$

$$\left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right)_{,l} = m_{kil}m_{ilk}. \tag{57}$$

Substituting (56) in the first equation of system (33), we get

$$\left[W^k_{,i} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right)W^k\right]_{,j} = 0\ ,$$

whose integration with respect to x_j, provides

$$W^k_{,i} + \left(A - \frac{m_{jik,i}}{m_{jik}}\right)W^k = D^j(x_i, x_l). \tag{58}$$

Using (57) in the second equation of system (33), we obtain

$$\left[W^k_{,i} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right)W^k\right]_{,l} = 0\ .$$

Therefore, integrating with respect to x_l, we get

$$W^k_{,i} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right)W^k = D^l(x_i, x_j). \tag{59}$$

From (58), (59) and (24), we conclude that

$$W^k = \frac{1}{T_{ijkl}}[D^l - D^j]. \tag{60}$$

Differentiating W^k, using (13) and the following derivatives

$$T_{ijkl,j} = \frac{m_{jik}m_{ljk}}{m_{kil}}T_{ijkl} \quad T_{ijkl,k} = 0 \quad T_{ijkl,l} = \frac{m_{kil}m_{jlk}}{m_{jik}}T_{ijkl},$$

we have

$$W^k_{,i} = -\frac{T_{ijkl,i}}{(T_{ijkl})^2}[D^l - D^j] + \frac{1}{T_{ijkl}}[D^l - D^j]_{,i} \tag{61}$$

$$W^k_{,j} = -\frac{m_{ljk}m_{jik}}{m_{kil}}\frac{1}{T_{ijkl}}[D^l - D^j] + \frac{1}{T_{ijkl}}D^l_{,j} \tag{62}$$

$$W^k_{,l} = -\frac{m_{jlk}m_{kil}}{m_{jik}}\frac{1}{T_{ijkl}}[D^l - D^j] - \frac{1}{T_{ijkl}}D^j_{,l} \tag{63}$$

$$\begin{aligned} W^k_{,ij} = \frac{1}{T_{ijkl}}D^l_{,ji} - \frac{T_{ijkl,i}}{(T_{ijkl})^2}D^l_{,j} - \Big[\frac{m_{ljk}}{m_{kil}} + \frac{m_{kji}}{T_{ijkl}} - \frac{m_{ljk}}{m_{kil}}\frac{T_{ijkl,i}}{(T_{ijkl})^2}\Big] \\ \times m_{jik}[D^l - D^j] - \frac{m_{ljk}m_{jik}}{m_{kil}}\frac{1}{T_{ijkl}}[D^l - D^j]_{,i} \end{aligned} \tag{64}$$

$$\begin{aligned} W^k_{,il} = -\frac{1}{T_{ijkl}}D^j_{,li} + \frac{T_{ijkl,i}}{(T_{ijkl})^2}D^j_{,l} - \Big[\frac{m_{klj}}{m_{jik}} + \frac{m_{ilk}}{T_{ijkl}} - \frac{m_{jlk}}{m_{jik}}\frac{T_{ijkl,i}}{(T_{ijkl})^2}\Big] \\ \times m_{kil}[D^l - D^j] - \frac{m_{jlk}m_{kil}}{m_{jik}}\frac{1}{T_{ijkl}}[D^l - D^j]_{,i} \end{aligned} \tag{65}$$

$$W^k_{,jl} = \frac{2m_{jlk}m_{ljk}}{T_{ijkl}}[D^l - D^j] - \frac{m_{kil}m_{jlk}}{m_{jik}}\frac{1}{T_{ijkl}}D^l_{,j} + \frac{m_{ljk}m_{jik}}{m_{kil}}\frac{1}{T_{ijkl}}D^j_{,l}. \tag{66}$$

The substitution of (60) and (61) into (58) and (59), gives

$$\Big(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\Big)[D^l - D^j] + [D^l - D^j]_{,i} = T_{ijkl}D^j, \tag{67}$$

$$\Big(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\Big)[D^l - D^j] + [D^l - D^j]_{,i} = T_{ijkl}D^l. \tag{68}$$

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Substituting (60), (62), (64) in the first equation of system (33), and using (67), we obtain

$$D^l_{,ji} + \left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) D^l_{,j} + \frac{m_{kjl}m_{jik}}{m_{kil}} T_{ijkl} D^l = 0. \tag{69}$$

Similarly, using (63) and (65) in the second equation of system (33), and using (68), we get

$$D^j_{,li} + \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) D^j_{,l} + \frac{m_{jlk}m_{kil}}{m_{jik}} T_{ijkl} D^j = 0. \tag{70}$$

The substitution of (62), (63) and (66) in the third equation of (33) gives an identity.

