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Abstract

We prove that the direct image complex for the D-twisted SL_n Hitchin fibration is determined by its restriction to the elliptic locus, where the spectral curves are integral. The analogous result for GL_n is due to Chaudouard and Laumon. Along the way, we prove that the Tate module of the relative Prym group scheme is polarizable, and we also prove δ-regularity results for some auxiliary weak abelian fibrations.

Contents

1. Introduction

Let C be a nonsingular projective and integral curve of genus g over an algebraically closed field of characteristic zero. Let D be a line bundle on C, with $d := \deg(D) > 2g-2$.

Fix a pair of coprime positive integers (n, e). The GL_n moduli space we consider is the moduli space [Nit91] of stable, rank n, degree e, D-twisted Higgs bundles $(E, \phi : E \to E(D))$ on C; it is an integral, quasi projective and nonsingular variety. There is the projective Hitchin morphism $h_n : M_n \to A_n = \bigoplus_{i=1}^{n} H^0(C, iD)$ onto the affine space of the possible characteristic polynomials of ϕ.

The decomposition theorem [BBD81] predicts that the direct image complex $Rh_{n*}\overline{\mathbb{Q}}_\ell$ splits into a finite direct sum of shifted simple perverse sheaves, each supported on an integral closed subvariety $S \subseteq A_n$. These subvarieties are called the supports of $Rh_{n*}\overline{\mathbb{Q}}_\ell$. The *socle of* $Rh_{n*}\overline{\mathbb{Q}}_\ell$, denoted by $\mathrm{Socle}(Rh_{n*}\overline{\mathbb{Q}}_\ell)$, is the finite subset of A_n of generic points η_S of the supports S of $Rh_{n*}\overline{\mathbb{Q}}_\ell$.

One of the main geometric ingredients of Ngô's proof [Ngô10] of the Langlands–Shelstad fundamental lemma for reductive Lie groups G, is his support theorem [Ngô10, Theorem 7.2.1].

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This is a statement concerning the socle of the direct image complex via the Hitchin morphism $M_G \to A_G$ associated with (G, C, D), after restriction to a certain large open subset of the target A_G. In the special case $G = \mathrm{GL}_n$, one considers the elliptic locus, i.e. the dense open subvariety $A_n^{\mathrm{ell}} \subseteq A_n$ corresponding to those points $a \in A_n$ for which the associated spectral curve is geometrically integral. Then the Ngô support theorem implies that $\mathrm{Socle}(Rh_{n,*}\overline{\mathbb{Q}}_\ell) \cap A_n^{\mathrm{ell}} = \{\eta_{A_n}\}$, the generic point of the target A_n. In other words, over the elliptic locus, the simple summands appearing in the decomposition theorem are the intermediate extensions to A_n^{ell} of the direct image lisse sheaves over the locus A_n^{smooth} of regular values of h_n. This has striking consequences for the handling of orbital integrals over the elliptic locus (for every G), which thus become more tractable: the ones corresponding to points in $A_n^{\mathrm{ell}} \backslash A^{\mathrm{smooth}}$ can be related to the ones over A_n^{smooth} by a principle of continuity on A_n^{ell}; this is precisely because there are no new supports on the boundary $A_n^{\mathrm{ell}} \backslash A_n^{\mathrm{smooth}}$ (cf. [Ngô11, § 1]).

Support-type theorems have been appearing in the related geometric contexts of relative Hilbert schemes and of relative compactified Jacobians of families of reduced planar curves in [MY14, MS13, MS13, MSV15, She12], also in connection with Bogomol'nyi–Prasad–Sommerfield (BPS) states.

It is thus interesting, important, and seemingly nontrivial, to 'go beyond the elliptic locus'. Chaudouard and Laumon have extended [CL12] Ngô's result on A_n^{ell} (which holds for every G), by proving that (and here we specialize their result to $G = \mathrm{GL}_n$) $\mathrm{Socle}(Rh_{n,*}\overline{\mathbb{Q}}_\ell) \cap A_n^{\mathrm{grss}} = \{\eta_{A_n}\}$, where $A_n^{\mathrm{ell}} \subseteq A_n^{\mathrm{grss}}$ is the larger open locus for which the associated spectral curves are reduced. They have also subsequently extended this result to the whole base A_n of the D-twisted GL_n Hitchin fibration in [CL16], where they prove the following.

THEOREM 1.0.1 (GL_n socle [CL16]). $\mathrm{Socle}(Rh_{n*}\overline{\mathbb{Q}}_\ell) = \{\eta_{A_n}\}$.

In particular, there are no new supports as one passes from the regular locus A_n^{smooth}, to the elliptic locus A_n^{ell}, to A_n^{grss} and, finally, to the whole of A_n. The decomposition theorem then takes the form of an isomorphism $Rh_{n*}\overline{\mathbb{Q}}_\ell \cong \bigoplus_{q \geqslant 0} \mathcal{IC}_{A_n}(R^q)[-q]$, where R^q is the lisse restriction of the $\overline{\mathbb{Q}}_\ell$-constructible sheaf $Rh_{n*}\overline{\mathbb{Q}}_\ell$ to A_n^{smooth}, and where $\mathcal{IC}$ denotes the intermediate extension functor shifted so as to 'start' in cohomological degree zero. Since the general fibers of h_n are (connected) abelian varieties, we even have $R^q \cong \bigwedge^q R^1$ for every $0 \leqslant q \leqslant 2d_{h_n}$, where d_{h_n} is the relative dimension of h_n.

When $G = \mathrm{SL}_n$, we have the following picture, which goes back, at least implicitly, to [Nit91]; see § 2.2. Our SL_n moduli space $\check{M}_n \subseteq M_n$ consists of those stable pairs with fixed $\epsilon = \det(E)$ and trivial trace $\mathrm{tr}(\phi) = 0$. Then $\check{M}_n$ is an integral, quasi projective and nonsingular variety. The restriction of the Hitchin morphism h_n, yields the Hitchin morphism $\check{h}_n : \check{M}_n \to \check{A}_n := \bigoplus_{i=2}^n H^0(X, iD)$, whose socle is the object of study of this paper.

This socle is known over the elliptic locus $\check{A}_n^{\mathrm{ell}} = \check{A}_n \cap A_n^{\mathrm{ell}}$: by work of Ngô [Ngô06, Ngô10], we have that $\mathrm{socle}(R\check{h}_{n*}\overline{\mathbb{Q}}_\ell) \cap \check{A}_n^{\mathrm{ell}}$ is given by the generic point $\eta_{\check{A}_n}$, union a finite set of points (66), directly related to the endoscopy theory of SL_n.

The purpose of this paper is to prove the following theorem, to the effect that there are no new supports in $\check{A}_n \backslash \check{A}_n^{\mathrm{ell}}$, beyond the ones (66) already known to dwell in $\check{A}_n^{\mathrm{ell}}$.

THEOREM 1.0.2 (SL_n socle). $\mathrm{Socle}(R\check{h}_{n*}\overline{\mathbb{Q}}_\ell) \subseteq \check{A}_n^{ell}$.

At first sight, the proof of our main Theorem 1.0.2 for the SL_n socle runs in parallel with the one of Theorem 1.0.1 for the GL_n socle in [CL16, § 9], where the authors use: Ngô support inequality over the whole base A_n; a multi-variable δ-regularity inequality for the Jacobi group scheme acting on the Hitchin fibers over the elliptic locus; the identity between the abelian variety parts of the Jacobian of an arbitrary spectral curve, and the Jacobian of the normalization of its reduction.

The situation over SL_n presents some substantial differences, which we now summarize.

(1) We need to prove the support inequality Theorem 3.4.1(1) over the whole SL_n base $\check{A}_n$. This had been known [Ngô10] over $\check{A}_n^{\mathrm{ell}}$ only.

(2) In order to achieve the SL_n support inequality, we need to establish the polarizability Theorem 4.7.2 of the Tate module of the Prym group scheme over $\check{A}_n$.

(3) In turn, this required that: we determine the explicit form (38) of a natural polarization of the Tate module of the Jacobian of an arbitrary spectral curve (see the GL_n polarizability Theorem 3.3.1); we combine the explicit (38) with the identification (47) of the affine parts of the fibers of the Jacobi and Prym groups schemes. At this juncture, the SL_n polarizability result follows by first exhibiting the Prym Tate module as a natural direct summand of the Jacobi Tate module, and then by using that pull-back and push-forward (norm) are adjoint for the cup product.

(4) The δ-regularity inequality over $\check{A}_n^{\mathrm{ell}}$ afforded by (58) is not useful towards proving our main result Theorem 1.0.2. However, the method of proof is: we use a product formula for the Hitchin fibration, and the identification (47) of the affine parts of the Jacobi and Prym varieties, to show that the codimensions of the δ-loci are preserved when passing from the elliptic locus A_n^{ell}, to the traceless elliptic locus $\check{A}_n^{\mathrm{ell}}$, so that (58) holds.

(5) We pursue the same line of argument to reach the correct SL_n replacement (76) of the GL_n multi-variable δ-regularity inequality used in [CL16, § 9]. This is done by first considering a multi-variable Hitchin base, then by slicing it using linear weighted conditions on the traces, and finally by verifying that the codimensions of the δ-loci are un-effected by the slicing.

(6) We fix a minor inaccuracy in [CL16]. See Remark 5.4.3.

As to the structure of the paper, we refer the reader to the summaries at the beginning of each of the five sections.

2. Preliminaries

This section is a collection of preliminary constructions, results and definitions. Sections 2.1, 2.2 introduce the D-twisted SL_n Hitchin morphism $\check{h}_n : \check{M}_n \to \check{A}_n$ which is the focus of this paper. The GL_n case plays an important role, and is thus discussed as well. Section 2.3 discusses spectral curves and covers: diagram (2) plays a recurrent role in the paper. Spectral curves afford an important alternative interpretation of the fibers of the Hitchin morphism via the Hitchin, Beauville–Narasimhan–Ramanan, Schaub correspondence, which is discussed in § 2.4, together with some essential properties of the Hitchin morphism and of its fibers: connectivity, action of the Prym variety (8), irreducible components over the elliptic locus. This leads to a discussion in § 2.5 of the endoscopic locus for SL_n, which can be described with the aid of the n-torsion in $\mathrm{Pic}^0(C)$. Section 2.6 discusses Ngô's notion of δ-regular weak abelian fibration, which is a very important tool in the study of Hitchin systems, and an essential one for this paper; two highlights are Ngô support inequality, and its 'opposite', the δ-regularity inequality.

Unless otherwise mentioned, we work with varieties, separated schemes of finite type, over a field of characteristic zero. Let C be an integral and nonsingular curve of genus g and let $D \in \mathrm{Pic}^d(C)$ be a fixed line bundle on C of degree $d > 2g-2$. We fix two coprime integers (n, e)

and a degree e line bundle $\epsilon \in \mathrm{Pic}^e(C)$. Recall that the coprimality condition ensures that the two notions of stability and of semistability coincide, so that the (coarse = fine) moduli spaces of Higgs bundles we consider are nonsingular.

2.1 GL_n and SL_n Hitchin fibrations

A standard reference for what follows is [Nit91].

The GL_n *case.* Let $\mathscr{M}$ be the moduli space of stable, D-twisted, GL_n Higgs bundles of rank n and degree e on the curve C. Then $\mathscr{M}$ is a nonsingular and quasi-projective variety of pure dimension n^2d+1. It parameterizes stable pairs (E,ϕ), where: E is a rank n and degree e vector bundle on the curve C, and $\phi : E \to E(D)$ is a morphism of $\mathcal{O}_C$-modules. The notion of stability is the usual one: for every ϕ-invariant proper sub-bundle $F \subseteq E$, the slopes $\mu := \deg/\mathrm{rk}$ satisfy the inequality $\mu(F) < \mu(E)$. There is the projective characteristic morphism

$$h : \mathscr{M} \to \mathscr{A} := \bigoplus_{i=1}^{n} H^0(C, iD),$$

sending (E,ϕ) to the coefficients $(-\mathrm{tr}(\phi), +\mathrm{tr}(\wedge^2\phi), \dots, (-1)^n \det(\phi))$ of the characteristic polynomial of ϕ. The elements of $\mathscr{A}$ are called characteristics.

The pure-dimensional nonsingular variety $\mathscr{M}$ is connected, hence irreducible. One way to see this, is to couple the fact that the proper characteristic morphism is of pure relative dimension [CL16, Corollaire 8.2] with the fact (Remark 2.4.4) that the general fiber, being the Jacobian of a nonsingular and connected spectral curve, is connected. I thank the anonymous referee for bringing this to my attention.

The moduli space N of rank n and degree e vector bundles on C sits naturally in $\mathscr{M}$ (take $\phi := 0$). It is well known that N is integral, nonsingular, projective and of dimension $n^2(g-1)+1$. We have inclusions $\mathscr{M} = \overline{T} \supseteq T \supseteq N$, where T is the total space of the vector bundle of rank $n^2[d-(g-1)]$ over N with fiber at E given by $H^0(C, \mathrm{End}(E)(D))$; see [Nit91, Proposition 7.1 and the formula above it]. Then T is integral, nonsingular, of dimension n^2d+1, and it is a Zariski-dense open subvariety of $\mathscr{M}$; see [Nit91, pp. 297–298].

The GL_n *traceless case.* We need the following simple traceless variant of the D-twisted GL_n moduli space: geometrically, it is the pre-image via the morphism $h : \mathscr{M} \to \mathscr{A}$ of the locus $\mathscr{A}(0) \subseteq \mathscr{A}$ of traceless characteristics. Let $\mathscr{M}(0) \subseteq \mathscr{M}$ be the moduli space of stable pairs (E,ϕ) as above, subject to the additional traceless constraint $\mathrm{tr}(\phi) = 0$. By repeating the arguments in [Nit91] concerning $\mathscr{M}$, but with the traceless constraint, we see that $\mathscr{M}(0)$ is a nonsingular and quasi-projective variety, of pure dimension $nd^2 + 1 - h^0(D)$. Moreover, we have a natural isomorphism $\mathscr{M} \cong H^0(C,D) \times \mathscr{M}(0)$ (see § 4.3, (51)), implying that the nonsingular $\mathscr{M}(0)$ is connected and irreducible.

As above, we have inclusions $\mathscr{M}(0) = \overline{T(0)} \supseteq T(0) \supseteq N$, with the same properties listed above, except that we take traceless endomorphisms, and the rank of the corresponding vector bundle on N equals $h^0(C, \mathrm{End}^0(E)(D)) = n^2[d-(g-1)] - h^0(D)$. We have the projective characteristic morphism

$$h(0) : \mathscr{M}(0) \to \mathscr{A}(0) := \bigoplus_{i=2}^{n} H^0(C, iD).$$

The SL_n *case.* Finally, we introduce the moduli space to which this paper is devoted. Fix a line bundle $\epsilon \in \mathrm{Pic}^e(C)$ on C, of degree e. Let $\mathscr{M}(0,\epsilon) \subseteq \mathscr{M}(0) \subseteq \mathscr{M}$ be the moduli space of stable pairs (E,ϕ) as above, subject to $\mathrm{tr}(\phi)=0$ and to $\det(E)=\epsilon$. By repeating the arguments in [Nit91], but with the traceless and fixed-determinant constraints, we see that the variety $\mathscr{M}(0,\epsilon)$ is nonsingular and quasi-projective, of pure dimension $n^2d+1-h^0(D)-g$. We have the projective characteristic map

$$h(0,\epsilon):\mathscr{M}(0,\epsilon)\to\mathscr{A}(0):=\bigoplus_{i=2}^{n}H^0(C,iD).$$

Let $\mathscr{M}(0,\epsilon)_o$ be the irreducible (also a connected) component containing the moduli space $N(\epsilon)$ of stable rank n and degree e bundles on C with fixed determinant $\epsilon\in\mathrm{Pic}^e(C)$. It is well known that the variety $N(\epsilon)$ is integral, nonsingular, projective, and of dimension $(n^2-1)(g-1)$. As above, we have inclusions $\mathscr{M}(0,\epsilon)_o=\overline{T(0,\epsilon)}\supseteq T(0,\epsilon)\supseteq N(\epsilon)$, with the same properties listed above (again, we take traceless endomorphisms).

Note that $\mathscr{M}(0,\epsilon)=\mathscr{M}(0,\epsilon)_o$ and that the isomorphism class of $\mathscr{M}(0,\epsilon)_o$ is independent of $\epsilon\in\mathrm{Pic}^e(C)$. This can be seen as in the proof of the following simple lemma.

Lemma 2.1.1. *The variety $\mathscr{M}(0,\epsilon)$ is connected, i.e. $\mathscr{M}(0,\epsilon)=\mathscr{M}(0,\epsilon)_o$. The variety $\mathscr{M}(0,\epsilon)$ is the fiber over $\epsilon\in\mathrm{Pic}^e(C)$ of the determinant map $\det:\mathscr{M}(0)\to\mathrm{Pic}^e(C)$, as well as the fiber over $(0,\epsilon)\in H^0(C,D)\times\mathrm{Pic}^e(C)$ of the trace-determinant map $\mathrm{tr}\times\det:\mathscr{M}\to H^0(C,D)\times\mathrm{Pic}^e(C)$.*

Proof. The map det is equivariant with respect to the action of $\mathrm{Pic}^0(C)$ given by $L\cdot(E,\phi):=(E\otimes L,\phi\otimes\mathrm{Id}_L)$ on the domain, and by $L\cdot M:=M\otimes L^{\otimes n}$ on the target. It follows that det is smooth of relative dimension $\dim(\mathscr{M}(0,\epsilon))$, and that all of its fibers are mutually isomorphic to each other. The same is true of the restriction of det to the $\mathrm{Pic}^0(C)$-invariant open subvariety $T(0)\subseteq\mathscr{M}(0)$. Let $Z:=\mathscr{M}(0)\backslash T(0)$ be the closed complement. The resulting map $Z\to\mathrm{Pic}^0(C)$ is also $\mathrm{Pic}^0(C)$-invariant, so that all of its fibers have the same dimension, which must be strictly smaller than $\dim(\mathscr{M}(0,\epsilon))$. It is clear that $\mathscr{M}(0,\epsilon)_o$ is contained in $\det^{-1}(\epsilon)=\mathscr{M}(0\,e)$ and that, by the smoothness of det, it must constitute a connected component of such fiber. Since the fiber $\det^{-1}(\epsilon)$ is of pure dimension $\dim(\mathscr{M}(0,\epsilon))$, the variety Z cannot contain any other connected component of the smooth fiber $\det^{-1}(\epsilon)$. We have thus proved that $\det^{-1}(\epsilon)=\mathscr{M}(0,\epsilon)=\mathscr{M}(0,\epsilon)_o$, which are thus all connected, for the third one is by construction. The assertion concerning $\mathrm{tr}\times\det$ is proved in a similar way. □

2.2 Simplified notation for Hitchin fibrations

We want to simplify our notation, while emphasizing the role of the rank n.

