ON ERGODIC THEOREM FOR A BANACH VALUED RANDOM SEQUENCE

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In this paper we shall deal with a probability space (S, Σ, P), a separable Banach space X having its strong dual X^* and a strictly stationary random sequence $\{Y_k\}_{k=1}^{+\infty}$ defined as in [7], where Y_k's are X-valued, Gelfand-Pettis (weakly) integrable [6], [9], and strongly measurable random variables. In the case when Y_k's are Bochner (strongly) integrable random variables one can find the ergodic theorem for such a sequence and, with respect to strong convergence in X, in the papers [7], [8].

Here we are going to present the ergodic theorem for the sequence $\{Y_k\}_{k=1}^{+\infty}$ described above, using the representation for a Gelfand-Pettis integrable function obtained by J. K. Brooks in [3].

The setting. Let $Y : S \to X$ be a (strongly) measurable and weakly integrable random variable. Then by [3], Y has the representation:

$$Y = \sum_{j=1}^{+\infty} y_j I_{E_j} \tag{1}$$

where $y_j \in X$, $E_j \in \Sigma$, $j = 1, 2, \cdots$, where the series in (1) is absolutely convergent a.e. $[P]$ and, moreover, the series

$$\sum_{j=1}^{+\infty} y_j P(E_j \cap E) \tag{2}$$

is unconditionally convergent [5], for every $E \in \Sigma$. In [3] was given the following formal definition for the conditional expectation $E^{\mathscr{B}}(Y)$($\mathscr{B}$ is a σ-subalgebra of Σ), for a weakly integrable random variable Y having the representation (1) and satisfying (2):

$$E^{\mathscr{B}}(Y) = \sum_{j=1}^{+\infty} y_j P^{\mathscr{B}}(E_j), \tag{3}$$

where $P^{\mathscr{B}}(E_j) = E^{\mathscr{B}}(I_{E_j})$, and whenever the series in (3) is unconditionally convergent a.e. $[P]$ ($E^{\mathscr{B}}(I_{E_j})$ is a real valued conditional expectation).

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Let $(X, \mathscr{B}_X)$ be a Borel space defined as in [8] and let $(X^\infty, \mathscr{B}^\infty) = \prod_{i=1}^{+\infty} (X, \mathscr{B}_X)$, i.e., $(X^\infty, \mathscr{B}^\infty)$ is a measurable space obtained as the product of countable number of copies of $(X, \mathscr{B}_X)$. Then, according to [8], $(X^\infty, \mathscr{B}^\infty)$ is a standard Borel space, and one can always define the regular conditional probability $P^{\mathscr{B}}$ of P ($\mathscr{B}$ is a σ-subalgebra of $\mathscr{B}^\infty$) such that $P^{\mathscr{B}}(A; x)$ is a $\mathscr{B}$-measurable function for a fixed $A \in \mathscr{B}^\infty$, and a probability measure on $\mathscr{B}^\infty$ for a fixed $x \in X^\infty$. Let T be a shift operator which maps X^∞ into itself such that $Y_i(T^j x) = Y_{i+j}$, for $i, j = 1, 2, \cdots$, and $x \in X^\infty$, $x = (x_1, x_2, \cdots)$, with $x_j \in X, j = 1, 2, \cdots$. Let μ be a probability measure on $(X^\infty, \mathscr{B}^\infty)$ which is induced by P, and which is invariant with respect to T, i.e., $\mu = \mu T^{-1}$, as a consequence of the fact that $\{Y_k\}_{k=1}^{+\infty}$ is a strictly random sequence. Finally, put

$$\mathscr{F} = \{A; A \in \mathscr{B}^\infty, T^{-1}A = A \text{ a.e. } [\mu]\}.$$

Then, $\mathscr{F}$ is a σ-subalgebra of $\mathscr{B}^\infty$. Now we state

PROPOSITION. *Let $Y: X^\infty \to X$ be a measurable and weakly integrable random variable having the representation (according to (1) as the result from [3]):*

$$Y = \sum_{j=1}^{+\infty} x_j I_{E_j} \tag{4}$$

where $x_j \in X$, $E_j \in \mathscr{B}^\infty$, $j = 1, 2, \cdots$, and series in (4) is absolutely convergent a.e. $[\mu]$, with the corresponding series in (2) unconditionally convergent a.e. $[\mu]$. Furthermore, let $\mathscr{F}$ be the σ-subalgebra of $\mathscr{B}^\infty$ defined as before, and assume that $\mathscr{F}$ is generated by a countable partition of X^∞. Then:

$$E^{\mathscr{F}}(Y)(x) = \sum_{j=1}^{+\infty} x_j \mu^{\mathscr{F}}(E_j; x), \tag{5}$$

where $\mu^{\mathscr{F}}$ is a regular conditional probability for μ and the series in (5) is unconditionally convergent a.e. $[\mu]$.