Next we compute the Laplace invariant $\bar{m}_{ji}$ of equation (69),

$$\bar{m}_{ji} = \left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)_{,j} - \frac{m_{kjl}m_{jik}}{m_{kil}} T_{ijkl}.$$

Using (38), (13) and the relation

$$T_{ijkl,ij} = T_{ijkl,j}\left(T_{ijkl} + \frac{T_{ijkl,i}}{T_{ijkl}}\right) + m_{jik}m_{kji}T_{ijkl}$$

we conclude that

$$\bar{m}_{ji} = 0.$$

Therefore, the solution of equation (69) is given by,

$$D^l(x_i, x_j) = m_{jik}T_{ijkl}e^{-\int A\,dx_i}\left[\int \frac{e^{\int A dx_i} G_i(x_i)}{m_{jik}T_{ijkl}}\,dx_i + G_j(x_j)\right].$$

where $G_i(x_i)$ e $G_j(x_j)$ are vector valued functions in $\mathbb{R}^5$.

Let us compute the Laplace invariant $\tilde{m}_{li}$ of equation (70):

$$\tilde{m}_{li} = \left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right)_{,l} - \frac{m_{jlk}m_{kil}}{m_{jik}} T_{ijkl}.$$

Using (39), (13) and the relation

$$T_{ijkl,il} = -T_{ijkl,l}\left(T_{ijkl} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) + m_{kil}m_{ilk}T_{ijkl}$$

we obtain

$$\tilde{m}_{li} = 0.$$

Therefore, the solution of equation (70) is given by,

$$D^j(x_i, x_l) = m_{kil}T_{ijkl}e^{-\int (A+m_{jil})\,dx_i}\left[\int \frac{e^{\int (A+m_{jil})\,dx_i}\bar{G}_i(x_i)}{m_{kil}T_{ijkl}}\,dx_i + G_l(x_l)\right]. \tag{71}$$

where $\bar{G}_i(x_i)$ and $\bar{G}_l(x_l)$ are vector valued functions in $\mathbb{R}^5$.

We will now show that $G_i(x_i) = \bar{G}_i(x_i)$. Differentiating D^l and D^j with respect to x_i, we obtain, respectively

$$D^l_{,i} = -\left(A - \frac{m_{jik,i}}{m_{jik}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) D^l + G_i(x_i),$$
$$D^j_{,i} = -\left(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}} - \frac{T_{ijkl,i}}{T_{ijkl}}\right) D^j + \bar{G}_i(x_i).$$

Therefore

$$[D^l - D^j]_{,i} = G_i(x_i) - \bar{G}_i(x_i) - \left(A - \frac{T_{ijkl,i}}{T_{ijkl}}\right)[D^l - D^j] + \frac{m_{jik,i}}{m_{jik}} D^l + \left(m_{jil} - \frac{m_{kil,i}}{m_{kil}}\right) D^j.$$

It follows from (67), that

$$G_i(x_i) = \bar{G}_i(x_i).$$

Now using (26) and (29) we get

$$D^l(x_i, x_j) = \frac{P_j^4}{Q_j^4}\Big[\int \frac{Q_j^4 G_i(x_i)}{P_j^4}\, dx_i + G_j(x_j)\Big]. \tag{72}$$

On the other hand, from equations (13) we obtain,

$$m_{kil} T_{ijkl} = m_{jil} U_{ijkl}. \tag{73}$$

Therefore from (73) and (71), we have that

$$D^j(x_i, x_l) = m_{jil} U_{ijkl} e^{-\int (A + m_{jil})\, dx_i}\Big[\int \frac{e^{\int (A+m_{jil})\, dx_i} G_i(x_i)}{m_{jil} U_{ijkl}}\, dx_i + G_l(x_l)\Big].$$

Using (26) and (29) we obtain

$$D^j(x_i, x_l) = \frac{P_l^4}{Q_l^4}\Big[\int \frac{Q_l^4 G_i(x_i)}{P_l^4}\, dx_i + G_l(x_l)\Big]. \tag{74}$$