Fix (n,e,ϵ,D). Denote the characteristic Hitchin morphisms

$$h:\mathscr{M}\to\mathscr{A},\quad h(0):\mathscr{M}(0)\to\mathscr{A}(0),\quad h(0,e):\mathscr{M}(0,\epsilon)\to\mathscr{A}(0)$$

as follows:

$$h_n:M_n\to A_n,\quad h_n(0):M_n(0)\to A_n(0),\quad \check{h}_n:\check{M}_n\to\check{A}_n:=A_n(0).\tag{1}$$

We are denoting the same object $\check{A}_n=A_n(0)$ in two different ways: we prefer to use the notation $A_n(0)$ when dealing with $M_n(0)$, and to use $\check{A}_n$ when dealing with $\check{M}_n$.

The projective morphisms h_n and $\check{h}_n$ are known as the D-twisted, Hitchin GL_n and SL_n fibrations. The morphism $h_n(0)$ plays an important auxiliary role in this paper.

We shall also need to consider two several-variable-variants of these Hitchin fibrations, namely $h_{n_\bullet}:M_{n_\bullet}\to A_\bullet$, and $h_{n_\bullet m_\bullet}(0):M_{n_\bullet m_\bullet}(0)\to A_{n_\bullet m_\bullet}(0)$ (cf. §§ 5.1 and 5.3).

An important locus inside the base of the Hitchin fibration is the elliptic locus. In the case of GL_n and SL_n we define it as follows.

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DEFINITION 2.2.1 (Elliptic locus). The elliptic loci $A_n^{\mathrm{ell}} \subseteq A_n$ and $\check{A}_n^{\mathrm{ell}} \subseteq \check{A}_n$ are the respective Zariski-dense open subvarieties of points such that the associated spectral curves are geometrically integral.

Clearly, $\check{A}_n^{\mathrm{ell}} = A_n^{\mathrm{ell}} \cap \check{A}_n$.

2.3 Spectral covers and the norm map

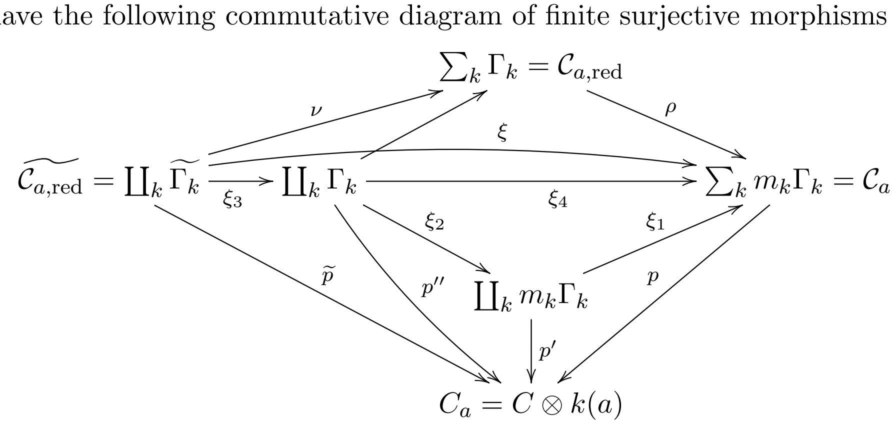
Let $\pi : V(D) \to C$ be the surface total space of the line bundle D on C. Let t be the universal section of π^*D, with zero set on $V(D)$ given by C, viewed as the zero section on $V(D)$. Let $\mathcal{C} = \mathcal{C}_n \subseteq V(D) \times A_n$ be the universal spectral curve, that is the relative curve over A_n with fiber $\mathcal{C}_a$ over a closed point $a = (a(1), \dots, a(n)) \in A_n$, given by the zero set in $V(D) \times \{a\}$ of the section $P_a(t) := t^n + \pi^* a(1) t^{n-1} + \pi^* a(2) t^{n-2} + \cdots + \pi^* a(n)$ of the line bundle $\pi^*(nD)$ on $V(D) \times \{a\}$. Note that A_n is an affine space inside the projective space given by the linear system $|nC|$ on the standard projective completion $\mathbb{P}_C(\mathcal{O}_C \oplus \mathcal{O}_C(-D))$ of $V(D)$, where C sits as the zero section. Let $p : \mathcal{C} \to A_n$ be the natural ensuing morphism. For $a \in A_n$, the spectral curve $\mathcal{C}_a$ is geometrically connected and maps $n:1$ onto $C_a := C \otimes k(a)$ via the flat finite morphism $p_a := p_{|\mathcal{C}_a} : \mathcal{C}_a \to C_a$. The total space of the family $\mathcal{C}$ is integral and nonsingular, and the natural morphism $\mathcal{C} \to C \times A_n$ is finite, flat and of degree n.

When we view each spectral curve $\mathcal{C}_a$ over a geometric point a of A_n, as an effective Cartier divisor on $V(D) \otimes k(a)$, we may write $\mathcal{C}_a = \sum_{k=1}^{s} m_{k,a} \mathcal{C}_{k,a}$, where each $\mathcal{C}_{k,a}$ is geometrically integral, each integer $m_{k,a} > 0$, and the expression is unique. Each curve $\mathcal{C}_{k,a}$ maps finitely onto C_a; denote the corresponding degree by $n_{k,a}$. Clearly, $n = \sum_k m_{k,a} n_{k,a}$. By considering the coefficients $a(i)$ above as the ith symmetric functions of the D-valued roots of the polynomial equation $P_a(t)$, we obtain the unique factorization $P_a(t) = \prod_k P_{a_k}^{m_{k,a}}(t)$, where each $a_k(i) \in H^0(C, iD)$, $1 \leqslant i \leqslant n_k$, is the ith symmetric function of the D-valued roots of $P_a(t)$ that lie on $\mathcal{C}_{k,a}$. In particular, we have that a_k is a geometric point of A_{n_k} (base of the Hitchin fibration for (n_k, e, D)), and that $\mathcal{C}_{k,a}$ is a spectral curve for the D-twisted GL_{n_k} Hitchin fibration.

Let $a \in A_n$. We need to list the various covers of the curve $C_a = C \otimes k(a)$ that arise from the given spectral cover $p_a : \mathcal{C}_a \to C_a$. In doing so, we also simplify and abuse the notation a little bit. We do not assume the point $a \in A_n$ to be a geometric one, so that the intervening integral curves may not be geometrically integral.

We denote the curve $\mathcal{C}_a = \sum_k m_k \Gamma_k$, where: each Γ_k is a spectral curve, zero-set of a section $\mathfrak{s}_k$ of the line bundle $\pi^*(n_k D)$ on the surface $V(D) \otimes k(a)$; the $n_k > 0$ are uniquely-determined positive integers, and we have $n = \sum n_k m_k$. Scheme-theoretically, $m_k \Gamma_k$ is the zero set of the m_kth power $\mathfrak{s}^{m_k}$, and $\mathcal{C}_a = \sum_k m_k \Gamma_k$ is the zero set of the product $\prod_k \mathfrak{s}_k^{m_k}$. We denote by $\xi_{3,k} : \widetilde{\Gamma_k} \to \Gamma_k$ the normalization morphism.

We have the following commutative diagram of finite surjective morphisms of curves.



(2)

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Fact 2.3.1 (The Jacobian of a spectral curve). Let $\overline{a}$ be a geometric point of A_n with underlying Zariski point $a \in A_n$. Then the identity connected component $\mathrm{Pic}^0(\mathcal{C}_a)$ of the degree zero component of $\mathrm{Pic}(\mathcal{C}_a)$ consists of the isomorphism classes of line bundles on the spectral curve $\mathcal{C}_a$ whose restriction to each irreducible component of $\mathcal{C}_{\overline{a}}$ have degree zero; see [BLR90, § 9.3, Corollary 13].

Each of the morphisms to C_a in diagram (2) comes with an associated norm morphism into $\mathrm{Pic}(C_a)$, and with an associated pull-back morphism from $\mathrm{Pic}(C_a)$. Similarly, if we replace Pic with Pic^0. For the definition and properties of the norm morphism, see [EGAII, § 6.5] and [EGAIV.4, § 21.5]. For a quick reference for the facts we use in this paper, see also [HP12, § 3]. See also Fact 2.4.5. We have the norm morphism

$$N_p : \mathrm{Pic}(\mathcal{C}_a) \longrightarrow \mathrm{Pic}(C), \quad \mathrm{Pic}^0(\mathcal{C}_a) \longrightarrow \mathrm{Pic}^0(C_a). \tag{3}$$

We also have the norm morphisms $N_{\widetilde{p}}, N_{p'}$ and $N_{p''}$, as well as the pull-back morphisms $\widetilde{p}^*, p'^*$ and p''^*; similarly, for each of their kth components.

We end this section with the following consideration that will play a role later.

Fact 2.3.2. Since D has positive degree $d > 2g-2$ on C, we have that, on each Γ_k, the line bundle $(p''^* n_k D)_{|\Gamma_k}$ admits some nontrivial section z_k with zero subscheme ζ_k supported at a closed finite nonempty subset of Γ_k. We fix such a section, and we obtain the short exact sequences of $\mathcal{O}_{\Gamma_k}$-modules

$$0 \longrightarrow \mathcal{O}_{\Gamma_k}(-\Gamma_k) \longrightarrow \mathcal{O}_{\Gamma_k} \longrightarrow \mathcal{O}_{\zeta_k} \longrightarrow 0. \tag{4}$$

2.4 The fibers of the Hitchin fibrations

Let $a \in A_n$ and let $p : \mathcal{C}_a =: \Gamma = \sum_k m_k \Gamma_k \to C_a$ be the corresponding spectral cover, with $n = n_\Gamma = \deg(p) = \sum_k n_k m_k = \mathrm{rk}_C(p_* \mathcal{O}_\Gamma)$; see (2). Let $j_k : \eta_k \to \Gamma$ be the finitely many generic points in Γ, one for each irreducible component $m_k \Gamma_k$. A coherent sheaf $\mathcal{E}$ on Γ is torsion free if and only if the natural map $\mathcal{E} \to \prod_k \mathcal{E}_k$ is injective, where $\mathcal{E}_k = j_{k*} j_k^* \mathcal{E}$; see [Sch98, Definition 1.1. and Proposition 1.1]. A torsion-free $\mathcal{E}$ is said to have $\mathrm{Rk}_\Gamma(\mathcal{E}) = r$ if its lengths at the generic points satisfy $l_k(\mathcal{E}) := l_{\mathcal{O}_{\eta_k}}(\mathcal{E}_{\eta_k}) = r\, m_k$, for every k; such a rank is then a nonnegative rational number, which is zero if and only if $\mathcal{E} = 0$. A torsion free $\mathcal{E}$ may fail to have a well-defined $\mathrm{Rk}_\Gamma(\mathcal{E})$. When this rank is well defined, one defines the degree by setting $\mathrm{Deg}_\Gamma(\mathcal{E}) := \chi(\mathcal{E}) - \mathrm{Rk}_\Gamma(\mathcal{E}) \chi(\mathcal{O}_\Gamma)$.

Let $P_\Gamma = \prod_k P_{\Gamma_k}^{m_k}$ be the characteristic equation defining Γ. A torsion-free coherent sheaf $\mathcal{E}$ on Γ corresponds, via p_*, to a pair $(E, \phi : E \to E(D))$ on C, where: $E = p_* \mathcal{E}$ is locally free of rank $\mathrm{rk}_C(E) = \sum_k n_k l_k$; ϕ is the twisted endomorphism corresponding to multiplication by t on $\mathcal{E}$. Then ϕ has characteristic polynomial $P_\phi = \prod_k P_{\Gamma_k}^{l_k}$. It follows that $P_\phi = P_\Gamma$ if and only if $\mathrm{Rk}_\Gamma(\mathcal{E})$ is well defined and equals 1 (this is the content of [Sch98, Proposition 2.1]).

Note that [HP12, § 3.3] introduces, via the Riemann–Roch theorem, a different notion of rank and degree for every coherent $\mathcal{O}_\Gamma$-module, even for those torsion-free ones for which the notion of degree given above is not well defined. In this paper, we use the notion of rank and degree given above [Sch98], not the one in [HP12]. The forthcoming modular description of the fibers of the Hitchin fibration is given in terms of the notions employed in this paper, and the torsion-free sheaves on spectral curves that arise are, by necessity, the ones for which the rank is well defined and it has value one.

Example 2.4.1. Let $nC = \mathcal{C}_0$ be the spectral curve for the characteristic polynomial t^n, i.e. for $a = 0 \in A_n$. See § 2.3 for the notation.

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For $1 \leqslant m \leqslant n$, we consider the curves mC, their structural sheaves $\mathcal{O}_{mC}$ and their ideal sheaves $\mathcal{I}_{mC,nC} \subseteq \mathcal{O}_{nC}$. We have $\chi(\mathcal{O}_{mC}) = -\binom{m}{2}d - m(g-1)$; see (7). We then have: $\mathrm{Rk}_{nC}(\mathcal{O}_{mC}) = m/n$; $\mathrm{Rk}_{nC}(\mathcal{I}_{mC,nC}) = 1-m/n$; $\mathrm{Deg}_{nC}(\mathcal{O}_{mC}) = (m/2)(n-m)d$; $\mathrm{Deg}_{nC}(\mathcal{I}_{mC,nC}) = -(m/2)(n-m)d$. We have $P(\mathcal{O}_{mC}) = P_C^m$, $P(\mathcal{I}_{mC,nC}) = P_C^{m-n}$.

Let E be a stable vector bundle of rank n and degree e on C; let $i: C \to nC$ be the natural map induced by the zero section $C \to C \subseteq V$, followed by the closed embedding $C = (nC)_{\mathrm{red}} \to nC$; we have that $\mathrm{Rk}_{nC}(i_*E) = 1$ and $\mathrm{Deg}_{nC}(i_*E) = e + \binom{n}{2}d$. We have $P(i_*E) = P_{nC} = P_C^n$.

It is easy to show that in the context of torsion-free and $\mathrm{Rk}_\Gamma(-) = r$ coherent sheaves on Γ, the notion of slope in [Sim94, p. 55] and [Sim95, Corollary 6.9] and the notion of slope $\mathrm{Deg}_\Gamma/\mathrm{Rk}_\Gamma$ yield coinciding notions of slope stability. In turn, this coincides with the notion of slope-stable Higgs pair $(p_*\mathcal{E}, \phi)$, with slopes defined by taking $\deg_C/\mathrm{rk}_C$. By working with quotients, instead of with subobjects, the stability condition takes the form (6) below. Define

$$e' := e + \binom{n}{2}d. \tag{5}$$

Remark 2.4.2. As pointed out in [CL16, Remark 4.2], the statement of [Sch98, Theorem 3.1], which characterizes stability, needs to be slightly modified (cf. (6)).

Remark 2.4.3. Let us point out that one has also to correct some minor inaccuracies at the end of the proof of [Sch98, Proposition 2.1, p. 303, from the top, to the end of the proof]: the degrees on the finite maps from the reduced irreducible components of the spectral curve are omitted from the first two displayed equalities; the inequality on the lengths implying that the rank should be 1 is not justified. One remedies this minor inaccuracies by means of the discussion at the beginning of this section involving the role of the characteristic polynomials.

Modular description of the Hitchin fiber $M_{n,a} := h_n^{-1}(a)$, $a \in A_n$. The discussion that follows does not require that one first proves that M_n is irreducible; in particular, it can be used in order to establish this fact, as it has been done in § 2.1. The Hitchin fiber $M_{n,a} := h_n^{-1}(a)$, i.e. the moduli space of stable D-twisted Higgs pairs with rank n and degree e and with characteristic $a \in A_n$, is isomorphic to the moduli space of torsion-free sheaves $\mathcal{E}$ on the spectral curve $\mathcal{C}_a$ with $\mathrm{Rk}_{\mathcal{C}_a}(\mathcal{E}) = 1$ (and hence with associated characteristic polynomial $P_\phi = P_{\mathcal{C}_a}$) and $\mathrm{Deg}_{\mathcal{C}_a}(\mathcal{E}) = e'$, subject to the following stability condition: for every closed subscheme $i_Z : Z \to \mathcal{C}_a$ of pure dimension one, for every torsion-free quotient $\mathcal{O}_Z$-module $i_Z^*E \twoheadrightarrow \mathcal{E}_Z$ with $\mathrm{Rk}_Z(\mathcal{E}_Z) = 1$, we have

$$\frac{\mathrm{Deg}_Z(\mathcal{E}_Z)}{\mathrm{Rk}_C(p_*\mathcal{O}_Z)} + \frac{1}{2}(n - \mathrm{Rk}_C(p_*\mathcal{O}_Z))d > \frac{e'}{n}. \tag{6}$$

The isomorphism is given by the push-forward morphism p_{a*} on coherent sheaves under the finite, flat, degree n, spectral cover morphism $p_a : \mathcal{C}_a \to C_a = C \otimes k(a)$.

Remark 2.4.4. If the spectral curve $\mathcal{C}_a$ is smooth, i.e. for $a \in A_n$ general, then the fiber $M_{n,a}$ is geometrically connected, for, in view of its modular description, it coincides with $\mathrm{Pic}^{e'}(\mathcal{C}_a)$.

Let us record the properties of the norm map that we need.