PROOF. From the result in [8], it follows that there exists a regular conditional probability $\mu^{\mathscr{F}}$ of μ, as a consequence of the fact that $(X^\infty, \mathscr{B}^\infty)$ is a standard Borel space. Using the relation (3) one can immediately write the formal representation (5) for the conditional expectation $E^{\mathscr{F}}(Y)$. Let us show that the series in (5), with $\mathscr{F}$ defined as before, is unconditionally convergent a.e. $[\mu]$. From the assumption that $(X^\infty, \mathscr{B}^\infty)$ is a Borel space generated by a countable partition of X^∞, it follows that there exists a countable family, say $\{F_k\}_{k=1}^{+\infty}$ of $\mathscr{B}^\infty$, F_k's being pairwise disjoint, which generates $\mathscr{F}$. Then, one can write:

$$\mu^{\mathscr{F}}(A; x) = \sum_{k=1}^{+\infty} \frac{\mu(A \cap F_k)}{\mu(F_k)} I_{F_k}(x), \tag{6}$$

or, using (5) and (6),

$$E^{\mathscr{F}}(Y)(x) = \sum_{j=1}^{+\infty} x_j \sum_{k=1}^{+\infty} \frac{\mu(E_j \cap F_k)}{\mu(F_k)} I_{F_k}(x), \tag{7}$$

wherefrom, for every $x \in X^\infty$, there exists a positive integer $k_0(x)$ such that $x \in F_{k_0(x)}$, and (7) becomes:

$$E^{\mathscr{F}}(Y)(x) = \frac{1}{\mu(F_{k_0(x)})} \sum_{j=1}^{+\infty} x_j \mu(E_j \cap F_{k_0(x)}), \tag{8}$$

where the series in (8) is unconditionally convergent a.e. $[\mu]$, due to the relation (2).

REMARK 1. The assumption that the σ-algebra $\mathscr{F}$ is generated by a countable partition is a rather strong restriction. It will be of interest to replace this condition with some weaker one implying that the series in (5) is unconditionally convergent.

REMARK 2. In the case when the series in (5) is absolutely convergent a.e. $[\mu]$, one gets the strong conditional expectation which was the subject of the studies by the various authors (see [4], [10], for example).

THEOREM. *Let $(X^\infty,\ \mathscr{B}^\infty, \mu)$ be a probability space and let $\mathscr{F}$ be a σ-subalgebra of $\mathscr{B}^\infty$ satisfying the assumptons of the previous Proposition. Let $\{Y_k\}_{k=1}^{+\infty}$ be a strictly stationary random sequence of X-valued weakly integrable random variables, i.e., $Y_k : X^\infty \to X$, $k = 1, 2, \cdots$, having the representations:*

$$Y_k = \sum_{j=1}^{+\infty} x_j^{(k)} I_{E_j^{(k)}},\ k = 1, 2, \cdots,$$

satisfying the properties described in (1) *and* (2). *Put $S_n = (Y_1 + Y_2 + \cdots + Y_n)/n$, $n = 1, 2, \cdots$. If*

$$\lim_{n \to +\infty} \sum_{j \in M} x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})} \tag{9}$$

exists for every subset M of the set of positive integers, then:

$$\|S_n - E^{\mathscr{F}}(Y_1)\| \to 0, \text{ a.e.}[\mu], \text{ as } n \to +\infty.$$

PROOF. First, it follows from the properties of the shift operator T described earlier, that

$$S_n = \frac{1}{n} \sum_{j=0}^{n-1} Y_1(T^j x), \text{ or, } S_n = \sum_{j=1}^{+\infty} x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})},\ n = 1, 2, \cdots,$$

where

$$\sum_{j=1}^{+\infty} \left\| x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})} \right\| < +\infty, n = 1, 2, \cdots. \tag{10}$$

Indeed, for $j = 1, 2, \cdots$, and a fixed positive integer n one gets that

$$(*)\quad x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}(x) = \frac{1}{n}\, x_j^{(1)}(I_{E_j^{(1)}}(x) + I_{E_j^{(1)}}(Tx) + \cdots + I_{E_j^{(1)}}(T^{n-1}x)).$$

From (*) and the fact that each of the series $\sum_{j=1}^{+\infty} x_j^{(k)} I_{E_j}(k)$, $k = 1, 2, \cdots, n$ is absolutely convergent a.e. $[\mu]$, it follows that the series

$$\sum_{j=1}^{+\infty} x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}$$

is absolutely convergent a.e. $[\mu]$, which represents the assertion in (10).