Substituting (72) and (74) into (60), we conclude that

$$W^k = \frac{1}{T_{ijkl}}\Big\{\frac{P_j^4}{Q_j^4}\Big[\int \frac{Q_j^4 G_i(x_i)}{P_j^4}\, dx_i + G_j(x_j)\Big] - \frac{P_l^4}{Q_l^4}\Big[\int \frac{Q_l^4 G_i(x_i)}{P_l^4}\, dx_i + G_l(x_l)\Big]\Big\}.$$

It follows from (73) and (29) that $P_l^4 = m_{jil} U_{ijkl} = m_{kil} T_{ijkl}$. Using this relation in the expression above, we obtain (55). ■

Lemma 3.4 *The solution of* (34) *is given by*

$$W^j = \frac{m_{jik}}{Q_k^4}\Big[\int \frac{Q_k^4 F(x_i)}{P_k^4}\,dx_i + G_k(x_k)\Big] - \frac{m_{jil}}{Q_l^4}\Big[\int \frac{Q_l^4 F(x_i)}{P_l^4}\,dx_i + \bar{G}_l(x_l)\Big]. \tag{75}$$

Proof Using (13) and (38), we have that

$$\Big(A + m_{jir} - \frac{m_{jir,i}}{m_{jir}}\Big)_{,r} = m_{jir}m_{irj}, \quad r = k, l. \tag{76}$$

The substitution of (76) in the first two equations of (34), gives

$$\Big[W^j_{,i} + \Big(A + m_{jir} - \frac{m_{jir,i}}{m_{jir}}\Big)W^j\Big]_{,r} = 0, \quad r = k, l.$$

Therefore, we have that

$$W^j_{,i} + \Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\Big)W^j = L^k(x_i, x_l) \tag{77}$$

and

$$W^j_{,i} + \Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}}\Big)W^j = L^l(x_i, x_k). \tag{78}$$

From (77), (78) and (25), we conclude that

$$W^j = \frac{1}{U_{ijkl}}[L^l - L^k]. \tag{79}$$

Differentiating W^j, using (13) and the following derivatives

$$U_{ijkl,j} = 0, \quad U_{ijkl,k} = \frac{m_{jik}m_{jkl}}{m_{jil}}U_{ijkl}, \quad U_{ijkl,l} = \frac{m_{jil}m_{jlk}}{m_{jik}}U_{ijkl},$$

we get

$$W^j_{,i} = -\frac{U_{ijkl,i}}{(U_{ijkl})^2}[L^l - L^k] + \frac{1}{U_{ijkl}}[L^l - L^k]_{,i} \tag{80}$$

$$W^j_{,k} = -\frac{m_{jkl}m_{jik}}{m_{jil}}\frac{1}{U_{ijkl}}[L^l - L^k] + \frac{1}{U_{ijkl}}L^l_{,k} \tag{81}$$

$$W^j_{,l} = -\frac{m_{jlk}m_{jil}}{m_{jik}}\frac{1}{U_{ijkl}}[L^l - L^k] - \frac{1}{U_{ijkl}}L^k_{,l} \tag{82}$$

$$\begin{aligned} W^j_{,ki} = \frac{1}{U_{ijkl}}L^l_{,ki} - \frac{U_{ijkl,i}}{(U_{ijkl})^2}L^l_{,k} - \Big[\frac{m_{jkl}}{m_{jil}} + \frac{m_{ikj}}{U_{ijkl}} - \frac{m_{jkl}}{m_{jil}}\frac{U_{ijkl,i}}{(U_{ijkl})^2}\Big] \\ \times m_{jik}[L^l - L^k] - \frac{m_{jkl}m_{jik}}{m_{jil}}\frac{1}{U_{ijkl}}[L^l - L^k]_{,i} \end{aligned} \tag{83}$$