Fact 2.4.5. Let $p_a : \mathcal{C}_a \to C_a$ be a spectral cover (of degree n) with norm map $N_{p_a} : \mathrm{Pic}^0(\mathcal{C}_a) \to \mathrm{Pic}^0(C_a)$ and pull-back map $p_a^* : \mathrm{Pic}^0(C_a) \to \mathrm{Pic}^0(\mathcal{C}_a)$. For what follows, see [HP12, Corollary 1.3 and § 3].

(1) For every $L \in \mathrm{Pic}(C_a)$, we have $N_{p_a}(p_a^* L) = L^{\otimes n}$; in particular, N_{p_a} is surjective.
(2) Let $\mathcal{E}$ be a torsion-free $\mathcal{O}_{\mathcal{C}_a}$-module of some integral rank $\mathrm{Rk}_{\mathcal{C}_a}(\mathcal{E}) =: r$ and let $\mathcal{L} \in \mathrm{Pic}(\mathcal{C}_a)$; then $\det(p_{a*}(\mathcal{E} \otimes \mathcal{L})) = \det(p_{a*}\mathcal{E}) \otimes N_{p_a}(\mathcal{L})^{\otimes r}$.
(3) If $a \in A_n$ is general, then $\mathrm{Ker}(N_{p_a})$ is a (connected) abelian variety (see § 2.5).

Proposition 2.4.6. *The projective, D-twisted,* GL_n *Hitchin morphism* $h_n : M_n \to A_n$ *is surjective, with geometrically connected fibers, flat of pure relative dimension*

$$d_{h_n} = \binom{n}{2} d + n(g-1) + 1. \tag{7}$$

Let $a \in A_n$*. Then* $\mathrm{Pic}^0(\mathcal{C}_a)$ *acts on the Hitchin fiber* $M_{n,a}$*. If the spectral curve* $\mathcal{C}_a$ *is smooth, then the corresponding Hitchin fiber* $M_{n,a} \cong \mathrm{Pic}^{e'}(\mathcal{C}_a)$ *is smooth, and a* $\mathrm{Pic}^0(\mathcal{C}_a)$*-torsor via tensor product.*

Proof. In view of the modular description of $M_{n,a}$, it is clear that Fact 2.4.5(2) implies that, for every $a \in A_n$, $\mathrm{Pic}^0(\mathcal{C}_a)$ acts on $M_{n,a}$ via tensor product (degree and stability are preserved), and that, when $\mathcal{C}_a$ is smooth, this action turns $M_{n,a}$ into the $\mathrm{Pic}^0(\mathcal{C}_a)$-torsor $\mathrm{Pic}^{e'}(\mathcal{C}_a)$. Since the locus of characteristics in A_n yielding a smooth spectral curve is open and dense in A_n, we conclude that h_n is dominant. Since h_n is projective, it is also surjective. The same line of argument implies that the general fiber of h_n is geometrically connected. On the other hand, since A_n is nonsingular, hence normal, Zariski's main theorem implies that h_n has geometrically connected fibers. In view of [CL16, § 8, Corollary], the morphism $h_n : M_n \to A_n$ is of pure relative dimension equal to the arithmetic genus of the spectral curves, which can be easily shown to be (7). Since M_n and A_n are nonsingular, the pure-relative-dimension morphism h_n is flat. □

Remark 2.4.7 (No line bundles in the nilpotent cone when $(e, n) = 1$). The fiber $M_{n,0}$ over the origin does not contain line bundles. In fact, the spectral curve is of the form nC (given by $t^n = 0$ on the surface $V(D)$), a nonreduced curve with multiple structure of multiplicity n, and with reduced curve C; it follows that every line bundle on it has degree Deg_{nC} a multiple of n; since the required degree is $e' = e + \binom{n}{2} d$ and $(e, n) = 1$, in general, there is no such line bundle (e.g. if n is odd or if d is even). By way of contrast, if the spectral curve $\mathcal{C}_a$ is geometrically integral, then $\mathrm{Pic}^{e'}(\mathcal{C}_a) \subseteq M_{n,a}$ is an integral, Zariski-dense, open subvariety. Finally, if we arrange for $e' = 0$ (in which case, we may not have the coprimality of the pair (e, n)), one sees that, for every $a \in A_n$, the variety $\mathrm{Pic}^0(\mathcal{C}_a)$ is open in $M_{n,a}$; see [Sch98, Corollary 5.2].

Modular description of the Hitchin fiber $\check{h}_n^{-1}(a)$, $a \in \check{A}_n$. The description in question is the same as the modular description given above, except for the added constraint on the determinant $\det(p_{a*}\mathcal{E}) = \epsilon$, where $\epsilon \in \mathrm{Pic}^e(C)$ is the fixed line bundle involved in the definition (1) of $\check{M}_n$.

Definition 2.4.8 (The Prym variety of a spectral cover). Let $a \in A_n$ and set

$$\mathrm{Prym}_a := \mathrm{Ker}\{N_{p_a} : \mathrm{Pic}^0(\mathcal{C}_a) \to \mathrm{Pic}^0(C_a)\}. \tag{8}$$

In general, the Prym variety Prym_a is a disconnected group scheme with finitely many components; see [HP12] for a description of these components at geometric points of A_n. We also call Prym variety the corresponding identity connected component. In a given context, we shall make it clear which Prym variety we are using.

If $a \in A_n$ is general, then Prym_a is geometrically connected (Fact 2.4.5(3)).

PROPOSITION 2.4.9. *The projective, D-twisted,* SL_n *Hitchin morphism* $\check{h}_n : \check{M}_n \to \check{A}_n$ *is surjective, with geometrically connected fibers, flat of pure relative dimension*

$$d_{\check{h}_n} = d_{h_n} - g = \binom{n}{2} d + (n-1)(g-1). \tag{9}$$

Let $a \in \check{A}_n$. *Then* Prym_a *acts on the Hitchin fiber* $\check{M}_{n,a}$. *If the spectral curve* $\mathcal{C}_a$ *is smooth, then* Prym_a *is connected, the corresponding* SL_n *Hitchin fiber* $\check{M}_{n,a}$ *is smooth, and a* Prym_a*-torsor via tensor product.*

Proof. By Proposition 2.4.6, for every $a \in \check{A}_n$, the GL_n Hitchin fiber $M_{n,a} \neq \emptyset$. There is the natural morphism

$$\mathfrak{p}_a := \det \circ p_{a*} : M_{n,a} \longrightarrow \mathrm{Pic}^e(C). \tag{10}$$

In view of the modular description of the SL_n Hitchin fiber $\check{M}_{n,a}$, we have that $\check{M}_{n,a} = \mathfrak{p}_a^{-1}(\epsilon)$.

The morphism $\mathfrak{p}_a$ is equivariant for the $\mathrm{Pic}^0(C)$-actions given by $L \cdot \mathcal{E} := \mathcal{E} \otimes L$ on the domain and by $L \cdot M := M \cdot L^{\otimes n}$ on the target (Fact 2.4.5(2),(1)). It follows that $\mathfrak{p}_a$ is surjective. In particular, for every $a \in \check{A}_n$, $\check{M}_{n,a} \neq \emptyset$, so that $\check{h}_n$ is surjective.

By Zariski's main theorem, in order to check that $\check{h}_n$ has geometrically connected fibers, it is enough to do so at a general point. We do this next.

Since $\check{M}_{n,a} = \mathfrak{p}_a^{-1}(\epsilon)$, Fact 2.4.5(2) implies that $\mathrm{Prym}_a \subseteq \mathrm{Pic}^0(\mathcal{C}_a)$ is the largest subgroup acting on $\check{M}_{n,a}$. More precisely, if $\mathcal{E} \in \check{M}_{n,a}$ and $L \in \mathrm{Pic}^0(\mathcal{C}_a)$, then $\mathcal{E} \otimes \mathcal{L} \in \check{M}_{n,a}$ if and only if $\mathcal{L} \in \mathrm{Prym}_a$.

Let $a \in \check{A}_n$ be a traceless characteristic yielding a nonsingular spectral curve $\mathcal{C}_a$. Since $M_{n,a}$ is a $\mathrm{Pic}^0(\mathcal{C}_a)$-torsor by Proposition 2.4.6, we deduce that $\check{M}_{n,a}$ is s a Prym_a-torsor. For $a \in \check{A}_n$ general, Prym_a is geometrically connected by Fact 2.4.5(3), so that so is the general fiber $\check{M}_{n,a}$, and, as anticipated, $\check{h}_n$ has thus geometrically connected fibers.

Since all fibers of $\check{h}_n$ are now known to be geometrically connected, so is the fiber $\check{M}_{n,a}$ corresponding to a smooth spectral curve. Since such a fiber is a Prym_a-torsor, the Prym variety Prym_a is also geometrically connected.

Finally, since the morphism $\mathfrak{p}_a$ is flat, and the morphism h_n is of pure dimension (7), all the fibers of $\check{h}_n$ are of pure dimension (7) minus g, and hence (9) holds. The flatness of $\check{h}_n$ follows by this and by the smoothness of $\check{M}_n$ and of $\check{A}_n$. □

2.5 Endoscopy loci of the Hitchin SL_n fibration

Let a be a geometric point of A_n^{ell}, so that the spectral curve $\mathcal{C}_a$ is (geometrically) integral. The D-twisted, GL_n Hitchin fiber $M_{n,a}$ is also integral: it is isomorphic to the compactified Jacobian of the integral locally planar spectral curve, parameterizing rank one and degree e' torsion-free coherent sheaves on it. In particular, the regular part $\mathrm{Pic}^{e'}(\mathcal{C}_a) \cong M_{n,a}^{\mathrm{reg}} \subseteq M_{n,a}$ of this fiber is integral, Zariski open and dense in the whole fiber, and it is a $\mathrm{Pic}^0(\mathcal{C}_a)$-torsor.

Let a be a geometric point of $\check{A}_n$. Then the D-twisted, SL_n Hitchin fiber $\check{M}_{n,a} = \mathfrak{p}_a^{-1}(\epsilon)$ (cf. (10)), and it is (geometrically) connected. Since the morphism $\check{h}_n$ is flat and $\check{A}_n$ is nonsingular, every fiber of $\check{h}_n$ is a local complete intersection (l.c.i).

Assume, in addition, that a is a geometric point of $\check{A}_n^{\mathrm{ell}}$. By the $\mathrm{Pic}^0(C)$-equivariance of $\mathfrak{p}_a$, the regular part of $\check{M}_{n,a}$ satisfies $\check{M}_{n,a}^{\mathrm{reg}} = M_{n,a}^{\mathrm{reg}} \cap \check{M}_{n,a}$, and it is Zariski open and dense. Since the fiber $\check{M}_{n,a}$ is a l.c.i., we have that, being smooth on a Zariski-dense open subset, it is also reduced. The regular part $\check{M}_{n,a}^{\mathrm{reg}}$ is made of line bundles $\mathcal{E}$ on the spectral curve with $\mathfrak{p}_a(\mathcal{E}) = \epsilon$. It is clear that $\check{M}_{n,a}^{\mathrm{reg}}$ is then a Prym_a-torsor.

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Fact 2.5.1. Let $a \in \check{A}_n^{\mathrm{ell}}$. The discussion above implies that the number of irreducible components of the pure dimensional and reduced $\check{M}_{n,a}$ coincides with the number of connected components of Prym_a.

For every $a \in A_n$, the group of connected components $\pi_0(\mathrm{Prim}_a)$ is described in [HP12, Theorem 1.1]. The locus $\check{A}_{n,\mathrm{endo}} \subseteq \check{A}_n$ over which Prym_a is disconnected is called the endoscopic locus of the SL_n Hitchin fibration and it is described in [HP12, § 5, especially Lemmas 5.1; 7.1]:

$$\check{A}_{n,\mathrm{endo}} = \bigcup_{\Gamma} \check{A}_{n,\Gamma}, \tag{11}$$

where Γ ranges over the finite set of cyclic subgroups of $\mathrm{Pic}^0(C)[n]$ of prime number order. Each $\check{A}_{n,\Gamma} \subseteq \check{A}_n$ is a geometrically integral subvariety. The codimension of each $\check{A}_{n,\Gamma}$ can be computed in the same way as in the proof of [HP12, Lemma 7.1], whose proof in the case $D = K_C$, remains valid for D: we need the knowledge of $d_{\check{A}_n}$ (78), obtained by the Riemann–Roch theorem, and the formula directly above [HP12, Lemma 5.1]. The resulting value

$$\mathrm{codim}_{\check{A}_n}(\check{A}_{n,\Gamma}) = \tfrac{1}{2}(n-\nu)\{(n+\nu)d + [d - 2(g-1)]\}, \quad (\nu := n/\#(\Gamma)), \tag{12}$$

is strictly positive in view, for example, of our assumption $d > 2(g-1)$.

The subvarieties $\check{A}_{n,\Gamma}^{\mathrm{ell}} := \check{A}_{n,\Gamma} \cap \check{A}_n^{\mathrm{ell}} \subseteq \check{A}_n^{\mathrm{ell}}$ are nonsingular and mutually disjoint [Ngô06, Proposition 10.3]. By construction, the number

$$o(\Gamma) := \#(\pi_0(\mathrm{Prym}_a)) \tag{13}$$

of connected components of Prym_a is independent of $a \in \check{A}_{n,\Gamma}^{\mathrm{ell}}$.

A point $a \in \check{A}_{n,\Gamma}^{\mathrm{ell}}$ if and only if the spectral cover $p_a : \mathcal{C}_a \to C$ has the property that the induced morphism from the normalization of the integral spectral curve $\widetilde{p_a} = \widetilde{\mathcal{C}_a} \to C$ factors through the étale cyclic cover of C associated with Γ (cf. [HP12, Proof of Theorem 5.3]).

The locus

$$\check{A}_{n,\mathrm{endo}}^{\mathrm{ell}} = \coprod_{\Gamma} \check{A}_{n,\Gamma}^{\mathrm{ell}} \tag{14}$$

is the $G = \mathrm{SL}_n$ endoscopic locus introduced by Ngô in [Ngô06, § 10] for D-twisted, G Hitchin fibrations (G reductive). It determines the socle $\mathrm{Socle}(R\check{h}_{n*}\overline{\mathbb{Q}}_\ell) \cap \check{A}_n^{\mathrm{ell}}$ over the elliptic locus; see § 4.9.

2.6 Weak abelian fibrations and δ-regularity

The notion of δ-regular weak abelian fibration has been introduced in [Ngô10] as an encapsulation of some important features of the Hitchin fibration over the elliptic locus: presence of the action of a commutative smooth group scheme with affine stabilizers, polarizability of the associated Tate module, and δ-regularity of the group scheme. See also [Ngô11] for an introduction to this circle of ideas.

In this section, let $g : J \to A$ be a smooth commutative group scheme over an irreducible variety A such that g has geometrically connected fibers.

Chevalley devissage. References for what follows are, for example, [Mil, Theorem 10.25, Propositions 10.24, 10.5 (and its proof), Proposition 10.3] and [Con02].

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Let $\overline{a}$ be a geometric point on A with underlying point a Zariski point $a \in A$. Let $J_{\overline{a}}$ be the fiber of J at $\overline{a}$. There is a canonical short exact sequence of commutative connected group schemes over the residue field of $\overline{a}$:

$$0 \to J_{\overline{a}}^{\mathrm{aff}} \to J_{\overline{a}} \to J_{\overline{a}}^{\mathrm{ab}} \to 0, \tag{15}$$

where $J_{\overline{a}}^{\mathrm{aff}} \subseteq J_{\overline{a}}$ is the maximal connected affine linear subgroup of $J_{\overline{a}}$, and $J_{\overline{a}}^{\mathrm{ab}}$ is an abelian variety. The dimensions of these varieties depend only on the Zariski point $a \in A$, and are denoted by $d_a^{\mathrm{aff}}(J)$ and $d_a^{\mathrm{ab}}(J)$, respectively. Clearly,

$$d_a(J) = d_a^{\mathrm{aff}}(J) + d_a^{\mathrm{ab}}(J). \tag{16}$$

The notion of δ-regularity. The function

$$\delta : A \longrightarrow \mathbb{Z}^{\geqslant 0}, \quad a \mapsto \delta_a := d_a^{\mathrm{aff}} \tag{17}$$

is upper semicontinuous (jumps up on closed subsets); see [SGA3.II, X, Remark 8.7]. We have the disjoint union decomposition

$$A = \coprod_{\delta \geqslant 0} S_\delta, \quad S_\delta = S_\delta(J/A) := \{a \in A \mid \delta_a = \delta\} \tag{18}$$

of A into locally closed subvarieties of A. We call S_δ the δ-locus of J/A.

DEFINITION 2.6.1 (δ-regularity). We say that $g : J \to A$ is δ-regular if

$$\operatorname{codim}_A(S_\delta) \geqslant \delta, \quad \forall \delta \geqslant 0, \tag{19}$$

where one requires the inequality to hold for every irreducible component of S_δ.

The following lemma is an immediate consequence of the upper-semicontinuity of the function δ and of the identity (16).