Using the Birkhoff's Ergodic theorem [2], it follows that:

$$\lim_{n\to+\infty} \left(x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}(x) \right) = x_j^{(1)} \mu^{\mathscr{F}}(E_j^{(1)}; x) \text{ a.e. } [\mu], \tag{11}$$

for $j = 1, 2, \cdots$.

Further, we need the following lemma which is equivalent to the result due to S. Banach in [1] pp. 138-139, and which can be restated as follows:

Let $\{x_{in}\}$ be a double sequence in a Banach space such that:

(i) $$\sum_{n=1}^{+\infty} \|x_{in}\| < +\infty$$

for all $i = 1, 2, \cdots$;

(ii) $$\lim_{i\to+\infty} \sum_{n\in M} x_{in} = 0$$

for every subset M of the set of positive integers N; then

$$\lim_{i\to+\infty} \sum_{n=1}^{+\infty} \|x_{in}\| = 0.$$

Indeed, if as assume that in the contrary there exists a subsequence $i_1 < i_2 < \cdots$, such that for some $\varepsilon > 0$, $\sum_{n=1}^{+\infty} \|x_{in}\| > \varepsilon > 0$, then it is possible to construct two subsequences, say, $j_1 < j_2 \cdots$, and $p_1 < p_2 \cdots$, such that:

$$\sum_{n=1}^{p_k} \|x_{j_k n}\| < \varepsilon/8, \sum_{n=p_k+1}^{p_{k+1}} \|x_{j_k n}\| > (3\varepsilon)/4, \text{ and, } \sum_{n=p_{k+1}}^{+\infty} \|x_{j_k n}\| < \varepsilon/8.$$

If we take $M = \{p_k\}$ we have that:

$$\|\sum_{n\in M} x_{in}\| \geqq \sum_{n=p_k+1}^{p_{k+1}} \|x_{j_k n}\| - \sum_{n=1}^{p_k} \|x_{j_k n}\| - \sum_{n=p_{k+1}}^{+\infty} \|x_{j_k n}\| > (3\varepsilon)/4 - \varepsilon/8 - \varepsilon/8 = \varepsilon,$$

which contradicts the assumption (ii).

Therefore, according to this lemma, it follows from (9), (10) and (11) that

$$\lim_{n\to+\infty} \sum_{j=1}^{+\infty} \left\| x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})} - x_j^{(1)} \mu^{\mathscr{F}}(E_j^{(1)}; x) \right\| = 0, \text{ a.e. } [\mu],$$

which implies that

$$\|S_n - E^{\mathscr{F}}(Y_1)\| = \left\| \frac{1}{n}\sum_{i=0}^{n-1}\sum_{j=1}^{+\infty} x_j^{(1)} I_{T^{-i}(E_j^{(1)})}(x) - \sum_{j=1}^{+\infty} x_j^{(1)}\mu^{\mathscr{F}}(E_j^{(1)}; x)\right\|$$
$$\leqq \sum_{j=1}^{+\infty}\left\| x_j^{(1)}\frac{1}{n}\sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}(x) - x_j^{(1)}\mu^{\mathscr{F}}(E_j^{(1)}; x)\right\| \to 0,$$

a.e. $[\mu]$, as $n \to +\infty$.

COROLLARY 1. *Let $\{Y_k\}_{k=1}^{+\infty}$ be a strictly stationary weakly integrable random sequence, and assume that X is a Hilbert space. If, moreover,*

$$\lim_{n\to+\infty}\left\|\sum_{j\in M} x_j^{(1)}\frac{1}{n}\sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}(x)\right\| = \left\|\sum_{j\in M} x_j^{(1)}\mu^{\mathscr{F}}(E_j^{(1)}; x)\right\| \tag{12}$$

for every subset M of the set of positive integers N, then: $\|S_n - E^{\mathscr{F}}(Y_1)\| \to 0$, a.e. $[\mu]$, as $n \to +\infty$.