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$$W^{j}_{,li} = -\frac{1}{U_{ijkl}}L^{k}_{,li} + \frac{U_{ijkl,i}}{(U_{ijkl})^2}L^{k}_{,l} - \Big[\frac{m_{klj}}{m_{jik}} + \frac{m_{ilj}}{U_{ijkl}} - \frac{m_{jlk}}{m_{jik}}\frac{U_{ijkl,i}}{(U_{ijkl})^2}\Big] \times m_{jil}[L^l - L^k] - \frac{m_{jlk}m_{jil}}{m_{jik}}\frac{1}{U_{ijkl}}[L^l - L^k]_{,i} \tag{84}$$

$$W^{j}_{,lk} = 2\frac{m_{jlk}m_{jkl}}{U_{ijkl}}[L^l - L^k] - \frac{m_{jlk}m_{jil}}{m_{jik}}\frac{1}{U_{ijkl}}L^{l}_{,k} + \frac{m_{jkl}m_{jik}}{m_{jil}}\frac{1}{U_{ijkl}}L^{k}_{,l}. \tag{85}$$

The substitution of (79) and (80) in (77) and (78), gives

$$\Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\Big)[L^l - L^k] + [L^l - L^k]_{,i} = U_{ijkl}L^k, \tag{86}$$

$$\Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}}\Big)[L^l - L^k] + [L^l - L^k]_{,i} = U_{ijkl}L^l. \tag{87}$$

Using (79), (81) and (83) in the first equation of (34), as a consequence of (86), we get

$$L^{l}_{,ki} + \Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\Big)L^{l}_{,k} + \frac{m_{lkj}m_{jik}}{m_{jil}}U_{ijkl}L^l = 0. \tag{88}$$

Similarly, it follows from (79), (82), (84) substituted in the second equation of (34), as a consequence of (87), that

$$L^{k}_{,li} + \Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}}\Big)L^{k}_{,l} + \frac{m_{jlk}m_{jil}}{m_{jik}}U_{ijkl}L^k = 0. \tag{89}$$

The expressions (81), (82) and (85) substituted in the third equation of (34) provide an identity.

The Laplace invariant $\bar{\bar{m}}_{ki}$ of equation (88) is given by

$$\bar{\bar{m}}_{ki} = \Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}}\Big)_{,k} - \frac{m_{lkj}m_{jik}}{m_{jil}}U_{ijkl}.$$

Using (39), (13) and the relation

$$U_{ijkl,ik} = U_{ijkl,k}\Big(U + \frac{U_{ijkl,i}}{U_{ijkl}}\Big) + m_{jik}m_{ikj}U_{ijkl}$$

we obtain

$$\bar{\bar{m}}_{ki} = 0.$$

Therefore, the solution of equation (88) is given by

$$L^l(x_i, x_k) = m_{jik}U_{ijkl}e^{-\int (A+m_{jik})\,dx_i}\Big[\int \frac{e^{\int (A+m_{jik})\,dx_i}F(x_i)}{m_{jik}U_{ijkl}}\,dx_i + G_k(x_k)\Big],$$

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where $F(x_i)$ and $G_k(x_k)$ are vector valued functions in $\mathbb{R}^5$.

Similarly, we compute the Laplace invariant $\tilde{\tilde{m}}_{li}$ of equation (89),

$$\tilde{\tilde{m}}_{li} = \Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}} \Big)_{,l} - \frac{m_{jlk} m_{jil}}{m_{jik}} U_{ijkl}.$$

Using (39), (13) and the relation

$$U_{ijkl,il} = -U_{ijkl,l} \Big(U_{ijkl} - \frac{U_{ijkl,i}}{U_{ijkl}} \Big) + m_{jil} m_{ilj} U_{ijkl}$$

we obtain

$$\tilde{\tilde{m}}_{li} = 0.$$

We conclude that the solution of (89) is given by,

$$L^k(x_i, x_l) = m_{jil} U_{ijkl} e^{-\int (A+m_{jil})\, dx_i} \Big[\int \frac{e^{\int (A+m_{jil})\, dx_i} \bar{F}(x_i)}{m_{jil} U_{ijkl}} \, dx_i + \bar{G}_l(x_l) \Big],$$

where $\bar{F}(x_i)$ and $\bar{G}_l(x_l)$ are vector valued functions in $\mathbb{R}^5$.