LEMMA 2.6.2. *A group scheme $g : J \to A$ as above is δ-regular if and only if either of the two following equivalent conditions hold.*

(1) *For every closed irreducible subvariety $Z \subseteq A$, let δ_Z be the minimum value of δ on Z (it is attained at general points of Z, as well as at the generic point of Z); then $\operatorname{codim}_A(Z) \geqslant \delta_Z$.*
(2) *For every point $a \in A$, let $d_a := \dim \overline{\{a\}}$ and let $d_A := \dim(A)$; then*

$$d_a^{\mathrm{ab}}(J) \geqslant d_a(J) - d_A + d_a. \tag{20}$$

The Tate module $T_{\overline{\mathbb{Q}}_\ell}(J)$ and the notion of its polarizability. Let $g : J \to A$ be as above and let $d_g := \dim(J) - \dim(A)$ be the pure relative dimension of g. The Tate module of J is the $\overline{\mathbb{Q}}_\ell$-adic sheaf [Ngô10, § 4.12]

$$T_{\overline{\mathbb{Q}}_\ell}(J) := R^{2d_g-1} g_! \overline{\mathbb{Q}}_\ell(d_g). \tag{21}$$

Its stalk at any geometric point $\overline{a}$ of A is given by the Tate module $T_{\overline{\mathbb{Q}}_\ell}(J_{\overline{a}})$, i.e. the inverse limit, with respect to $i \in \mathbb{N}$, of the ℓ^i-torsion points on $J_{\overline{a}}$, tensored with $\overline{\mathbb{Q}}_\ell$ over $\mathbb{Z}_\ell$. The Chevalley devissage at the stalks yields the natural short exact sequence

$$0 \to T_{\overline{\mathbb{Q}}_\ell}(J_{\overline{a}}^{\mathrm{aff}}) \to T_{\overline{\mathbb{Q}}_\ell}(J_{\overline{a}}) \to T_{\overline{\mathbb{Q}}_\ell}(J_{\overline{a}}^{\mathrm{ab}}) \to 0. \tag{22}$$

The Tate module $T_{\overline{\mathbb{Q}}_\ell}(J)$ is said to be polarizable if it admits a polarization, i.e. an alternating bilinear pairing

$$\psi : T_{\overline{\mathbb{Q}}_\ell}(J) \otimes_{\overline{\mathbb{Q}}_\ell} T_{\overline{\mathbb{Q}}_\ell}(J) \longrightarrow \overline{\mathbb{Q}}_\ell(1), \tag{23}$$

such that, for every geometric point $\overline{a}$ of A, we have that the kernel of $\psi_{\overline{a}}$ is exactly $T_{\overline{\mathbb{Q}}_\ell}(J^{\mathrm{aff}}_{\overline{a}})$. In this case, the pairings $\psi_{\overline{a}}$ descend to nondegenerate, alternating, bilinear parings on the $T_{\overline{\mathbb{Q}}_\ell}(J^{\mathrm{ab}}_{\overline{a}})$.

Note that by general principles (cf. [SGA7.I, VIII, Corollary 4.10]), the alternating bilinear pairings we consider in this paper are automatically trivial on the 'affine' part, and do descend to the 'abelian' part. We do verify this fact along the way to proving the key fact that, in the cases we deal with, they in fact descend to nondegenerate pairings.

Affine stabilizers. Let $h : M \to A$ be a morphism of varieties and let $J \to A$ be a group scheme acting on M/A. We say that the action has affine stabilizers if for every geometric point m of M, we have that the stabilizer subgroup $\mathrm{St_m} \subseteq J_{h(m)}$ is affine.

δ-regular weak abelian fibrations. See [Ngô10, Ngô11]. Let $h : M \to A \leftarrow J : g$ be a pair of morphisms of varieties, where g is as in the beginning of this section (smooth commutative group scheme, with geometrically connected fibers over an irreducible A), h is proper, and J/A acts on M/A. We denote this situation simply by (M, A, J); the context will make it clear which morphisms h, g are being used.

Definition 2.6.3 (Weak abelian fibration). We say that (M, A, J) is a weak abelian fibration if g and h have the same pure relative dimension, the Tate module $T_{\overline{\mathbb{Q}}_\ell}(J)$ is polarizable and the action has affine stabilizers. (*δ-regular weak abelian fibration*) A weak abelian fibration (M, A, J) is said to be δ-regular if $g : J \to A$ is δ-regular as in Definition 2.6.1 (equation (19)), or equivalently as in Lemma 2.6.2 (equation (20)).

Ngô support inequality. The following is a remarkable, and remarkably useful, topological restriction on the dimensions of the supports appearing in the context of weak abelian fibrations. If $a \in A$, then $d_a := \dim \overline{\{a\}}$ is the dimension of the closed subvariety of A with generic point a. For the notion of socle, see § 1. The celebrated Ngô support theorem [Ngô10, Theorem 7.2.1] is a more refined restriction on the geometry of the supports, and it is proved also by using the support inequality.

Theorem 2.6.4 (Nĝo's support inequality [Ngô10, Theorem 7.2.2]). *Let (M, A, J) be a weak abelian fibration with M and A nonsingular and with h projective of pure relative dimension d_h. If $a \in \mathrm{Socle}(Rh_*\overline{\mathbb{Q}}_\ell)$, then*

$$d_h - d_A + d_a \geqslant d_a^{\mathrm{ab}}(J). \tag{24}$$

Given that we are assuming $d_h = d_g$, we may re-formulate (24) as follows via (16):

$$d_a^{\mathrm{aff}}(J) \geqslant \mathrm{codim}(\overline{\{a\}}). \tag{25}$$

3. The GL_n weak abelian fibration

This section is devoted to a detailed study of the δ-regular weak abelian fibration (M_n, A_n, J_n), arising from the action of the Jacobi group scheme J_n/A_n, associated with the family of spectral curves of the GL_n Hitchin fibration M_n/A_n. Section 3.1 introduces the Jacobi group scheme J_n/A_n and its action on M_n/A_n: its fibers are the Jacobians of the spectral curves. Section 3.2

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shows that the stabilizers for this action are affine. I am not aware of an explicit reference in the literature for this result over the whole base A_n; [Ngô10, 4.15.2] deals with a suitable open proper subset of A_n, and for every G reductive. Section 3.3 is devoted to the lengthy proof that the Tate module associated with J_n/A_n is polarizable over the whole base A_n. Again, I am not aware of an explicit reference in the literature for this result over the whole base A_n; the standard reference for this important-for-us technical fact is [Ngô10, § 4.12], which deals with the situation over the elliptic locus $A_n^{\mathrm{ell}} \subseteq A_n$. Following this preparation, § 3.4 contains the main result of this section, namely Theorem 3.4.1, to the effect that (M_n, A_n, J_n) is a weak abelian fibration that is δ-regular over the elliptic locus A_n^{ell}; this affords the support inequality over the whole A_n, and the δ-regularity inequality over the elliptic locus A_n^{ell}. We need some of these explicit details of this GL_n, especially in connection with nonreduced spectral curves, in view of our main Theorem 1.0.2 on the SL_n socle.

3.1 The action of the Jacobi group scheme J_n

For what follows, see [CL16, § 5]. Let $J_n \to A_n$ be the identity connected component of the degree zero component of the relative Picard stack $\mathrm{Pic}_{\mathcal{C}/A_n}$. This is a connected, smooth, commutative group scheme over A_n, whose fiber $J_{n,a}$ over a point $a \in A_n$ is $\mathrm{Pic}^0(\mathcal{C}_a)$; see Fact 2.3.1 for a description of this group. In particular, the structural morphism $g_n : J_n \to A_n$ is of pure relative dimension, call it d_{g_n}, the arithmetic genus of the spectral curves, which coincides with the pure relative dimension d_{h_n} (7) of $h_n : M_n \to A_n$, i.e. we have

$$d_{g_n} = d_{h_n}. \tag{26}$$

The group scheme J_n/A_n acts on the Hitchin fibration M_n/A_n; see Proposition 2.4.9.

3.2 Affine stabilizers for the action of the Jacobi group scheme

PROPOSITION 3.2.1. *The action of J_n/A on M_n/A_n has affine stabilizers.*

Proof. Let a be a geometric point of A_n and let $\mathcal{E} \in M_{n,a}$. Recall that $\mathrm{Rk}_{\mathcal{C}_a}(\mathcal{E}) = 1$ means that, with the notation of § 2.3, if $\mathcal{C}_a = \sum_k m_k \Gamma_k$, with $m_k \geqslant 1$ for every k, then the length of $\mathcal{E}$ at the stalk of the generic point of Γ_k is m_k, for every k. Let $\xi : \widetilde{\mathcal{C}_{a,\mathrm{red}}} = \coprod_k \widetilde{\Gamma_k} \to \mathcal{C}_a$ be the morphism from the normalization of $\mathcal{C}_{a,\mathrm{red}}$ (cf. (2)). Let $0 \to \mathrm{Tors}(\xi^*\mathcal{E}) \to \xi^*\mathcal{E} \to \xi^*\mathcal{E}/\mathrm{Tors}(\xi^*\mathcal{E}) =: \mathscr{E} \to 0$ be the canonical devissage of the torsion of $\xi^*\mathcal{E}$ on the nonsingular projective curve $\coprod_k \widetilde{\Gamma_k}$. Let $\mathcal{L} \in \mathrm{Pic}^0(\mathcal{C}_a)$. Assume that $\mathcal{L}$ stabilizes $\mathcal{E}$. Then $\xi^*\mathcal{L}$ stabilizes every term in the canonical torsion devissage of $\xi^*(\mathcal{E}) \otimes \xi^*\mathcal{L}$. In particular, $\xi^*\mathcal{L}$ stabilizes the vector bundle $\mathscr{E}$, which has rank m_k on each $\widetilde{\Gamma_k}$. By considerations of determinants, we see that $\xi^*\mathcal{L} \in \prod_k \mathrm{Pic}^0(\widetilde{\Gamma_k})[m_k]$, a finite group. The natural morphism $\xi^* : \mathrm{Pic}^0(\mathcal{C}_{a,\mathrm{red}}) \to \mathrm{Pic}^0(\widetilde{\mathcal{C}_a}) = \prod_k \mathrm{Pic}^0(\widetilde{\Gamma_k})$ is surjective, with affine (connected) kernel (cf. [BLR90, § 9]). It follows that the stabilizer of $\mathcal{E}$ is an extension of a finite group by an affine subgroup, so that it is affine. □

3.3 The Tate module of the Jacobi group scheme is polarizable

We refer to § 2.6 for the terminology. Let $g_n : J_n \to A_n$ be the structural morphism for Picard. The Tate module is the $\overline{\mathbb{Q}}_\ell$-adic sheaf (22) $T_{\overline{\mathbb{Q}}_\ell}(J_n) := R^{2d_{h_n}-1} g_{n!} \overline{\mathbb{Q}}_\ell(d_{h_n})$. If a is a geometric point of A_n, then the Chevalley devissage yields the natural short exact sequences

$$0 \to T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{aff}}) \to T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}) \to T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) \to 0. \tag{27}$$

Note that: (i) $\dim_{\overline{\mathbb{Q}}_\ell} T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) = 2 \dim J_{n,a}^{\mathrm{ab}}$; (ii) $\dim_{\overline{\mathbb{Q}}_\ell} T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{aff}}) \leqslant \dim J_{n,a}^{\mathrm{aff}}$, and that the strict inequality can occur: this is due to the fact that the affine part $J_{n,a}^{\mathrm{aff}}$ is an iterated extension of the

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additive and of the multiplicative group $\mathbb{G}_a$ and $\mathbb{G}_m$ [BLR90, § 9], and only the latter contribute to the Tate module.

The goal of this section is to prove the following polarizability result, which has been proved over the elliptic locus A_n^{ell} in [Ngô10], and is stated implicitly over the whole base A_n and then used in [CL16, § 9].

Theorem 3.3.1. *The Tate module $T_{\overline{\mathbb{Q}}_\ell}(J_n)$ on A_n is polarizable.*

Proof. Let $p : \mathcal{C} \to A_n$ be the family of spectral curves: it is proper, flat, with geometrically connected fibers, with nonsingular total space, and with nonsingular general fiber. As in [Ngô10, § 4.12], the pairing is defined by constructing it over the strict henselianization of the local ring of any Zariski point $a \in A_n$, for the construction yields a canonical outcome. We denote these new shrunken families by $p : \mathcal{C} \to A$, $g : J \to A$. For a coherent sheaf F on $\mathcal{C}$, set $\Delta(F) := \det(Rp_* F)$, where we are taking the determinant of cohomology [Del87, Sou92, especially, § 1.4] and the result is a graded line bundle on A. If F is $\mathcal{O}_A$-flat, then the degree of this graded line bundle is the Euler characteristic of F along the fibers $\mathcal{C}_a$. The Weil pairing construction associates with $L, M \in \mathrm{Pic}^0(\mathcal{C}/A)$ the graded line bundle on A given by the formula

$$\langle L, M\rangle_{\mathcal{C}/A} := P(L, M) := \Delta(L \otimes M) \otimes \Delta(\mathcal{O}_A) \otimes \Delta(L)^\vee \otimes \Delta(M)^\vee. \qquad (28)$$

Note that both of the terms defined by (28) make sense for any pair of coherent sheaves on $\mathcal{C}$. However, we shall use $\langle -, -\rangle$ when dealing with line bundles, whereas we shall use $P(-, -)$ also for other coherent sheaves, and hence the two distinct pieces of notation.

Let $L, M \in \mathrm{Pic}^0(\mathcal{C}/A)[\ell^i]$ be ℓ^i-torsion line bundles. The formalism of the determinant of cohomology yields two distinguished isomorphisms $i_L, i_M : \langle L, M\rangle_{\mathcal{C}/A}^{\otimes \ell^i} \longrightarrow \mathcal{O}_S$ whose difference $\epsilon_{L,M}$ is an ℓ^ith root of unity in the ground field and which depends only on the isomorphism classes of L and of M. By taking inverse limits with respect to i, and then by tensoring with $\overline{\mathbb{Q}}_\ell$, we obtain a pairing, let us call it the Tate–Weil pairing

$$TW : T_{\overline{\mathbb{Q}}_\ell}(J) \otimes_{\overline{\mathbb{Q}}_\ell} T_{\overline{\mathbb{Q}}_\ell}(J) \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(\mathbb{G}_m) = \overline{\mathbb{Q}}_\ell(1), \quad \{L_i, M_i\}_{i\in\mathbb{N}} \mapsto \{\epsilon_{L_i,M_i}\}_{i\in\mathbb{N}} \in \mathbb{Z}_\ell(1). \qquad (29)$$

The Weil and the Tate–Weil pairing are compatible with base change.

Let a be a geometric point of A. Consider the diagram (2) of maps of curves, and extract the following morphisms:

$$\xi = \textstyle\coprod_k \xi_k : \coprod_k \widetilde{\Gamma_k} \xrightarrow{\xi_3 = \coprod_k \xi_{3,k}} \coprod_k \Gamma_k \xrightarrow{\xi_4 = \coprod_k \xi_{4,k}} \sum_k m_k \Gamma_k, \qquad (30)$$

$$\xi : \widetilde{\mathcal{C}_{a,\mathrm{red}}} \xrightarrow{\nu} \mathcal{C}_{a,\mathrm{red}} \xrightarrow{\rho} \mathcal{C}_a, \quad \xi = \coprod_k \xi_k : \coprod_k \widetilde{\Gamma_k} \xrightarrow{\nu = \coprod_k \nu_k} \sum_k \Gamma_k \xrightarrow{\rho} \sum_k m_k \Gamma_k. \qquad (31)$$

Claim. *Let $L, M \in J_a = \mathrm{Pic}(\sum_k m_k \Gamma_k)$. Then*

$$\langle L, M\rangle_{\sum_k m_k \Gamma_k} = \bigotimes_k \langle \xi_{4,k}^* L, \xi_{4,k}^* M\rangle_{\Gamma_k}^{\otimes m_k} = \bigotimes_k \langle \xi_k^* L, \xi_k^* M\rangle_{\widetilde{\Gamma_k}}^{\otimes m_k}. \qquad (32)$$

In order to prove this claim, we first list the three short exact sequences below.

The ideal sheaf of $\mathcal{C}_a$ in $\mathcal{O}_{V(D)\otimes k(a)}$ is locally generated by the product $\prod_{k=1}^s \mathfrak{s}_k^{m_k}$ (cf. § 2.3) of powers of sections of the line bundle $\pi^* D$ on the surface $V(D) \otimes k(a)$. Fix any index $1 \leqslant k_o \leqslant s$; fix any sequence $\{\mu_k\}_{k=1}^s$, with $0 \leqslant \mu_k \leqslant m_k$ for every k, with $1 \leqslant \mu_{k_o}$, and with $\sum_k \mu_k \geqslant 2$

(these conditions are simply to ensure that (33) below is meaningful as written). We have the following system of short exact sequences on the curve $\sum_k \mu_k \Gamma_k$ (see [Rei97, Lemma 3.10], for example)

$$0 \longrightarrow \mathcal{O}_{\Gamma_{k_o}}(-\Gamma_{k_o}) \longrightarrow \mathcal{O}_{\sum_k \mu_k \Gamma_k} \longrightarrow \mathcal{O}_{(\mu_{k_o}-1)\Gamma_{k_o}+\sum_{k\neq k_o}\mu_k\Gamma_k} \longrightarrow 0. \tag{33}$$

We have the short exact sequences (4) on the curves Γ_k

$$0 \longrightarrow \mathcal{O}_{\Gamma_k}(-\Gamma_k) \longrightarrow \mathcal{O}_{\Gamma_k} \longrightarrow \mathcal{O}_{\zeta_k} \longrightarrow 0. \tag{34}$$

We have a natural short exact sequence on $\coprod_k \Gamma_k$ arising from the normalization map ξ_3

$$0 \longrightarrow \mathcal{O}_{\coprod_k \Gamma_k} \longrightarrow \mathcal{O}_{\coprod_k \widetilde{\Gamma_k}} \longrightarrow \Sigma \longrightarrow 0, \tag{35}$$

where Σ is supported at finitely many points on $\sum_k \Gamma_k$.

Since Σ is supported at finitely many points, it follows from the definition that, for every pair of line bundles L, M on the curve $\sum_k \Gamma_k$, we have that $P(\Sigma \otimes L, \Sigma \otimes M)$ is canonically isomorphic to the trivially trivialized, trivial line bundle on the spectrum of the residue field of a; see [Ngô10, proof of Lemma 4.12.2]. We call this circumstance the P-triviality property of Σ. The same holds true for $P(\mathcal{O}_{\zeta_k} \otimes L, \mathcal{O}_{\zeta_k} \otimes M)$, i.e. we have the P-triviality property of ζ_k.

By what we have said above, and by using the multiplicativity property of the determinant of cohomology with respect to short exact sequences, and hence of the operation $P(-,-)$, we see that the second equality of the claim (32) follows from the short exact sequence (35) on $\coprod_k \Gamma_k$, by using the P-triviality property of Σ, and the fact that $\xi_k^* = \xi_{3,k}^* \circ \xi_{4,k}^*$; in fact, we get, the identity

$$P(\xi_{3,k}^*\xi_{4,k}^*, \xi_{3,k}^*\xi_{4.k}^*M) = P(\xi_{4,k}^*L, \xi_{4,k}^*M) \otimes P(\Sigma \otimes \xi_{4,k}^*L, \Sigma \otimes \xi_{4,k}^*M) = P(\xi_{4,k}^*L, \xi_{4,k}^*M).$$

(N.B. there is no need for the exponents m_k, for this second equality in (32).)