PROOF. Using (10) and the Proposition, one gets that

$$\sum_{j\in M} |x^*(x_j^{(1)})| \frac{1}{n}\sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})} < +\infty, \text{ for } n = 1, 2, \cdots,$$

and

$$\sum_{j\in M} |x^*(x_j^{(1)})|\mu^{\mathscr{F}}(E_j^{(1)}; x) < +\infty, \text{ for every } x^* \in X^*, \text{ i.e.,}$$

$$\left\{x^*(x_j^{(1)})\frac{1}{n}\sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}\right\} \in l_1 \text{ for } n = 1, 2, \cdots,$$

and $\{x^*(x_j^{(1)})\mu^{\mathscr{F}}(E_j^{(1)}; x)\} \in l_1$, where from it follows, by Birkhoff's Ergodic theorem that:

$$x^*\left(\sum_{j\in M} x_j^{(1)}\frac{1}{n}\sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}\right) \to x^*\Big(\sum_{j\in M} x_j^{(1)}\mu^{\mathscr{F}}(E_j^{(1)}; x)\Big), \text{ a.e. } [\mu], \tag{13}$$

as $n \to +\infty$, for every $x^* \in X^*$. Taking into account (12) and (13) and the fact that X is a Hilbert space, it follows that:

$$\lim_{n\to+\infty}\sum_{j\in M} x_j^{(1)}\frac{1}{n}\sum_{i=0}^{n-1} I_{T^{-i}(E_j^{(1)})}$$

exists for every $M \subseteq N$, which is the condition (9) in the previous theorem. Therefore, this together with (10) and (11) implies that $\|S_n - E^{\mathscr{F}}(Y_1)\| \to 0$, a.e. $[\mu]$, as $n \to +\infty$.

Now we have the following result from [8] as a consequence of the previous theorem:

COROLLARY 2. *If $\{Y_k\}_{k=1}^{+\infty}$ are Bochner integrable, then $\|S_n - E^{\mathscr{F}}(Y_1)\| \to 0$, a.e. $[\mu]$, as $n \to +\infty$.*

PROOF. In the case when Y_k's are Bochner integrable the series

$$\sum_{j=1}^{+\infty} x_j^{(1)} \mu^{\mathscr{F}}(E_j^{(1)}; x)$$

is absolutely convergent a.e. $[\mu]$, that is,

$$\sum_{j=1}^{+\infty} ||x_j^{(1)}|| \mu^{\mathscr{F}}(E_j^{(1)}; x) < +\infty, \text{ a.e. } [\mu].$$

Consider:

$$\left\| \sum_{j \in M} x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j{}^{(1)})} - \sum_{j \in M} x_j^{(1)} \mu^{\mathscr{F}}(E_j^{(1)}; x) \right\| \tag{14}$$
$$= \sup_{||x^*|| \leqq 1} \left| \sum_{j \in M} x^*(x_j^{(1)}) \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j{}^{(1)})} - \sum_{j \in M} x^*(x_j^{(1)}) \mu^{\mathscr{F}}(E_j^{(1)}; x) \right|,$$

where $M \subset N$, and $x^* \in X^*$. Let $Y_1^\#$ be a new random variable which is obtained from Y_1 by the formula: $Y_1^\# = \sum_{j \in M} x_j^{(1)} I_{E_j(1)}$. Then, $Y_1^\#$ is a Bochner integrable provided that Y_1 is a Bochner integrable, and (14) can be written as:

$$\left\| \sum_{j \in M} x_j^{(1)} \frac{1}{n} \sum_{i=0}^{n-1} I_{T^{-i}(E_j{}^{(1)})} - \sum_{j \in M} x_j^{(1)} \mu^{\mathscr{F}}(E_j^{(1)}; x) \right\| \tag{15}$$
$$= \sup_{||x^*|| \leqq 1} \left| \int_{X^\infty} x^*(Y_1^\#(y)) \mu_n(dy, x) - \int_{X^\infty} x^*(Y_1^\#(y)) \mu^{\mathscr{F}}(dy, x) \right|,$$

where μ_n is a measure which has its masses concentrated in the points: $x, Tx, \cdots, T^{n-1}x$, of X^∞, and equal $1/n$ at each of these points, and $\mu^{\mathscr{F}}(. , x)$ is a probability measure for a fixed $x \in X^\infty$, as was assumed earlier. From the fact that $|x^*(Y_1^\#)| \leqq ||Y_1^\#||$, for $||x^*|| \leqq 1$, and Bochner integrability of $Y_1^\#$, it follows that

$$\int_{X^\infty} ||Y_1^\#(y)|| \mu^{\mathscr{F}}(dy; x) \mu(dx) < +\infty,$$

which implies, as was shown in [8], that the right-hand side X^∞ of (15) tends to zero a.e. $[\mu]$, as $n \to -\infty$, and therefore from (14) one gets the condition (9) in the theorem, which together with (10) and (11) implies that $||S_n - E^{\mathscr{F}}(Y_1)|| \to 0$, a.e. $[\mu]$, as $n \to +\infty$.

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