As in the proof of the previous lemma, we show that $F(x_i) = \bar{F}(x_i)$. Differentiating L^l and L^k with respect to x_i we obtain respectively

$$L^l_{,i} = -\Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}} - \frac{U_{ijkl,i}}{U_{ijkl}} \Big) L^l + F(x_i), \tag{90}$$

$$L^k_{,i} = -\Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}} - \frac{U_{ijkl,i}}{U_{ijkl}} \Big) L^k + \bar{F}(x_i). \tag{91}$$

From (90), (91) and (86) we conclude that

$$F(x_i) = \bar{F}(x_i).$$

Now using (26) and (29) we have that

$$L^l(x_i, x_k) = \frac{P^4_k}{Q^4_k} \Big[\int \frac{Q^4_k F(x_i)}{P^4_k} \, dx_i + G_k(x_k) \Big],$$

$$L^k(x_i, x_l) = \frac{P^4_l}{Q^4_l} \Big[\int \frac{Q^4_l F(x_i)}{P^4_l} \, dx_i + \bar{G}_l(x_l) \Big].$$

Substituting the last two expressions in (79) we get

$$W^j = \frac{1}{U_{ijkl}} \Big\{ \frac{P^4_k}{Q^4_k} \Big[\int \frac{Q^4_k F(x_i)}{P^4_k} \, dx_i + G_k(x_k) \Big] - \frac{P^4_l}{Q^4_l} \Big[\int \frac{Q^4_l F(x_i)}{P^4_l} \, dx_i + \bar{G}_l(x_l) \Big] \Big\}.$$

Finally we obtain (75), using (26). ∎

We can now prove our main result

Proof of Theorem 3.1 It follows from Lemmas 3.2–3.4 that a Dupin hypersurface is given by (32), where W^k and W^j are given for (55) and (75), respectively. Differentiating (55) with respect to x_i and using (73), we obtain

$$W^k_{,i} = -\frac{m_{jik}}{Q_j^4}\Big(A - \frac{m_{jik,i}}{m_{jik}}\Big)\Big[\int \frac{Q_j^4 G_i(x_i)}{P_j^4}\,dx_i + G_j(x_j)\Big] + \frac{m_{kil}}{Q_l^4}\Big(A + m_{jil} - \frac{m_{kil,i}}{m_{kil}}\Big)\Big[\int \frac{Q_l^4 G_i(x_i)}{P_l^4}\,dx_i + G_l(x_l)\Big]. \tag{92}$$

Differentiating (75) with respect to x_i, we get

$$W^j_{,i} = -\frac{m_{jik}}{Q_k^4}\Big(A + m_{jik} - \frac{m_{jik,i}}{m_{jik}}\Big)\Big[\int \frac{Q_k^4 F(x_i)}{P_k^4}\,dx_i + G_k(x_k)\Big] + \frac{m_{jil}}{Q_l^4}\Big(A + m_{jil} - \frac{m_{jil,i}}{m_{jil}}\Big)\Big[\int \frac{Q_l^4 F(x_i)}{P_l^4}\,dx_i + \bar{G}_l(x_l)\Big]. \tag{93}$$

The substitution of (55), (75), (92) and (93) into equation (44), gives

$$\frac{m_{kil}}{Q_l^4}\Big(m_{jil} + \Big[\log\Big(\frac{m_{jik}}{m_{kil}}\Big)\Big]_{,i}\Big)\Big[\int \frac{Q_l^4 G_i(x_i)}{P_l^4}\,dx_i + G_l(x_l)\Big] = \frac{m_{jil}}{Q_l^4}\Big(m_{jil} - m_{jik} + \Big[\log\Big(\frac{m_{jik}}{m_{jil}}\Big)\Big]_{,i}\Big)\Big[\int \frac{Q_l^4 F(x_i)}{P_l^4}\,dx_i + \bar{G}_l(x_l)\Big].$$

From (24) and (25), we get

$$m_{kil}T_{ijkl}\Big[\int \frac{Q_l^4 G_i(x_i)}{P_l^4}\,dx_i + G_l(x_l)\Big] = m_{jil}U_{ijkl}\Big[\int \frac{Q_l^4 F(x_i)}{P_l^4}\,dx_i + \bar{G}_l(x_l)\Big]$$

and it follows from (73), that

$$\int \frac{Q_l^4 G_i(x_i)}{P_l^4}\,dx_i + G_l(x_l) = \int \frac{Q_l^4 F(x_i)}{P_l^4}\,dx_i + \bar{G}_l(x_l). \tag{94}$$

Differentiating (94) with respect to x_i, we get

$$G_i(x_i) = F(x_i),$$

and therefore

$$G_l(x_l) = \bar{G}_l(x_l).$$

The substitution of these two equalities in (55), (75) and in (43), gives

$$\bar{X} = \frac{1}{m_{jik}}\big[m_{jik}B_j^4 - m_{kil}B_l^4 - m_{jik}B_k^4 + m_{jil}B_l^4\big],$$

where we have used (28). It follows from the fourth equation of (13) that

$$\bar{X} = B^4_j - B^4_k + B^4_l,$$

which substituted into (36), implies (27).