The first equality of the claim (32), and here the exponents m_k are essential, follows in the same way (by using the P-triviality property for ζ_k) from (33) and (34) by means of a simple descending induction on the multiplicities $\mu_k \leqslant m_k$, based on the following equalities (where we denote line bundles and their restrictions in the same way, and we instead add a subfix to $P(-,-)$)

$$\begin{gathered} P_{\sum_k \mu_k \Gamma_k}(L,M) = P_{(\mu_{k_o}-1)\Gamma_{k_o}+\sum k\neq k_o \mu_k \Gamma_k}(L,M) \otimes P_{\Gamma_{k_o}}(L-\Gamma_{k_o}, M-\Gamma_{k_o}), \\ P_{\Gamma_{k_o}}(L-\Gamma_{k_o}, M-\Gamma_{k_o}) = P_{\Gamma_{k_o}}(L,M) \otimes P_{\zeta_{k_o}}(L,M) = P_{\Gamma_{k_o}}(L,M). \end{gathered}$$

We now use the just-proved claim (32) to verify that the Tate–Weil pairing TW (29) has, at every geometric point a of A_n, kernel given *precisely* by the 'affine part' $T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{aff}})$, so that it descends to a nondegenerate pairing on $T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}})$.

By [BLR90, § 9.3, Corollary 11], we have the canonical short exact sequence

$$0 \longrightarrow \operatorname{Ker} \xi^* \longrightarrow J_{n,a} = \operatorname{Pic}^0\left(\mathcal{C}_a = \sum_k m_k \Gamma_k\right) \longrightarrow \operatorname{Pic}^0(\widetilde{\mathcal{C}_{a,\mathrm{red}}}) = \prod_k \operatorname{Pic}^0(\widetilde{\Gamma_k}) \longrightarrow 0, \tag{36}$$

with quotient an abelian variety and with affine and *connected* $\operatorname{Ker} \xi^*$, an iterated extension of groups of type $\mathbb{G}_a$ and $\mathbb{G}_m$. It follows that the above short exact sequence is the 'abelian-by-affine' Chevalley devissage (§ 2.6) of $J_{n,a}$. By passing to Tate modules, we get the short exact sequence

$$0 \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(\operatorname{Ker} \xi^*) = T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{aff}}) \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}) \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) = \bigoplus_k T_{\overline{\mathbb{Q}}_\ell}(\operatorname{Pic}^0(\widetilde{\Gamma_k})) \longrightarrow 0. \tag{37}$$

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In view of (32), and of the definition of the Tate–Weil pairing via the Weil pairing, we see that the kernel of the Tate–Weil pairing contains $T_{\overline{\mathbb{Q}}_\ell}(\operatorname{Ker}\xi^*)$, so that the Tate–Weil pairing $TW := TW_{\sum_k m_k \Gamma_k}$ descends to a paring $TW^{\rm ab}$ on the abelian part $T_{\overline{\mathbb{Q}}_\ell}(J^{\rm ab}_{n,a})$ where, again in view of (32), it is the direct sum of the Tate–Weil pairing $TW_{\widetilde{\Gamma_k}}$ on the individual nonsingular projective curves $\widetilde{\Gamma_k}$, multiplied by the integer m_k

$$TW^{\rm ab} = \sum_k m_k TW_{\widetilde{\Gamma_k}}. \tag{38}$$

Each $TW_{\widetilde{\Gamma_k}}$ is nondegenerate: in fact, it is the Tate–Weil pairing on the Tate module of the Jacobian of a nonsingular projective curve over an algebraically closed field, which, in turn, can be identified with the cup product on the first étale $\overline{\mathbb{Q}}_\ell$-adic cohomology group of the curve; see [Mil80, ch. V, Remark 2.4(f), and references therein]. It follows that their m_k-weighted direct sum $TW^{\rm ab}$ is nondegenerate as well. □

Remark 3.3.2. [BLR90, § 9.2, Theorem 11] gives a precise structure theorem for the Jacobians of curves which immediately yields the following description of their abelian variety parts. Let a be a geometric point of A_n and let $\mathcal{C}_a = \sum_k m_k \Gamma_k$ be corresponding spectral curve. Then we have natural isomorphisms of abelian varieties

$$\operatorname{Pic}^0(\mathcal{C}_a)^{\rm ab} = \operatorname{Pic}^0(\mathcal{C}_{a,\rm red})^{\rm ab} = \prod_k \operatorname{Pic}^0(\Gamma_k)^{\rm ab} = \prod_k \operatorname{Pic}^0(\widetilde{\Gamma_k}). \tag{39}$$

3.4 δ-regularity of the action of the Jacobi group scheme over the elliptic locus

Theorem 3.4.1. *The triple (M_n, A_n, J_n) is a weak abelian fibration and its restriction over $A_n^{\rm ell}$ is a δ-regular weak abelian fibration. In particular, we have the following.*

(i) *If $a \in \operatorname{Socle}(Rh_{n*}\overline{\mathbb{Q}}_\ell)$, then*

$$d_{h_n} - d_{A_n} + d_a \geqslant d_a^{\rm ab}(J_n) \quad (\textit{Ngô support inequality}). \tag{40}$$

(ii) *If $a \in A_n^{\rm ell}$, then*

$$d_a^{\rm ab}(J_n) \geqslant d_{h_n} - d_{A_n} + d_a \quad (\mathrm{GL}_n\ \delta\textit{-regularity inequality}). \tag{41}$$

Proof. The two morphisms h_n and g_n have the same pure relative dimension (26). The stabilizers of the action are affine by Proposition 3.2.1. The Tate module is polarizable by Theorem 3.3.1. It follows that the triple is indeed a weak abelian fibration. Since h_n is projective and M_n is nonsingular, (40) follows from Ngô support inequality Theorem 2.6.4. The inequality (41) is known as 'Severi's inequality'; see [CL16, Theorem 7.3] for references; see also [FGvS99, the paragraph following Theorem 2 on p. 3]. The δ-regularity assertion (41) then follows from Lemma 2.6.2, equation (20). □

4. The SL$_n$ weak abelian fibration

Section 4 is devoted to proving Theorem 4.8.1, i.e. the SL_n counterpart to Theorem 3.4.1 for GL_n. Section 4.1 introduces the group scheme $\check{J}_n/\check{A}_n$ of identity components of the Prym group scheme, which, in turn, has fibers (8) that become disconnected precisely over the endoscopic locus (11). Section 4.2 establishes the precise relation between the abelian-variety-parts of the fibers of the Jacobi group scheme J_n/A_n, and the ones of the Prym-like group scheme $\check{J}_n/\check{A}_n$; this is a key step in establishing the δ-regularity of $\check{J}_n$ over the elliptic locus. Section 4.3 establishes the expected product structure of M_n, with factors $M_n(0)$ (traceless Higgs bundles) and $H^0(C, D)$ (space of possible traces); this is another key step towards the δ-regularity above. These factorizations are further pursued in § 4.4, where one factors J_n in the same way. Section 4.5 establishes the δ-regularity of $\check{J}_n/\check{A}_n$ over the elliptic locus $\check{A}_n^{\mathrm{ell}}$. Section 4.6 studies in detail the norm morphism associated with arbitrary (not necessarily irreducible, nor reduced) spectral curves. Section 4.7 establishes the key polarizability of the Tate module of $\check{J}_n$ over the whole base $\check{A}_n$ by using: the explicit form (38) of the polarization of the Tate module of J_n; the explicit form (61) of the norm map; a formal reduction of the SL_n polarizability result to the classical fact that, at the level of Tate modules of Jacobians, the maps induced by the pull-back and by the norm are adjoint for the Tate–Weil pairing. Section 4.8 is devoted to binding-up the results of this section by establishing Theorem 4.8.1, i.e. the SL_n counterpart to Theorem 3.4.1 for GL_n, to the effect that $(\check{M}_n, \check{A}_n, \check{J}_n)$ is a weak abelian fibration which is δ-regular over the elliptic locus; this yields the support inequality over the whole base $\check{A}_n$, and the δ-regularity inequality over the elliptic locus $\check{A}_n^{\mathrm{ell}}$. Section 4.9 is devoted to spelling-out the supports for the SL_n Hitchin fibration over the elliptic locus $\check{A}_n^{\mathrm{ell}}$; the results over the elliptic locus in § 4.9, and for every G, are due to Ngô [Ngô10].

4.1 The action of the Prym group scheme $\check{J}_n$

Let $p : \mathcal{C} \to A_n$ be the family of spectral curves as in § 2.3. The norm morphism (3) defines a morphism of group schemes over A_n (cf. [HP12, Corollary 3.12], for example)

$$N_p : J_n \longrightarrow \mathrm{Pic}^0(C) \times A_n, \quad L \mapsto \det(p_* L) \otimes [\det(p_*(\mathcal{O}_{\mathcal{C}}))]^{-1}. \tag{42}$$

The A_n-morphism $p : \mathcal{C} \to A_n$ induces the morphism $p^* : \mathrm{Pic}(C) \times A \to J_n$ of group schemes over A_n. One verifies that $N_p(p^*(-)) = (-)^{\otimes n}$; see Fact 2.4.5(1). In particular, the morphism N_p is surjective. The differential of the composition $N_p \circ p^*$ along the identity section is multiplication by n, so that the morphism N_p is smooth. The kernel $\mathrm{Ker}(N_p)$ of N_p is a closed subgroup scheme that is smooth over A_n. We call it the Prym group scheme. Its fibers are precisely the Prym varieties (8). Then, by [SGA3.I, Exp VI-B, Theorem 3.10], there is the open subgroup scheme over A_n

$$J'_n := (\mathrm{Ker}(N_p))^0 \tag{43}$$

of the kernel, which (set-theoretically) is the union of the identity connected components of the fibers of this kernel group scheme over A_n. Since this whole construction is compatible with arbitrary base change, the fiber $J'_{n,a}$ over $a \in A_n$ is precisely the identity connected component of the kernel of the norm morphism associated with the spectral cover $\mathcal{C}_a \to C_a = C \otimes k(a)$.

We restrict this whole picture to the SL_n Hitchin base $\check{A}_n = A_n(0) \subseteq A_n$ and set

$$\check{J}_n := J'_{n|\check{A}_n}, \tag{44}$$

which we also call the Prym group scheme.

Then $\check{J}_n/\check{A}_n$ is a smooth connected group scheme with connected fibers over $\check{A}_n$ that acts on $M_n(0)/A_n(0)$ (trace zero) preserving $\check{M}_n/\check{A}_n$ (trace zero and fixed determinant ϵ); see Fact 2.4.5(3) and the proof of Proposition 2.4.9. It follows that $\check{J}_n/\check{A}_n$ acts on $\check{M}_n/\check{A}_n$.

According to Proposition 2.4.9, on each fiber $\check{M}_{n,a}$, this action is free on the open part given by those rank one torsion-free sheaves which are locally free. The Hitchin fibers $\check{M}_{n,a}$ corresponding to nonsingular spectral curves are $\check{J}_{n,a}$-torsors via this action.

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4.2 The abelian variety parts

Let a be a geometric point of A_n ($\check{A}_n$, respectively). Recalling the Chevalley devissage § 2.6 for $J_{n,a}$ ($\check{J}_{n,a}$, respectively), we set, by taking dimensions as varieties over the algebraically closed residue field of a,

$$d_a^{\mathrm{ab}}(J_n) := \dim(J_{n,a}^{\mathrm{ab}}), \quad \check{d}_a^{\mathrm{ab}}(\check{J}_n) := \dim(\check{J}_{n,a}^{\mathrm{ab}});$$

these dimensions depend only on the Zariski point underlying a.

Lemma 4.2.1. *For every point $a \in \check{A}_n$, we have*

$$d_a^{\mathrm{ab}}(\check{J}_n) \geqslant d_a^{\mathrm{ab}}(J_n) - g. \tag{45}$$

Proof. Since $J_{n,a}^{\mathrm{aff}}$ is the biggest affine normal connected group subscheme inside $J_{n,a}$, we must have $d_a^{\mathrm{aff}}(\check{J}_n) \leqslant d_a^{\mathrm{aff}}(J_n)$. Since $\dim(J_{n,a}) = \dim(\check{J}_{n,a}) + g$, the conclusion follows. □

In fact, as Proposition 4.2.2 below shows, the inequality of Lemma 4.2.1 is an equality.

Proposition 4.2.2. *For every geometric point a of $\check{A}_n$, we have that:*

$$d_a^{\mathrm{ab}}(\check{J}_n) = d_a^{\mathrm{ab}}(J_n) - g. \tag{46}$$

More precisely, we have

$$\check{J}_{n,a}^{\mathrm{aff}} = J_{n,a}^{\mathrm{aff}} \subseteq J_{n,a}, \tag{47}$$

$$J_{n,a}/\check{J}_{n,a} \cong J_{n,a}^{\mathrm{ab}}/\check{J}_{n,a}^{\mathrm{ab}}, \tag{48}$$

and a natural isogeny

$$J_{n,a}^{\mathrm{ab}}/\check{J}_{n,a}^{\mathrm{ab}} \longrightarrow \mathrm{Pic}^0(C_a). \tag{49}$$

Proof. Recall that we have the surjective norm morphism $N_p : J_{n,a} = \mathrm{Pic}^0(\mathcal{C}_a) \to \mathrm{Pic}^0(C_a)$ and that $\check{J}_{n,a} := (\mathrm{Ker}(N_p))^0$. We thus obtain the natural isogeny $J_{n,a}/\check{J}_{n,a} \to \mathrm{Pic}^0(C_a)$. In particular, $J_{n,a}/\check{J}_{n,a}$ is an abelian variety of dimension g.

In view of the Chevalley devissage construction, we have the commutative diagram of short exact sequences of morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \check{J}_{n,a}^{\mathrm{aff}} & \longrightarrow & \check{J}_{n,a} & \longrightarrow & \check{J}_{n,a}^{\mathrm{ab}} & \longrightarrow & 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w & & \\ 0 & \longrightarrow & J_{n,a}^{\mathrm{aff}} & \longrightarrow & J_{n,a} & \longrightarrow & J_{n,a}^{\mathrm{ab}} & \longrightarrow & 0, \end{array} \tag{50}$$

where: v is the natural inclusion; u, also an inclusion, arises from the fact that in the Chevalley devissage, $J_{n,a}^{\mathrm{aff}}$ is the biggest connected affine subgroup of $J_{n,a}$, so that it contains all other connected affine subgroups of $J_{n,a}$, so that it contains $\check{J}_{n,a}^{\mathrm{aff}}$; w is the natural map induced by the commutativity of the left-hand square.

The snake lemma yields a natural exact sequence:

$$0 \to \mathrm{Ker}\, u \to \mathrm{Ker}\, v \to \mathrm{Ker}\, w \to \mathrm{Coker}\, u \to \mathrm{Coker}\, v \to \mathrm{Coker}\, w \to 0,$$

which, in view of the fact that u, v are injective, reduces to

$$0 \to \mathrm{Coker}\, u/\mathrm{Ker}\, w \to J_{n,a}/\check{J}_{n,a} \to J_{n,a}^{\mathrm{ab}}/(\check{J}_{n,a}^{\mathrm{ab}}/\mathrm{Ker}\, w) \to 0.$$

Since $\mathrm{Ker}\, w$ sits inside the abelian variety $\check{J}^{\mathrm{ab}}$ and inside the affine $\mathrm{Coker}\, u$, it is a finite group.

Since $\mathrm{Coker}\, u/\mathrm{Ker}\, w$ is affine, connected, and sits inside the abelian variety $J_{n,a}/\check{J}_{n,a}$, it is trivial. It follows that $\mathrm{Coker}\, u = \mathrm{Ker}\, w$, and since $\mathrm{Coker}\, u$ is connected, so is the finite $\mathrm{Ker}\, w$ which is thus trivial. In particular, $\mathrm{Coker}\, u$ is also trivial and $\check{J}_{n,a}^{\mathrm{aff}} = J_{n,a}^{\mathrm{aff}}$.

It follows that $J_{n,a}/\check{J}_{n,a} = J_{n,a}^{\mathrm{ab}}/\check{J}_{n,a}^{\mathrm{ab}}$, and we are done. □

4.3 Product structures

LEMMA 4.3.1. *There is the following cartesian diagram with q, q' isomorphisms.*

$$\begin{array}{ccc} H^0(C,D)\times M_n(0) & \xrightarrow[\sim]{q'} & M_n \\ {\scriptstyle \mathrm{Id}\times h_n(0)}\downarrow & & \downarrow{\scriptstyle h_n} \\ H^0(C,D)\times A_n(0) & \xrightarrow[\sim]{q} & A_n \end{array} \tag{51}$$

Proof. The map q' is defined by the assignment $(\sigma,(E,\phi))\mapsto(E,\phi+\sigma\mathrm{Id}_E)$. Since ϕ preserves a subsheaf of E if and only if $\phi+\sigma\mathrm{Id}_E$ does the same, we have that q' preserves stability. The inverse assignment to q' is $(E,\phi)\mapsto(\mathrm{tr}(\phi)/n,(E,\phi-(\mathrm{tr}(\phi)/n)\mathrm{Id}_E))$.

Let $p(M)(t)=\det(t\mathrm{Id}-M)=\sum_{i=0}^{n}(-1)^i m_i t^{n-i}$ be the characteristic polynomial of an $n\times n$ matrix M. Let s be a scalar. Then a simple calculation shows that

$$p(M+s\mathrm{Id})(t)=\sum_{i=0}^{n}(-1)^i\left[m_i+\sum_{j=1}^{i-1}\binom{n-i+j}{j}m_{i-j}s^j+\binom{n}{i}s^i\right]t^{n-i}, \tag{52}$$

where we have broken up the summation in square bracket to emphasize that the coefficients of t^{n-i} is linear in m_i, and to identify the coefficient of s^i.