Considering α^i and α^s, $s = j, k, l$ defined by (30), it follows from (16)–(18), (22) and (12) that

$$X_{,r} = V\alpha^r, \quad r = i, j, k, l. \tag{95}$$

Differentiating (95), we have

$$X_{,rr} = V_{,r}\alpha^r + V\alpha^r_{,r}, \quad r = i, j, k, l. \tag{96}$$

It follows from (95) that the metric of $X_{,r}$ is given by

$$g_{rr} = (V)^2|\alpha^r|^2, \quad g_{rt} = 0,\ r \neq t. \tag{97}$$

A unit vector field normal to X is given by

$$N = \frac{\alpha^i \times \alpha^j \times \alpha^k \times \alpha^l}{|\alpha^i|\,|\alpha^j|\,|\alpha^k|\,|\alpha^l|}. \tag{98}$$

Since X is a Dupin hypersurface parametrized by orthogonal curvature lines, with λ_s, as principal curvature we have, for $1 \leq r \neq s \leq 4$

$$\langle N, X_{,rs}\rangle = 0, \quad \lambda_s = \frac{\langle X_{,rr}, N\rangle}{g_{rr}}.$$

Hence from (96) and (98) we obtain for $r = i, j, k, l$,

$$\lambda_r = \frac{\langle \alpha^r_{,r}, \alpha^i \times \alpha^j \times \alpha^k \times \alpha^l\rangle}{V|\alpha^r|^2|\alpha^i|\,|\alpha^j|\,|\alpha^k|\,|\alpha^l|}.$$

Therefore, we conclude that conditions (a), (b) and (c) are satisfied.

Conversely, let λ_r be real functions distinct at each point, such that $\lambda_{r,r} = 0$. Assume that the functions m_{rts}, defined by (31), satisfy (13) and suppose $G_r(x_r)$, $1 \leq r \leq 4$, are vector valued functions satisfying properties (a), (b) and (c). Defining X by (27), it follows from Lemma 2.3 and properties (a) and (b), that X is an immersion, whose coordinates curves are orthogonal. Moreover, the induced metric is given by (97) and a unit normal vector field by (98).

Differentiating (95) with respect to x_t, using Lemma 2.3, the expressions (13), (23) and (30) we obtain

$$X_{,rt} = V\Big(\frac{\lambda_{r,t}}{\lambda_t - \lambda_r}\alpha^r + \frac{\lambda_{t,r}}{\lambda_r - \lambda_t}\alpha^t\Big), \quad r \neq t.$$

From (98), it follows that $\langle X_{,rt}, N\rangle = 0$. Hence the second fundamental form is diagonal and therefore the coordinates curves are lines of curvature. Moreover, it follows from (96)–(98) and from property (c) that for $r = i, j, k, l$,

$$\frac{\langle X_{,rr}, N\rangle}{g_{rr}} = \frac{\langle \alpha^r_{,r}, \alpha^i \times \alpha^j \times \alpha^k \times \alpha^l\rangle}{V|\alpha^r|^2|\alpha^i|\,|\alpha^j|\,|\alpha^k|\,|\alpha^l|} = \lambda_r$$

which concludes the proof. ∎

4 Properties

In this section, we show that the vector valued functions which appear in Theorem 3.1 are invariant by inversions and homotheties.

Theorem 4.1 *Let $X\colon \Omega \subset \mathbb{R}^4 \to \mathbb{R}^5$ be a proper Dupin hypersurface, with four distinct principal curvatures λ_r, parametrized by lines of curvature as in Theorem 3.1. Then the vector valued functions $G_r(x_r)$, $1 \leq r \leq 4$ are invariants by inversions (assuming without loss of generality that $0 \notin X(\Omega)$) and homotheties.*

Proof (a) Let $\tilde{X} = I^5(X)$ be the immersion obtained by composing X with the inversion defined in (14). From Remark 2.2, we have that $\tilde{X}$ is a Dupin hypersurface parametrized by lines of curvature, with distinct principal curvatures given by

$$\tilde{\lambda}_r = \langle X, X\rangle \lambda_r + 2\langle X, N\rangle, \quad 1 \leq r \leq 4. \tag{99}$$

Using Theorem 3.1 for $\tilde{X}$, we have for i, j, k, l fixed distinct indices

$$\tilde{X} = \tilde{V}\left[\tilde{B}_j^4 - \tilde{B}_k^4 + \tilde{B}_l^4\right],$$

where

$$\begin{aligned} \tilde{B}_s^4 &= \frac{1}{\tilde{Q}_s^4}\Big[\int \frac{\tilde{Q}_s^4 \tilde{G}_i(x_i)}{\tilde{P}_s^4}\, dx_1 + \tilde{G}_s(x_s)\Big], \quad s \neq i, \\ \tilde{V} &= \frac{e^{\int \frac{\tilde{\lambda}_k - \tilde{\lambda}_j}{\tilde{\lambda}_j - \tilde{\lambda}_i} \tilde{m}_{jki}\, dx_k}}{\tilde{\lambda}_j - \tilde{\lambda}_i}, \\ \tilde{A} &= -\int \tilde{m}_{jki,i}\, dx_k, \end{aligned} \tag{100}$$

$\tilde{P}_s^4, \tilde{Q}_s^4, s \neq i$ are defined by (26) and (29), in terms of the higher-dimensional Laplace invariants $\tilde{m}$ and $\tilde{G}_r(x_r)$, $1 \leq r \leq 4$ are vector valued functions in $\mathbb{R}^5$.

From Remark 2.2, $\tilde{X}$ and X have the same higher-dimensional Laplace invariants. Therefore, it follows that

$$\tilde{A} = A, \quad \tilde{Q}_r^4 = Q_r^4, \quad \tilde{P}_r^4 = P_r^4 \neq 0, \ r \neq i. \tag{101}$$

Substituting (99) in (100), we have

$$\tilde{V} = \frac{V}{\langle X, X\rangle}. \tag{102}$$

On the other hand,

$$\tilde{X} = \frac{X}{\langle X, X\rangle}. \tag{103}$$

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We will show that $\tilde{G}_r(x_r) = G_r(x_r) r \neq i$. It follows from (102) and (103) that

$$\tilde{B}^4_j - B^4_j - (\tilde{B}^4_k - B^4_k) + \tilde{B}^4_l - B^4_l = 0. \tag{104}$$

We observe that

$$B^4_{j,i} = -AB^4_j + \frac{G_i(x_i)}{P^4_j}, \quad B^4_{s,i} = -(A + m_{jis})B^4_s + \frac{G_i(x_i)}{P^4_s}, \ \ s = k, l.$$

This fact follows from the equalities

$$Q^4_{j,i} = AQ^4_j, \quad Q^4_{s,i} = (A + m_{jis})Q^4_s, \ s = k, l.$$

Therefore differentiating (104) with respect to x_i, we get

$$-A\big(\tilde{B}^4_j - B^4_j\big) + \big(A + m_{jik}\big)\big(\tilde{B}^4_k - B^4_k\big) - \big(A + m_{jil}\big)\big(\tilde{B}^4_l - B^4_l\big) + (\tilde{G}_i - G_i)\Big[\frac{1}{P^4_j} - \frac{1}{P^4_k} + \frac{1}{P^4_l}\Big] = 0.$$

Using (104) and the fact that

$$\frac{1}{P^4_j} - \frac{1}{P^4_k} + \frac{1}{P^4_l} = 0,$$

we have

$$m_{jik}\big[\tilde{B}^4_k - B^4_k\big] - m_{jil}\big[\tilde{B}^4_l - B^4_l\big] = 0. \tag{105}$$

Differentiating this relation with respect to x_i, we obtain

$$\big[m_{jik,i} - m_{jik}(A + m_{jik})\big]\big[\tilde{B}^4_k - B^4_k\big] - \big[m_{jil,i} - m_{jil}(A + m_{jil})\big]\big[\tilde{B}^4_l - B^4_l\big] + (\tilde{G}_i(x_i) - G_i(x_i))\Big[\frac{m_{jik}}{P^4_k} - \frac{m_{jil}}{P^4_l}\Big] = 0.$$