The shape of q is dictated by the desire to have (51) commutative and by the relation (52) between the characteristic polynomial of ϕ and the one of $\phi+\sigma\mathrm{Id}_E$. We thus define q by the assignment (N.B. there is no u_1, so $j\neq i-1$, hence the upper bound $j=i-2$ in the summation below)

$$(\sigma,u_2,\ldots u_n)\longmapsto\left(n\sigma,\left\{u_i+\sum_{j=1}^{i-2}\binom{n-i+j}{j}\sigma^j u_{i-j}+\binom{n}{i}\sigma^i\right\}_{i=2}^{n}\right). \tag{53}$$

For example, $q:(\sigma,u_2,u_3)\mapsto(3\sigma,u_2+3\sigma^2,u_3+u_2\sigma+\sigma^3)$. A simple recursion, based on the fact that u_i appears linearly in the component labelled by i, shows that the assignment above can be inverted and that q is an isomorphism.

It is immediate to verify that the square diagram is commutative. Since the morphisms q and q' are isomorphisms, the diagram is cartesian. □

LEMMA 4.3.2. *There is a natural commutative diagram of proper morphisms with cartesian square*

$$\begin{array}{ccccc} \mathrm{Pic}^0(C)\times\check{M}_n & & \xrightarrow{\quad r \quad} & & M_n(0) \\ & \searrow^{\check{h}_n\circ\mathrm{pr}_2} & & \swarrow^{h_n(0)} & \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \check{A}_n=A(0) & & \downarrow{\scriptstyle \det} \\ \mathrm{Pic}^0(C) & & \xrightarrow{\quad r' \quad} & & \mathrm{Pic}^e(C) \end{array} \tag{54}$$

with r and r' proper Galois étale covers with Galois group the finite subgroup $\mathrm{Pic}^0[n]\subseteq\mathrm{Pic}^0(C)$ of line bundles of order n.

Proof. The map r is defined by the assignment $(L,(E,\phi))\mapsto(E\otimes L,\phi\otimes\mathrm{Id}_L)$. Since $\check{M}_n$ is the closure of the loci of stable Higgs pairs with stable underlying vector bundle, it is clear that, as indicated in (54), r maps into the closure $M_n(0)$ of the loci of stable Higgs pairs with stable underlying vector bundle.

The map r' is defined by the assignment $(L \mapsto \epsilon \otimes L^{\otimes n})$ (rem: $\epsilon \in \mathrm{Pic}^e(C)$ is the fixed line bundle used to define $\check{M}_n$). The map r' is finite, étale and Galois, with Galois group the subgroup $\mathrm{Pic}^0(C)[n] \subseteq \mathrm{Pic}(C)$ of n-torsion points.

By construction, (54) is commutative. We need to show that the square is cartesian.

Let F be the fiber product of r' and det. Since r' is étale and $M_n(0)$ is nonsingular, F is nonsingular. Since, by virtue of Lemma 2.1.1, det is smooth with integral fibers, then so is the natural projection $F \to \mathrm{Pic}^0(C)$, and F is integral. By the universal property of fibre products, we have a natural map $u : \mathrm{Pic}^0(C) \times \check{M}_n \to F$ making the evident diagram commutative. This map is bijective on closed points, where the inverse is given by $(L, (E, \phi)) \mapsto (L, (E \otimes L^{-1}, \phi \otimes \mathrm{Id}_{L^{-1}}))$. Since the domain and range of u are nonsingular and u is bijective, we conclude that u is an isomorphism: factor $u = f \circ j$, with j an open immersion and f finite and birational, so that f is necessarily an isomorphism, and j is bijective, hence an isomorphism as well. $\square$

4.4 Product structures, re-mixed

In analogy with Lemma 4.3.1, and keeping in mind the construction of spectral curves § 2.3 as the universal divisor inside of $V(D) \times A_n$, we have the cartesian square diagram with q, q'' isomorphisms

$$\begin{array}{ccc} H^0(C,D) \times A_n(0) \times V(D) & \xrightarrow[\sim]{q''} & A_n \times V(D) \\ \downarrow {\scriptstyle \mathrm{pr}_1 \times \mathrm{pr}_2} & & \downarrow {\scriptstyle \mathrm{pr}_1} \\ H^0(C,D) \times A_n(0) & \xrightarrow[\sim]{q} & A_n \end{array} \qquad (\sigma, u_\bullet, (x,v)) \mapsto (q(\sigma, u_\bullet), (x, v+\sigma)) \tag{55}$$

where $(\sigma, u_\bullet = (u_2, \ldots, u_n)) \in H^0(C,D) \times A(0)$ and $(x, v) \in V(D)_x$ is the line fiber of $V(D)$ over a point $x \in C$. For every fixed $(\sigma, u_\bullet)$, the resulting morphism $q'' : V(D) \xrightarrow{\sim} V(D)$ is, fiber-by-fiber, the translation in the line direction by the amount σ (linear change of coordinates $t \mapsto t + \sigma$) (cf. [HP12, Remark 2.5]).

Consider the spectral curve family $\mathcal{C} \subseteq A_n \times V(D)$ and the pre-image $\mathcal{C}(0)$ of $A_n(0)$. Then, by restricting q'' to $\mathcal{C}$, we obtain a cartesian square diagram

$$\begin{array}{ccc} H^0(C,D) \times \mathcal{C}(0) & \xrightarrow[\sim]{q'''} & \mathcal{C} \\ \downarrow {\scriptstyle \mathrm{Id} \times p(0)} & & \downarrow {\scriptstyle p} \\ H^0(C,D) \times A(0) & \xrightarrow[\sim]{q} & A \end{array} \tag{56}$$

with q, q''' isomorphisms. For every fixed $(\sigma, u_\bullet) \in H^0(C,D) \times A_n(0)$, we have the spectral curve $(\mathrm{Id} \times p(0))^{-1}(\sigma, u_\bullet) = p(0)^{-1}(u_\bullet) = \mathcal{C}_{0,u_\bullet}$. The morphism q'' maps $\mathcal{C}_{0,u_\bullet}$ isomorphically onto $\mathcal{C}_{q(\sigma,u_\bullet)}$, via the fiber-by-fiber translation by the amount σ.

By recalling that $J_n(0) = J_{n|A_n(0)}$, and by setting $q^{iv} := ((q''')^{-1})^*$, we obtain a cartesian square diagram with q, q^{iv} isomorphisms.

$$\begin{array}{ccc} H^0(C,D) \times J_n(0) & \xrightarrow[\sim]{q^{iv}} & J_n \\ \downarrow {\scriptstyle \mathrm{Id} \times g_n(0)} & & \downarrow {\scriptstyle g_n} \\ H^0(C,D) \times A_n(0) & \xrightarrow[\sim]{q} & A_n \end{array} \tag{57}$$

4.5 δ-regularity of Prym over the elliptic locus

Recall that the elliptic locus $A_n^{\mathrm{ell}} \subseteq A_n$ is the locus of characteristics $a \in A_n$ yielding geometrically integral spectral curves $\mathcal{C}_a$. We denote by $A_n^{\mathrm{ell}}(0)$ and by $\check{A}_n^{\mathrm{ell}}$ the restriction of the elliptic locus to $A_n(0) = \check{A}_n$. Recall Definition 2.6.1 (δ-regularity).

PROPOSITION 4.5.1. *The group scheme $J_n(0)/A_n(0)$ is δ-regular over $A_n^{\mathrm{ell}}(0)$. The group scheme $\check{J}_n/\check{A}_n$ is δ-regular over $\check{A}_n^{\mathrm{ell}}$, i.e. if $a \in \check{A}_n^{\mathrm{ell}}$, then*

$$d_a^{\mathrm{ab}}(\check{J}_n) \geqslant d_{\check{h}_n} - d_{\check{A}_n} + d_a \quad (\mathrm{SL}_n \ \delta\text{-regularity inequality}). \tag{58}$$

Proof. Consider the locally closed 'strata' with invariant $d_a^{\mathrm{aff}}(-) = \delta$:

$$S_\delta := S_\delta(J_n/A_n) \subseteq A_n, \quad S_\delta(0) := S_\delta(J_n(0)/A_n(0)) \subseteq A_n(0) \quad \check{S}_\delta := S_\delta(\check{J}_n/\check{A}_n) \subseteq \check{A}_n.$$

By Proposition 4.2.2(47), we have that $\check{S}_\delta = S_\delta \cap A_n(0) = S_\delta(0)$. It follows that the two conclusions of the proposition are equivalent to each other, and that it is enough to prove the codimension assertion for $S_\delta(0)$.

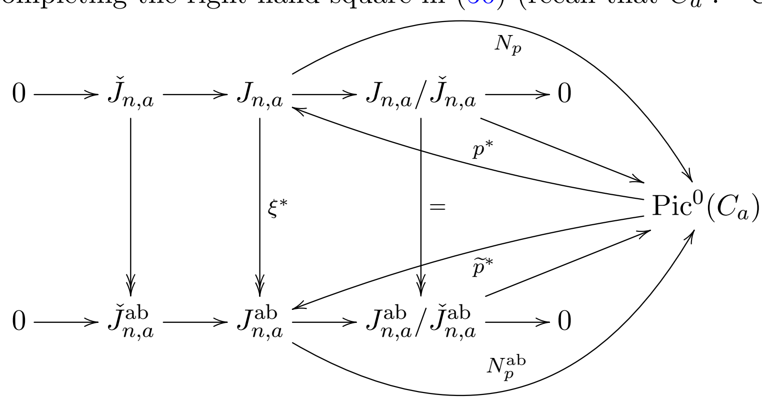
By Lemma 2.6.2, since we already know that $\mathrm{codim}_{A_n}(S_\delta) \geqslant \delta$, for every $\delta \geqslant 0$ (see Lemma 2.6.2(20) and the Severi inequality in the proof of Theorem 3.4.1), we need to make sure that intersecting with $A_n(0)$ does not spoil codimensions. This follows from (57), for it implies that

$$q^{-1}(S_\delta) = H^0(C, D) \times S_\delta(0), \tag{59}$$

so that the codimensions of S_δ in $A_n = H^0(C, D) \times A_n(0)$, and of $S_\delta(0)$ in $A_n(0)$, coincide. $\square$

4.6 The norm morphism N_p^{ab}

Fix a geometric point a of $\check{A}_n$. Recall the diagram (2) of finite morphisms of curves and let us focus on $\xi, p, \widetilde{p}$. We have the surjection (36) $\xi^* : J_{n,a} = \mathrm{Pic}^0(\mathcal{C}_a =: \sum_k m_k \Gamma_k) \to \mathrm{Pic}^0(\widetilde{\mathcal{C}_{a,\mathrm{red}}} = \coprod_k \widetilde{\Gamma_k})$. Keeping in mind the Chevalley devissage, we have the following commutative diagram of short exact sequences completing the right-hand square in (50) (recall that $C_a := C \otimes k(a)$)



(60)

where N_p^{ab} is the arrow induced by N_p, in view of the fact that, since N_p has target an abelian variety, it must be trivial when restricted to the connected and affine $\mathrm{Ker}\,\xi^* = J_{n,a}^{\mathrm{aff}} \subseteq J_{n,a}$.

The arrow N_p^{ab} is *not* the norm $N_{\widetilde{p}}$ associated with the morphism $\widetilde{p}$. In fact, we have the following lemma.

LEMMA 4.6.1. *For every $L \in J_{n,a}$, we have*

$$N_p^{\mathrm{ab}}(\xi^* L) = \bigotimes_k N_{\widetilde{p}_k}(\xi_k^* L)^{\otimes m_k}, \tag{61}$$

$$N_{\widetilde{p}}(\xi^* L) = \bigotimes_k N_{\widetilde{p}_k}(\xi_k^* L). \tag{62}$$

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Proof. Again, recall diagram (2). We have the following chain of identities:

$$N_p^{\mathrm{ab}}(\xi^* L) = N_p(L) = \bigotimes_k N_{p'_k}(\xi^*_{1,k} L) = \bigotimes_k N_{p''_k}(\xi^*_{2,k}\xi^*_{1,k} L)^{\otimes m_k} = \bigotimes_k N_{\widetilde{p}_k}(\xi^* L)^{\otimes m_k},$$

where: the first identity is by the definition of N_p^{ab}, for N_p has descended via the surjective $\xi^* : J_{n,a} \to J_{n,a}^{\mathrm{ab}}$, which has $\mathrm{Ker}\, \xi^* = J_{n,a}^{\mathrm{aff}}$; the second identity follows from [HP12, Lemma 3.5], applied to the morphisms $\xi_{1,k}$, by keeping in mind that the norm from a disjoint union is the tensor product of the norms from the individual connected components; the third identity follows from [HP12, Lemma 3.6], applied to the morphisms $\xi_{2,k}$; the fourth identity follows from [HP12, Lemma 3.4], applied to the morphisms $\xi_{3,k}$. This proves (61).

The identity (62) can be proved in the same way (without recourse to [HP12, Lemma 3.6]). □

4.7 The Tate module of Prym is polarizable

LEMMA 4.7.1. *Let a be any geometric point of $\check{A}_n$. Let $\widetilde{p}_a$, etc. be the corresponding morphisms in (2). Then we have:*

(1) *$T_{\overline{\mathbb{Q}}_\ell}(\widetilde{p}_a^*)$ and $T_{\overline{\mathbb{Q}}_\ell}(N_{p_a}^{\mathrm{ab}})$ are adjoint with respect to the bilinear forms TW_a^{ab} and TW_{C_a};*

(2) *$\mathrm{Ker}\,(T_{\overline{\mathbb{Q}}_\ell}(N_{p_a}^{\mathrm{ab}})) = T_{\overline{\mathbb{Q}}_\ell}(\check{J}_a^{\mathrm{ab}})$;*

(3) *$N_{p_a}^{\mathrm{ab}} \circ \widetilde{p}_a^* = n\, \mathrm{Id}_{\mathrm{Pic}^0(C_a)}$.*

Proof. Recall that we have the spectral cover $\mathcal{C}_a = \sum_k m_k \Gamma_k \to C_a = C \otimes k(a)$. We start with part (1). For every $\widetilde{\gamma} = \sum_k \widetilde{\gamma}_k \in T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) = \bigoplus_k T_{\overline{\mathbb{Q}}_\ell}(\mathrm{Pic}^0(\widetilde{\Gamma}_k))$, and for every $c \in T_{\overline{\mathbb{Q}}_\ell}(\mathrm{Pic}^0(C_a))$, we have that

$$\begin{aligned} TW^{\mathrm{ab}}(\widetilde{\gamma}, T_{\overline{\mathbb{Q}}_\ell}(\widetilde{p}^*)(c)) &= TW^{\mathrm{ab}}\Big(\sum_k \widetilde{\gamma}_k, \sum_k T_{\overline{\mathbb{Q}}_\ell}(\widetilde{p}_k^*)(c)\Big) = \sum_k m_k\, TW_{\widetilde{\Gamma}_k}(\widetilde{\gamma}_k, T_{\overline{\mathbb{Q}}_\ell}(\widetilde{p}_k^*)(c)) \\ &= \sum_k m_k\, TW_C(T_{\overline{\mathbb{Q}}_\ell}(N_{\widetilde{p}_k})(\widetilde{\gamma}_k), c) = TW_C\Big(\sum_k m_k\, T_{\overline{\mathbb{Q}}_\ell}(N_{\widetilde{p}_k})(\widetilde{\gamma}_k), c\Big) \\ &= TW_C(T_{\overline{\mathbb{Q}}_\ell}(N_p^{\mathrm{ab}})(\widetilde{\gamma}), c), \end{aligned}$$

where: the first identity follows simply by consideration of components; the second identity follows from the fact that TW^{ab} is obtained from TW, which is the direct sum of the individual $TW_{\widetilde{p}_k}$, weighted by m_k (see the end of the proof of Proposition 3.3.1); the third identity is the classical adjunction relation (cf. [Mum70, p. 186, equation I] and [LB92, Corollary 11.4.2, especially p. 331, equation (2)]) between norm and pull-back for the morphism $\widetilde{p}_k : \widetilde{\Gamma}_k \to C_a$; the last equality is obtained by applying the functor $T_{\overline{\mathbb{Q}}_\ell}$ to the identity (61), and part (1) is proved.

We prove part (2). The lower line in (60) yields, in view of the isogeny (49), the short exact sequence

$$0 \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(\check{J}_{n,a}^{\mathrm{ab}}) \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}/\check{J}_{n,a}^{\mathrm{ab}}) \cong T_{\overline{\mathbb{Q}}_\ell}(\mathrm{Pic}^0(C_a)) \longrightarrow 0$$

so that the resulting arrow $T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) \to T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}/\check{J}_{n,a}^{\mathrm{ab}})$ gets identified with

$$T_{\overline{\mathbb{Q}}_\ell}(N_p^{\mathrm{ab}}) : T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) \longrightarrow T_{\overline{\mathbb{Q}}_\ell}(\mathrm{Pic}^0(C_a)).$$

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We prove part (3). Recall the standard identity $N_{\widetilde{p}_k} \circ \widetilde{p}_k^* = n_k \mathrm{Id}$, and that $n = \sum_k n_k m_k$. Then part (3) follows from Lemma 4.6.1: for every $L \in \mathrm{Pic}^0(C_a)$, we have

$$N_p^{\mathrm{ab}}(\widetilde{p}^* L) = N_p^{\mathrm{ab}}(\xi^* p^* L) = \bigotimes_k N_{\widetilde{p}_k}(\xi_k^* p_k^* L)^{\otimes m_k}$$
$$= \bigotimes_k N_{\widetilde{p}_k}(\widetilde{p}_k^* L)^{\otimes m_k} = \bigotimes_k L^{\otimes n_k m_k} = L^{\otimes n}. \qquad \square$$

THEOREM 4.7.2 (Polarizability of the Tate module of Prym). *The restriction*

$$T\check{W} : T_{\overline{\mathbb{Q}}_\ell}(\check{J}_n) \otimes T_{\overline{\mathbb{Q}}_\ell}(\check{J}_n) \to \overline{\mathbb{Q}}_\ell(1)$$

of the Tate–Weil pairing $TW : T_{\overline{\mathbb{Q}}_\ell}(J_n) \otimes T_{\overline{\mathbb{Q}}_\ell}(J_n) \to \overline{\mathbb{Q}}_\ell(1)$ *is a polarization of the Tate module* $T_{\overline{\mathbb{Q}}_\ell}(\check{J}_n)$ *on* $\check{A}_n$.

Proof. We fix an arbitrary geometric point a of $\check{A}_n$. By Proposition 4.2.2, we have that $\check{J}_{n,a}^{\mathrm{aff}} = J_{n,a}^{\mathrm{aff}}$. We have already verified that TW is trivial on the 'affine part' $J_{n,a}^{\mathrm{aff}} = \check{J}_{n,a}^{\mathrm{aff}}$ (see the proof of Proposition 3.3.1 and (47)). It follows that TW is trivial on $T_{\overline{\mathbb{Q}}_\ell}(\check{J}_{n,a}^{\mathrm{aff}})$. We need to show that the descended nondegenerate TW^{ab} (cf. Theorem 3.3.1) on $T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}})$ stays nondegenerate on $T_{\overline{\mathbb{Q}}_\ell}(\check{J}_{n,a}^{\mathrm{ab}})$.

By Lemma 4.7.1(3), we have that

$$T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) = \mathrm{Ker}\,(T_{\overline{\mathbb{Q}}_\ell}(N_p^{\mathrm{ab}})) \oplus \mathrm{Im}\,(\widetilde{p}^*).$$

By Lemma 4.7.1(1), the direct sum decomposition is orthogonal with respect to TW^{ab}.

By Lemma 4.7.1(2), we may re-write the orthogonal direct sum decomposition above as

$$T_{\overline{\mathbb{Q}}_\ell}(J_{n,a}^{\mathrm{ab}}) = T_{\overline{\mathbb{Q}}_\ell}(\check{J}_{n,a}^{\mathrm{ab}}) \overset{\perp_{TW^{\mathrm{ab}}}}{\bigoplus} \mathrm{Im}\,(\widetilde{p}^*),$$

so that the nondegenerate form TW^{ab} restricts to a nondegenerate form on $T_{\overline{\mathbb{Q}}_\ell}(\check{J}_{n,a}^{\mathrm{ab}})$. $\square$

4.8 Recap for the SL_n weak abelian fibration

Theorem 3.4.1 tells us that in the D-twisted, GL_n case, the triple (M_n, A_n, J_n) is a weak abelian fibration that is δ-regular over the elliptic locus.

Proposition 4.5.1 implies that the analogous conclusion holds for $(M_n(0), A_n(0), J_n(0))$. In fact, the polarizability of the Tate module is automatic when restricting from A_n to $A_n(0)$: because $J_n(0) = J_{n|A_n(0)}$, and the Tate module is the restriction of the Tate module. Similarly, the stabilizers are the same and they are thus affine. Even though the Chevalley devissages are un-effected when passing from A_n to $A_n(0)$, it is not *a priori* evident that the δ-regularity should be preserved (intersecting may spoil codimensions), and this is precisely what Proposition 4.5.1 ensures.

The D-twisted SL_n case, i.e. $(\check{M}_n, \check{A}_n, \check{J}_n)$, is slightly trickier because, in addition to the discussion in the previous paragraph, the polarizability Theorem 4.7.2 for the Tate module $T_{\overline{\mathbb{Q}}_\ell}(\check{J}_n)$ did not follow immediately from the GL_n analogous Theorem 3.3.1.

We record for future use the following result.

THEOREM 4.8.1. *The triple* $(\check{M}_n, \check{A}_n, \check{J}_n)$ *is a weak abelian fibration which is* δ*-regular over* $\check{A}_n^{\mathrm{ell}}$. *In particular, we have the following.*

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(i) *If* $a \in \mathrm{Socle}(R\check{h}_{n*}\overline{\mathbb{Q}}_\ell)$, *then*

$$d_{\check{h}_n} - d_{\check{A}_n} + d_{\check{a}} \geqslant d^{\mathrm{ab}}_{\check{a}}(\check{J}_n) \quad (\textit{Ngô support inequality}). \tag{63}$$

(ii) *If* $a \in \check{A}^{\mathrm{ell}}_n$, *then*

$$d^{\mathrm{ab}}_{\check{a}}(\check{J}_n) \geqslant d_{\check{h}_n} - d_{\check{A}_n} + d_{\check{a}} \quad (\delta\textit{-regularity inequality}). \tag{64}$$

Proof. The projective morphism $\check{h}_n : \check{M}_n \to \check{A}_n$ is of pure relative dimension $d_{\check{h}_n} = d_{h_n} - g$ (Proposition 2.4.9). By (26), the pure relative dimension $d_{g_n} = d_{h_n}$. By the very construction § 4.1 of $\check{J}_n$, the pure relative dimension of $\check{g}_n : \check{J}_n \to \check{A}_n$ is $d_{\check{g}_n} = d_{g_n} - g$. It follows that $d_{\check{h}_n} = d_{\check{g}_n}$. The stabilizers of the $\check{J}_n$-action are affine because they are closed subgroups of the stabilizers of the J_n-action, which are affine by virtue of Proposition 3.2.1. The Tate module $T_{\overline{\mathbb{Q}}_\ell}(\check{J}_n)$ is polarizable by virtue of Theorem 4.7.2. We have thus verified that the triple is a weak abelian fibration. In particular, Ngô support inequality 2.6.4 implies (63). The δ-regularity assertion is contained in Proposition 4.5.1. □

4.9 Endoscopy and the SL_n socle over the elliptic locus

We employ the notation and results in § 2.5, especially Fact 2.5.1.

According to [Ngô10, Proposition 6.5.1], we have

$$(R^{2\check{h}_n}\check{h}_{n*}\overline{\mathbb{Q}}_\ell)_{|\check{A}^{\mathrm{ell}}_n} \cong \overline{\mathbb{Q}}_{\ell\,\check{A}^{\mathrm{ell}}_n} \bigoplus \bigoplus_{\Gamma} \overline{\mathbb{Q}}_{\ell\,\check{A}^{\mathrm{ell}}_{n,\Gamma}}^{\oplus o_\Gamma - 1} \quad (\Gamma,\, o_\Gamma \text{ as in (13)}). \tag{65}$$

In view of Theorem 4.8.1, the triple $(\check{M}_n, \check{A}_n, \check{J}_n)$ is a weak abelian fibration that is δ-regular over $\check{A}^{\mathrm{ell}}_n$, so that we may use Ngô support theorem [Ngô10, Theorem 7.2.1], to the effect that the supports over the elliptic locus must also be the supports appearing in (65), and conclude that

$$\mathrm{Socle}(R\check{h}_{n*}\overline{\mathbb{Q}}_\ell) \cap \check{A}^{\mathrm{ell}}_n = \{\eta_{\check{A}_n}\} \coprod \coprod_{\Gamma} \{\eta_{\check{A}_{n,\Gamma}}\}. \tag{66}$$

5. Multi-variable weak abelian fibrations

While the SL_n support inequality is used in the proof of our main Theorem 1.0.2 on the SL_n socle, the SL_n δ-regularity inequality is of no use in that respect. Section 5 is devoted to establish the δ-regularity-type inequality that we need instead, i.e. (76). To this end, § 5.1 introduces the multi-variable GL_n weak abelian fibration $(M_{n_\bullet}, A_{n_\bullet}, J_{n_\bullet})$. Section 5.3 introduces its $m_\bullet$-weighted-traceless counterpart $(M_{n_\bullet m_\bullet}(0), A_{n_\bullet m_\bullet}(0), J_{n_\bullet}(0))$, and establishes a series of product-decomposition-formulae of the form $H^0(C,D) \times (-)_{n_\bullet m_\bullet}(0) \cong (-)_{n_\bullet}$. This construction yields the group scheme $J_{n_\bullet m_\bullet}(0)/A_{n_\bullet m_\bullet}(0)$ with the useful δ-regularity-type inequality that we need. Extracting it, as it is done in § 5.4, is not *a priori* completely evident: one has trivially a δ-regularity-type inequality for the multi-variable Jacobi groups scheme $J_{n_\bullet m_\bullet}/A_{n_\bullet m_\bullet}$, which takes the form of an inequality for codimensions of δ-loci in $A_{n_\bullet m_\bullet}$; however, one needs instead to control the codimensions of the δ-loci *after* restriction to the linear subspace $A_{n_\bullet m_\bullet}(0)$, which is not meeting the δ-loci transversally.

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5.1 The weak abelian fibration $(M_{n_\bullet}, A_{n_\bullet}, J_{n_\bullet})$

Let $n_\bullet = (n_1, \ldots, n_s)$ be a finite sequence of positive integers. Define

$$(M_{n_\bullet}, A_{n_\bullet}, J_{n_\bullet}) := \left(\prod_k M_{n_k}, \prod_k A_{n_k}, \prod_k J_{n_k}\right) \tag{67}$$

$$A^{\mathrm{ell}}_{n_\bullet} := \prod_k A^{\mathrm{ell}}_{n_k}. \tag{68}$$

A geometric point of $A^{\mathrm{ell}}_{n_\bullet}$ correspond to an ordered s-tuple of geometrically integral spectral curves $(\Gamma_1, \ldots, \Gamma_s)$ of respective spectral degrees $(n_1, \ldots, n_s)$.

The requirements of Definition 2.6.3 (same pure relative dimensions, affine stabilizers, polarizability of Tate modules, δ-regularity on the elliptic locus) are met on each factor separately by virtue of Theorem 3.4.1. (In verifying δ-regularity, one needs a simple application of Lemma 2.6.2(2) to each factor: let $a \in A^{\mathrm{ell}}_{n_\bullet}$; let $x_\bullet$ be a closed general point in $\overline{\{a\}}$; let a_k be the projection of a to the kth factor; then $a_k \in A^{\mathrm{ell}}_{n_k}$, x_k is a closed general point of $\overline{\{a_k\}}$, and $\sum_k d_{a_k} \geqslant d_a$ (because $\overline{\{a\}} \subseteq \prod_k \overline{\{a_k\}}$); we have $d^{\mathrm{ab}}_a(J_{n_\bullet}) = d^{\mathrm{ab}}_{x_\bullet}(J_{n_\bullet}) = \sum_k d^{\mathrm{ab}}_{x_k}(J_{n_k}) = \sum_k d^{\mathrm{ab}}_{a_k}(J_{n_k}) \geqslant \sum_k (d_{a_k}(J_{n_k}) - d_{A_{n_k}} + d_{a_k}) = \sum_k d_{a_k}(J_{n_k}) - d_{A_{n_\bullet}} + \sum_k d_{a_k} \geqslant \sum_k d_{a_k}(J_{n_k}) - d_{A_{n_\bullet}} + d_a = \sum_k d_{x_k}(J_{n_k}) - d_{A_{n_\bullet}} + d_a = d_a(J_{n_\bullet}) - d_{A_{n_\bullet}} + d_a$.) It follows immediately that they are met on the product, so that (67) is a weak abelian fibration which is δ-regular over $A^{\mathrm{ell}}_{n_\bullet}$.

5.2 Stratification by type of the GL_n Hitchin base A_n

Let $n \in \mathbb{Z}^{\geqslant 1}$ and let $s \in \mathbb{Z}^{\geqslant 1}$ with $1 \leqslant s \leqslant n$. We consider the set $NM(s)$ of pairs $(n_\bullet, m_\bullet)$ subject to the following requirements: (1) $n_1 \geqslant \ldots \geqslant n_s$; (2) $m_k \geqslant m_{k+1}$ whenever $n_k = n_{k+1}$; (3) $\sum_{k=1}^s m_k n_k = n$. There is the partition of the integral variety

$$A_n = \coprod_{1 \leqslant s \leqslant n} \ \coprod_{(n_\bullet, m_\bullet) \in MN(s)} S_{n_\bullet m_\bullet} \tag{69}$$

into the locally closed integral subvarieties

$$S_{n_\bullet m_\bullet} := \left\{ a \in A_n \mid \mathcal{C}_{\overline{a}} = \sum_{k=1}^s m_k \mathcal{C}_{k,\overline{a}} \right\} \subseteq A_n, \tag{70}$$

where $\overline{a} \to a$ is given by an algebraic closure $k(a) \subseteq \overline{k(a)}$, and each spectral curve $\mathcal{C}_{k,\overline{a}}$ is irreducible of spectral curve degree n_k. The closure $\overline{S_{n_\bullet, m_\bullet}} \subseteq A_n$ is the image of the finite morphism [CL16, § 9]

$$\lambda_{m_\bullet n_\bullet} : A_{n_\bullet} \to A_n, \quad \mathrm{Im}\,(\lambda_{n_\bullet m_\bullet}) = \overline{S_{n_\bullet m_\bullet}} \subseteq A_n, \tag{71}$$

which on closed points is defined as follows: $(a_1, \ldots, a_s) \mapsto a$, where we view a_k as a characteristic polynomial P_{a_k} of degree n_k, we consider the degree n polynomial $\prod_{k=1}^s P_{a_k}^{m_k}$, and we take a to be the corresponding closed point on A_n. The stratum $S_{n_\bullet m_\bullet}$ is the image of a suitable Zariski-dense open subvariety inside the Zariski-dense open subvariety $\prod_{k=1}^s A^{\mathrm{ell}}_{n_k} \subseteq A_{n_\bullet}$. Given a point $a \in A_n$, we have $a \in S_{n_\bullet m_\bullet}$ for a unique triple $(s, (n_\bullet, m_\bullet))$, with $1 \leqslant s \leqslant n$ the number of irreducible components of $\mathcal{C}_{\overline{a}}$, and with $(n_\bullet, m_\bullet) \in NM(s)$, which we call the type of $a \in A_n$. Since the spectral curve $\mathcal{C}_a$ may have a strictly smaller number of components than $\mathcal{C}_{\overline{a}}$, the type of a is observed on $\mathcal{C}_{\overline{a}}$.

Geometrically, we may think of the morphism $\lambda_{n_\bullet m_\bullet}$ as sending an ordered s-tuple of integral curves $(\Gamma_1, \ldots, \Gamma_s)$, to the spectral curve denoted (§ 2.3) by $\sum_k m_k \Gamma_k$. As it is already clear in the case $s = 2$, with $(n_1, n_2; m_1, m_2) = (1, 1; 1, 1)$, in general, the finite morphisms $\lambda_{n_\bullet m_\bullet}$ are not birational.

The morphisms (71) are introduced in [CL16, § 9] in order to exploit the GL_n δ-regularity inequalities for each $J^{\mathrm{ell}}_{n_k}/A^{\mathrm{ell}}_{n_k}$, $k = 1, \dots, s$ (however, see Remark 5.4.3).

The resulting inequalities are of no use to us for the SL_n case: they are too weak. One may be tempted to replace them by taking the multi-variable counterpart to the SL_n δ-regularity inequality (64). As it turns out, these SL_n inequalities are also of no use to us towards the proof of Theorem 1.0.2 on the SL_n socle: they are not relevant in the proof given in § 6.2 of Theorem 1.0.2 (the SL_n support inequality (63) plays a crucial role, though).

The multi-variable δ-regularity inequalities that we need for the proof of Theorem 1.0.2 on the SL_n socle are given by Corollary 5.4.4(76), and are to be extracted from the constructions of § 5.3.

5.3 The weak abelian fibration $(M_{n_\bullet m_\bullet}(0), A_{n_\bullet m_\bullet}(0), J_{n_\bullet m_\bullet}(0))$

Define what we may call the subspace of multi-weighted-traceless characteristics by setting (recall that $a(1)$ is the trace-component of a characteristic)

$$A_{n_\bullet m_\bullet}(0) := \left\{ (a_1, \dots, a_s) \,\middle|\, \sum_k m_k a_k(1) = 0 \right\} \subseteq A_{n_\bullet}. \tag{72}$$

This is a vector subspace of codimension $h^0(C, D) = d - (g - 1)$. Define $M_{n_\bullet m_\bullet}(0) := h^{-1}_{n_\bullet}(A_{n_\bullet m_\bullet}(0)) \subseteq M_{n_\bullet}$ (given its reduced structure; we are about to verify the statement associated with (73), so that, a posteriori, this pre-image is indeed automatically reduced).

What follows is in direct analogy with the constructions in the proof of Lemma 4.3.1, and in its re-mixed version in § 4.4. We have the cartesian square diagram

$$\begin{array}{ccc} H^0(C,D) \times M_{n_\bullet m_\bullet}(0) & \xrightarrow[\sim]{\mathfrak{q}'} & M_{n_\bullet} \\ \Big\downarrow {\scriptstyle \mathrm{Id} \times h_\bullet(0)} & & \Big\downarrow {\scriptstyle h_\bullet} \\ H^0(C,D) \times A_{n_\bullet m_\bullet}(0) & \xrightarrow[\sim]{\mathfrak{q}} & A_{n_\bullet} \end{array} \tag{73}$$

with $\mathfrak{q}, \mathfrak{q}'$ isomorphisms, where we have the following.

(i) In analogy with (53), and by keeping in mind that here the entries u_{k1} are not necessarily zero, the map $\mathfrak{q}$ is given by the assignment sending

$$(\sigma, (u_{11}, \dots, u_{1n_1}), \dots (u_{s1}, \dots, u_{sn_s})), \quad \text{subject to } \sum_k m_k u_{k1} = 0,$$

to (having set $u_{k0} := 1$, for convenience)

$$\left(\left\{ \sum_{j=0}^{i} \binom{n_1 - i + j}{j} \sigma^j u_{1,i-j} \right\}_{i=1}^{n_1}, \dots, \left\{ \sum_{j=0}^{i} \binom{n_s - i + j}{j} \sigma^j u_{s,i-j} \right\}_{i=1}^{n_s} \right).$$

(ii) The isomorphism $\mathfrak{q}'$ is defined by the assignment

$$(\sigma, (E_1, \phi_1), \dots, (E_s, \phi_s)) \longmapsto ((E_1, \phi_1 + \sigma \mathrm{Id}), \dots, (E_s, \phi_s + \sigma \mathrm{Id})).$$

As in the proof Lemma 4.3.1, a simple recursion yields the map inverse to $\mathfrak{q}$, whereas the one inverse to $\mathfrak{q}'$ is given by the assignment (remember that $n = \sum_k m_k n_k$)

$$\{(E_k, \psi_k)\}_{k=1}^{s} \longmapsto \left(\frac{\sum_j m_j \mathrm{tr}(\psi_j)}{n}, \left\{ \left(E_k, \psi_k - \sum_j \frac{m_j}{n} \mathrm{tr}(\psi_j) \mathrm{Id} \right) \right\}_{k=1}^{s} \right).$$

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Finally, by setting $J_{n_\bullet m_\bullet}(0) := J_{n_\bullet|A_{n_\bullet m_\bullet}(0)}$, we have the cartesian square diagram with $\mathfrak{q}''$ and $\mathfrak{q}$ isomorphisms

$$\begin{array}{ccc} H^0(C,D)\times J_{n_\bullet m_\bullet}(0) & \xrightarrow{\mathfrak{q}''} & J_{n_\bullet} \\ \Big\downarrow{\scriptstyle \mathrm{Id}\times g_{n_\bullet m_\bullet}(0)} & & \Big\downarrow{\scriptstyle g_{n_\bullet m_\bullet}} \\ H^0(C,D)\times A_{n_\bullet m_\bullet}(0) & \xrightarrow{\mathfrak{q}} & A_{n_\bullet} \end{array} \tag{74}$$

obtained in the same way as (57).