It follows from (105) and from the fact that

$$\frac{m_{jik}}{P^4_k} - \frac{m_{jil}}{P^4_l} = 0,$$

that the expression above reduces to

$$\big[m_{jik,i} - (m_{jik})^2\big]\big[\tilde{B}^4_k - B^4_k\big] - \big[m_{jil,i} - (m_{jil})^2\big]\big[\tilde{B}^4_l - B^4_l\big] = 0.$$

Again, using (105), we obtain the relation

$$\Big(m_{jil} - m_{jik} + \frac{m_{jik,i}}{m_{jik}} - \frac{m_{jil,i}}{m_{jil}}\Big)\big[\tilde{B}^4_l - B^4_l\big] = 0,$$

which, as a consequence of (25), reduces to

$$U_{ijkl}\left[\tilde{B}_l^4 - B_l^4\right] = 0.$$

Since $U_{ijkl} \neq 0$, we obtain

$$\tilde{B}_l^4 = B_l^4. \tag{106}$$

Differentiating with respect to x_i, we get $\tilde{G}_i(x_i) = G_i(x_i)$, hence it follows that $\tilde{G}_l(x_l) = G_l(x_l)$. From (105) and (106), we have

$$\tilde{B}_k^4 = B_k^4, \tag{107}$$

and therefore $\tilde{G}_k(x_k) = G_k(x_k)$. Substituting (106) and (107) into (104), we obtain

$$\tilde{B}_j^4 = B_j^4.$$

and hence $\tilde{G}_j(x_j) = G_j(x_j)$, which concludes the proof of (a).

(b) Let $\bar{X} = aX$ be a homothety of X. From Remark 2.2, we have that $\bar{X}$ is a Dupin hypersurface parametrized by orthogonal curvature lines, with distinct principal curvatures given by

$$\bar{\lambda}_r = \frac{\lambda_r}{a}, \quad 1 \leq r \leq 4. \tag{108}$$

Using Theorem 3.1 for $\bar{X}$, we have for i, j, k, l distinct fixed indices

$$\bar{X} = \bar{V}\left[\bar{B}_j^4 - \bar{B}_k^4 + \bar{B}_l^4\right].$$

where

$$\begin{aligned} \bar{B}_s^4 &= \frac{1}{\bar{Q}_s^4}\left[\int \frac{\bar{Q}_s^4 \bar{G}_i(x_i)}{\bar{P}_s^4}\,dx_i + \bar{G}_s(x_s)\right], \quad s \neq i, \\ \bar{V} &= \frac{e^{\int \frac{\bar{\lambda}_k - \bar{\lambda}_j}{\bar{\lambda}_j - \bar{\lambda}_i} \bar{m}_{jki}\,dx_k}}{\bar{\lambda}_j - \bar{\lambda}_i}, \\ \bar{A} &= -\int \bar{m}_{jki,i}\,dx_k, \end{aligned} \tag{109}$$

$\bar{P}_s^4, \bar{Q}_s^4, s \neq i$ are defined by (26) and (29) in terms of the higher-dimensional Laplace invariants and $\bar{G}_r(x_r)$, $1 \leq r \leq 4$ are vector valued functions in $\mathbb{R}^5$. We will show that $\bar{G}_r(x_r) = G_r(x_r)$.

From Remark 2.2, $\bar{X}$ and X have the same Laplace invariants. Therefore, it follows that

$$\bar{A} = A \quad \bar{Q}_s^4 = Q_s^4 \quad \bar{P}_s^4 = P_s^4 \neq 0, \ s \neq i. \tag{110}$$

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Substituting (108) in (109), we have

$$\bar{V} = aV. \tag{111}$$

Since

$$X = V\left[B_j^4 - B_k^4 + B_l^4\right], \quad \bar{X} = \bar{V}\left[\bar{B}_j^4 - \bar{B}_k^4 + \bar{B}_l^4\right], \tag{112}$$

substituting the expressions (110), (111), (112) in $\bar{X} = aX$ we have,

$$\bar{B}_j^4 - B_j^4 - (\bar{B}_k^4 - B_k^4) + \bar{B}_l^4 - B_l^4 = 0.$$

The same argument of item (a) proves that $\bar{G}_r(x_r) = G_r(x_r)$, $\forall r$. ■

The results of this paper were announced in [15]. Higher dimensional generalizations of Theorem 3.1 and the non generic case will appear elsewhere.

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