5.4 Multi-variable δ-regularity over the elliptic loci

Recalling the definition of $S_\delta(J/A)$ in (19), we have the following identification of δ-loci

PROPOSITION 5.4.1. $\mathfrak{q}^{-1}(S_\delta(J_{n_\bullet}/A_{n_\bullet})) = H^0(C,D)\times S_\delta(J_{n_\bullet m_\bullet}(0)/A_{n_\bullet m_\bullet}(0))$.

Proof. Keeping in mind that $S_\delta(J_{n_\bullet}/A_{n_\bullet})$ is naturally stratified by products of individual $S_{\delta_k}(J_{n_k}/A_{n_k})$ with $\sum_k \delta_k = \delta$, the proof runs parallel to the one of Proposition 4.5.1, with (74) playing the role of (57). $\square$

THEOREM 5.4.2. *The weak abelian fibrations*

$$(M_n, A_n, J_n),\quad (M_n(0), A_n(0), J_n(0)),\quad (\check{M}_n, \check{A}_n, \check{J}_n),$$
$$(M_{n_\bullet}, A_{n_\bullet}, J_{n_\bullet}),\quad (M_{n_\bullet m_\bullet}(0), A_{n_\bullet m_\bullet}(0), J_{n_\bullet m_\bullet}(0))$$

are δ-regular when restricted to their respective elliptic loci

$$A_n^{\mathrm{ell}},\quad A_n^{\mathrm{ell}}(0) := A_n(0)\cap A_n^{\mathrm{ell}},\quad \check{A}_n^{\mathrm{ell}} := \check{A}_n \cap A_n^{\mathrm{ell}},$$
$$A_{n_\bullet}^{\mathrm{ell}} := \prod_k A_{n_k}^{\mathrm{ell}},\quad A_{n_\bullet m_\bullet}^{\mathrm{ell}}(0) := A_{n_\bullet m_\bullet}(0)\cap \prod_k A_{n_k}^{\mathrm{ell}}.$$

Proof. We have already proved all the conclusions in the single-variable case: we have displayed them for emphasis only. We have already observed in § 5.1 that the single-variable case implies that $(M_{n_\bullet}, A_{n_\bullet}, J_{n_\bullet})$ is a weak abelian fibration which is δ-regular over its elliptic locus $A_{n_\bullet}^{\mathrm{ell}}$.

By virtue of (73) and of (74), we see that $(M_{n_\bullet m_\bullet}(0), A_{n_\bullet m_\bullet}(0), J_{n_\bullet m_\bullet}(0))$ is a weak abelian fibration as well, which, by virtue of Proposition 5.4.1, is δ-regular over its elliptic locus $A_{n_\bullet m_\bullet}^{\mathrm{ell}}(0)$ (cf. the proof of Proposition 4.5.1). $\square$

Remark 5.4.3. The following claim *does not* hold: given a point $a \in S_{n_\bullet m_\bullet} \subseteq A_n$, we can write $a = \lambda_{n_\bullet m_\bullet}(a_1, \ldots a_s)$ for a suitable s-tuple $a_k \in A_{n_k}$. This is true if a is a closed point, but it fails in general. This claim has been used in [CL16, § 9, proof of main theorem].

Corollary 5.4.4, equation (75) below remedies the minor inaccuracy in the proof of [CL16, § 9, proof of main theorem] pointed out in Remark 5.4.3. It also establishes the SL_n-variant (76) that we need in the course of the proof of Theorem 1.0.2 in § 6.2.

COROLLARY 5.4.4 (Multi-variable δ-inequalities). *Let $a \in A_n$ and let $(n_\bullet, m_\bullet) \in NM(s)$ be its type (§ 5.3). Then we have the following multi-variable GL_n δ-inequality:*

$$d_a^{\mathrm{ab}}(J_n) \geqslant \sum_k (d_{h_{n_k}} - d_{A_{n_k}}) + d_a. \tag{75}$$

If, in addition, $a \in A_n(0) = \check{A}_n$, then we have the following multi-variable SL_n δ-inequality:

$$d_a^{\mathrm{ab}}(J_n) \geqslant \sum_k (d_{h_{n_k}} - d_{A_{n_k}}) + [d-(g-1)] + d_a. \tag{76}$$

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Proof. Let $a \in A_n$ and let $V(a) := \overline{\{a\}} \subseteq A_n$ be the associated integral closed subvariety. Let $(n_\bullet, m_\bullet) \in NM(s)$ be the type of a. Let α be any point in the non empty fiber $\lambda^{-1}_{n_\bullet m_\bullet}(a)$. Then $d_a := \dim(\overline{\{a\}}) = \dim(\overline{\{\alpha\}}) = d_\alpha$.

We choose an algebraic closure of $k(a)$ that contains the finite field extension $k(a) \subseteq k(\alpha)$. We can identify the curves $\mathcal{C}_{\overline{\alpha}} = \mathcal{C}_{\overline{a},\mathrm{red}}$, so that the two curves have the same number s of geometrically irreducible components. It follows that $\alpha \in A^{\mathrm{ell}}_{n_\bullet}$. By virtue of (39), it also follows that $d^{\mathrm{ab}}_a(J_n) = d^{\mathrm{ab}}_\alpha(J_{n_\bullet})$.

The δ-regularity inequality for $J_{n_\bullet}$ over $A^{\mathrm{ell}}_{n_\bullet}$ implies that $d^{\mathrm{ab}}_\alpha(J_{n_\bullet}) \geqslant d_{h_{n_\bullet}} - d_{A_{n_\bullet}} + d_\alpha$, and (75) follows.

Since a has type $(n_\bullet, m_\bullet)$, we have that α satisfies the weighted trace constraint (72) that defines $A^{\mathrm{ell}}_{n_\bullet m_\bullet}$, so that $\alpha \in A^{\mathrm{ell}}_{n_\bullet m_\bullet}(0)$. Then (76) is proved in the same way as (75) by using the δ-regularity of $J_{n_\bullet m_\bullet}(0)$ over $A^{\mathrm{ell}}_{n_\bullet m_\bullet}(0)$, and the facts that $d_{h_{n_\bullet}} = \sum_k d_{h_{n_k}}$, and (cf. (72)) $d_{A_{n_\bullet m_\bullet}}(0) = \dim(A_{n_\bullet}) - h^0(C, D) = \sum_k d_{A_{n_k}} - [d - (g-1)]$. □

6. Proof of the main Theorem 1.0.2 on the SL_n socle

This section is devoted to the proof of our main Theorem 1.0.2 on the SL_n socle. Section 6.1 collects some formulae. Section 6.2 contains the proof of Theorem 1.0.2.

6.1 A list of dimension formulae

We first list some dimensional formulae in the GL_n case. We set $d_{M_n} := \dim M_n$, $d_{A_n} := \dim A_n$, and $d_{h_n} := d_{M_n} - d_{A_n}$. The dimension of M_n is given by [Nit91, Proposition 7.1]; the dimension of $A_n = \bigoplus_{i=1}^n h^0(C, iD)$ is computed via Riemann–Roch; the relative dimension d_{h_n} is given by (7). We thus have

$$d_{M_n} = n^2 d + 1, \quad d_{A_n} = \frac{n(n+1)}{2} d - n(g-1),$$
$$d_{h_n} = \frac{n(n-1)}{2} d + n(g-1) + 1, \quad d_{h_n} - d_{A_n} = -nd + 2n(g-1) + 1. \tag{77}$$

The corresponding formula for SL_n follow easily, for example, from the above, remembering that, in view of lemmata 4.3.1 and 4.3.2, we have that $\dim(M_n) = \dim(\check{M}_n) + h^0(D) + g$:

$$d_{\check{M}_n} = n^2 d - d, \quad d_{\check{A}_n} = \frac{n(n+1)}{2} d - d - (n-1)(g-1),$$
$$d_{\check{h}_n} = \frac{n(n-1)}{2} d + (n-1)(g-1), \quad d_{\check{h}_n} - d_{\check{A}_n} = -(n-1)d + 2(n-1)(g-1). \tag{78}$$

Recall that, given $a \in A_n$, we have been denoting the dimensions of $J_{n,a}, J^{\mathrm{aff}}_{n,a}$ and $J^{\mathrm{ab}}_{n,a}$ by $d_a(J_n), d^{\mathrm{aff}}_a(J_n)$ and $d^{\mathrm{ab}}_a(J_n)$, respectively, and have been doing the same for $\check{a} \in \check{A}_n \subseteq A_n$, $\check{J}_{n,a}$ etc. (Recall that the Chevalley devissage is defined at geometric points, but the indicated dimensions depend only on the underlying Zariski point; as it is about to become clear, it is better to keep track of Zariski points.)

6.2 Proof of Theorem 1.0.2

Having done all the necessary preparation, the proof of Theorem 1.0.2 for the SL_n socle, can now proceed parallel to the proof of Theorem 1.0.1 for the GL_n socle in [CL16, § 9].

Let $a \in \check{A}_n$ belong to $\mathrm{Socle}(R\check{h}_{n*}\overline{\mathbb{Q}}_\ell)$. Apply the support inequality (63) for the Zariski points in the socle:

$$d_{\check{h}_n} - d_{\check{A}_n} + d_a \geqslant d^{\mathrm{ab}}_a(\check{J}_n). \tag{79}$$

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By Lemma (4.2.1), we have $d_a^{\mathrm{ab}}(\check{J}_n) = d_a^{\mathrm{ab}}(J_n) - g$, so that

$$d_{\check{h}_n} - d_{\check{A}_n} + d_a \geqslant d_a^{\mathrm{ab}}(J_n) - g. \tag{80}$$

By combining (80) with (76), we get

$$d_{\check{h}_n} - d_{\check{A}_n} \geqslant \sum_{k=1}^{s} (d_{h_{n_k}} - d_{A_{n_k}}) + [d - (g-1)] - g. \tag{81}$$

By using (78) for the left-hand side, and (77) for each n_k for the right-hand side, we re-write (81) as follows:

$$-(n-1)d + 2(n-1)(g-1) \geqslant \sum_k [-n_k d + 2n_k(g-1) + 1] + [d - (g-1)] - g, \tag{82}$$

i.e.

$$0 \geqslant \left(n - \sum_k n_k\right)[d - 2(g-1)] + (s-1). \tag{83}$$

Since $d > 2(g-1)$ (by assumption) and $s \geqslant 1$ (by construction), we must have $\sum_k m_k n_k = n = \sum_k n_k$ and $s = 1$.

The first condition forces all $m_k = 1$, so that the corresponding geometric spectral curve $\mathcal{C}_{\overline{a}}$ is reduced.

The second condition $s = 1$ means that in addition to being reduced, the geometric spectral curve $\mathcal{C}_{\overline{a}}$ must be integral, i.e. $a \in \check{A}_n^{\mathrm{ell}}$. □

We conclude with two remarks.

Remark 6.2.1 (Positive characteristic). Chaudouard has informed us that the main Theorem 1.0.1 for the GL_n socle in [CL16], should also hold over an algebraically closed field of positive characteristic bigger than n. This should be the case in view of the fact that one major obstacle in proving such theorem in positive characteristic had been the lack of the positive-characteristic analogue of the Severi inequality (41). At least as far as the corresponding inequality at the level of the semiuniversal (miniversal) deformation for integral (even reduced) locally planar curves, this obstacle has been removed in [RMV12, Theorem 3.3]. The restriction on the characteristic seems natural in view of the fact that the spectral covers have order n, and also because of formulae such as (53). We did not verify whether all of our arguments could be easily modified to yield the positive characteristic ($> n$) cases of Theorems 1.0.1 and 1.0.2, on the GL_n and SL_n socles.

Remark 6.2.2 ($D = K_C$). The methods of proofs of [CL16] for GL_n, and of this paper for SL_n, do not work in the very interesting case when $D = K_C$. There is even more geometry at play in that symplectic/integrable case. See [CL16, § 11] for a short discussion of the $D = K_C$ case.

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References

BBD81 A. Beilinson, I. N. Bernstein and P. Deligne, *Faisceaux pervers*, Astérisque, vol. 100 (Société Mathématique de France, Paris, 1981).

BLR90 S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21 (Springer, Berlin, 1990).

CL12 P.-H. Chaudouard and G. Laumon, *Le lemme fondamental pondéré. II. Énoncés cohomologiques*, Ann. of Math. (2) **176** (2012), 1647–1781.

CL16 P.-H. Chaudouard and G. Laumon, *Un théorème du support pour la fibration de Hitchin*, Ann. Inst. Fourier (Grenoble) **66** (2016), 711–727.

Con02 B. Conrad, *A modern proof of Chevalley's theorem on algebraic groups*, J. Ramanujan Math. Soc. **17** (2002), 1–18.

Del87 P. Deligne, *Le déterminant de la cohomologie*, in *Current trends in arithmetical algebraic geometry, Arcata, CA, 1985*, Contemporary Mathematics, vol. 67 (American Mathematical Society, Providence, RI, 1987), 93–177.

EGAII A. Grothendieck, *Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 5–205.

EGAIV.4 A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des scheémas et des morphismes de sch'emas IV*, Publ. Math. Inst. Hautes Études Sci. **32** (1967), 5–333.

FGvS99 B. Fantechi, L. Gottsche and D. van Straten, *Euler number of the compactified Jacobian and multiplicity of rational curves*, J. Algebraic Geom. **8** (1999), 115–133.

HP12 T. Hausel and C. Pauly, *Prym varieties of spectral covers*, Geom. Topol. **16** (2012), 1609–1638.

LB92 H. Lange and C. Birkenhake, *Complex abelian varieties*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 302 (Springer, Berlin, 1992).

MY14 D. Maulik and Z. Yun, *Macdonald formula for curves with planar singularities*, J. Reine Angew. Math. **694** (2014), 27–48.

MS13 L. Migliorini and V. Shende, *A support theorem for Hilbert schemes of planar curves*, J. Eur. Math. Soc. (JEMS) **15** (2013), 2353–2367.

MS13 L. Migliorini and V. Shende, *Higher discriminants and the topology of algebraic maps*, Preprint (2013), arXiv:1307.4059.

MSV15 L. Migliorini, V. Shende and F. Viviani, *A support theorem for Hilbert schemes of planar curves, II*, Preprint (2015), arXiv:1508.07602.

Mil J. S. Milne, *Algebraic groups, The theory of groups schemes of finite type over a field*, http://www.jmilne.org/math/CourseNotes/ala.html.

Mil80 J. S. Milne, *Étale cohomology*, Princeton Mathematical Series, vol. 33 (Princeton University Press, Princeton, NJ, 1980).

Mum70 D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, No. 5, Published for the Tata Institute of Fundamental Research, Bombay (Oxford University Press, London 1970).

Ngô06 B. C. Ngô, *Fibration de Hitchin et endoscopie*, Invent. Math. **164** (2006), 399–453.

Ngô10 B. C. Ngô, *Le lemme fondamental pour les algèbres de Lie*, Publ. Math. Inst. Hautes Études Sci. **111** (2010), 1–169.

Ngô11 B. C. Ngô, *Decomposition theorem and abelian fibration*, in *On the stabilization of the trace formula*, Stabilization of the Trace Formula, Shimura Varieties, and Arithmetic Applications, vol. 1 (International Press, Somerville, MA, 2011), 253–264.

Nit91 N. Nitsure, *Moduli space of semistable pairs on a curve*, Proc. Lond. Math. Soc. (3) **62** (1991), 275–300.

RMV12 A. Rapagnetta, M. Melo and F. Viviani, *Fourier–Mukai and autoduality for compactified Jacobians, I*, Preprint (2012), arXiv:1207.7233.

Rei97 M. Reid, *Chapters on algebraic surfaces*, in *Complex algebraic geometry, Park City, UT, 1993*, IAS/Park City Mathematics Series, vol. 3 (American Mathematical Society, Providence, RI, 1997), 3–159.

Sch98 D. Schaub, *Courbes spectrales et compactifications de jacobiennes*, Math. Z. **227** (1998), 295–312.

SGA3.I A. Grothendieck and M. Demazure, *Séminaire de Géométrie Algébrique, Schémas en Groupes (SGA 3), vol. I*, Lecture Notes in Mathematics, vol. 151 (Springer, Heidelberg, 1970).

SGA3.II A. Grothendieck and M. Demazure, *Séminaire de Géométrie Algébrique, Schémas en Groupes (SGA 3), vol. II*, Lecture Notes in Mathematics, vol. 152 (Springer, Heidelberg, 1970).

SGA7.I A. Grothendieck, M. Raynaud and D. S. Rim, *Groups de Monodromie en Géométrie Agébrique (SGA 7), vol. I*, Lecture Notes in Mathematics, vol. 288 (Springer, Heidelberg, 1972).

She12 V. Shende, *Hilbert schemes of points on a locally planar curve and the Severi strata of its versal deformation*, Compos. Math. **148** (2012), 531–547.

Sim94 C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Publ. Math. Inst. Hautes Études Sci. **79** (1994), 47–129.

Sim95 C. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Publ. Math. Inst. Hautes Études Sci. **80** (1995), 5–79.

Sou92 C. Soulé, *Lectures on Arakelov geometry*, Cambridge Studies in Advanced Mathematics, vol. 33 (Cambridge University Press, Cambridge, 1992).

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