Can. J. Math., Vol. XXXVI, No. 1, 1984, pp. 95-130

DIRECTIONALLY LIPSCHITZIAN MAPPINGS ON BAIRE SPACES

J. M. BORWEIN AND H. M. STRÒJWAS

0. Introduction. Studies of optimization problems have led in recent years to definitions of several types of generalized directional derivatives. Those derivatives of primary interest in this paper were introduced and investigated by F. M. Clarke ([**5**], [**6**], [**7**], [**8**]), J. B. Hiriart-Urruty ([**12**]), Lebourg ([**16**], [**17**]), R. T. Rockafellar ([**23**], [**24**], [**26**], [**27**]), Penot ([**21**], [**22**]) among others.

In an attempt to explore in more detail relationships between various types of generalized directional derivatives we discovered some unexpected results which were not observed by the above mentioned authors. We are able to give simple conditions which characterize directionally Lipschitzian functions defined on a Baire metrizable locally convex topological vector space.

In Section 1 of this paper we present a brief summary of the properties of generalized derivatives which have been described in the literature. Connections with some types of tangent cones are also explained. For more details and proofs the reader is referred to [**23**], [**24**], [**25**], [**27**].

The statement and proofs of the results for Baire metrizable spaces are given in Section 2.

In Section 3 we furnish a variety of useful examples which show that in a general space we cannot expect similar properties to those met in finite dimensional spaces or in Baire metrizable spaces.

Whenever possible we keep the same notation as in [**23**] and [**24**].

1. Generalized directional derivatives and subgradients. Let E be a real locally convex (Hausdorff) space with a continuous dual space E^* and let f be an extended-real-valued function on E. For a point x at which the function f is finite and lower semicontinuous we consider the following generalized directional derivatives:

(a) *Lower one sided Hadamard derivative* of f at x with respect to y

Received August 15, 1982 and in revised form July 7, 1983. The research of the first author was partially supported on NSERC Grant A5116 and that of the second author by NSF Grant ECS 8300214.

https://doi.org/10.4153/CJM-1984-008-7 Published online by Cambridge University Press

$$(1.1)\quad f_{+}(x; y) := \liminf_{\substack{y' \to y \\ t \downarrow 0}} \frac{f(x + ty') - f(x)}{t},$$

(b) *upper subderivative* of f at x with respect to y (Rockafellar [**23**])

$$(1.2)\quad f^{\uparrow}(x; y) := \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \inf_{y' \to y} \frac{f(x' + ty') - f(x')}{t}$$

$$:= \sup_{Y \in \mathfrak{n}(y)} \inf_{\substack{X \in \mathfrak{n}(x) \\ \delta > 0 \\ \lambda > 0}} \sup_{\substack{t \in (0, \lambda) \\ x' \in X \\ f(x') \leqq f(x) + \delta}} \inf_{y' \in Y} \frac{f(x' + ty') - f(x')}{t},$$

(We use the notation:

$$x' \to_f x \Leftrightarrow x' \to x \text{ and } f(x') \to f(x);$$

$\mathfrak{n}(x)$ and $\mathfrak{n}(y)$ denote bases of neighbourhoods at x and y respectively),

(c) *Clarke derivative* of f at x with respect to y (Rockafellar [**23**], Clarke [**6**], [7] for locally Lipschitzian functions on normed spaces)

$$(1.3)\quad f^{\circ}(x; y) := \limsup_{\substack{x' \to_f x \\ t \downarrow 0}} \frac{f(x' + ty) - f(x')}{t}.$$

(d) Recall that the ordinary *one-sided directional derivative* of f at x with respect to y is defined as

$$(1.4)\quad f'(x; y) := \lim_{t \downarrow 0} \frac{f(x + ty) - f(x)}{t}$$

if the limit above (finite or not) exists.

For each type of the generalized derivative the corresponding *set* (possibly empty) *of subgradients of the function f at the point x* is defined by

$$\partial_{+} f(x) := \{z \in E^{*} | \langle y, z \rangle \leqq f_{+}(x; y) \text{ for all } y \in E\},$$

$$\partial^{\circ} f(x) := \{z \in E^{*} | \langle y, z \rangle \leqq f^{\circ}(x; y) \text{ for all } y \in E\},$$

$$\partial^{\uparrow} f(x) := \{z \in E^{*} | \langle y, z \rangle \leqq f^{\uparrow}(x; y) \text{ for all } y \in E\}.$$

It may be verified that the function $f_{+}(x; \cdot)$ is lower semicontinuous but generally not sublinear. The Clarke derivative $f^{\circ}(x; \cdot)$ is always sublinear ([7]) with

$$f^{\circ}(x; 0) = 0$$

but is not necessarily lower semicontinuous. The upper subderivative $f^{\uparrow}(x; \cdot)$ is sublinear and lower semicontinuous ([**24**]); therefore one of the following alternatives must hold

(i) $f^{\uparrow}(x; 0) = 0$ and $f^{\uparrow}(x; \cdot)$ is a proper function,

(ii) $f^{\uparrow}(x; 0) = -\infty$ and $f^{\uparrow}(x; \cdot)$ has no finite values.

This implies the following result proven in [**24**].

PROPOSITION 1.1. *The following are equivalent*

(a) $\partial^{\uparrow} f(x) = \emptyset$,

(b) $f^{\uparrow}(x; 0) = -\infty$.

Comparison of definitions (1.1), (1.2), (1.3) and (1.4) shows that for any y one has:

$$(1.5) \qquad f_{+}(x; y) \leqq f^{\uparrow}(x; y) \leqq f^{\circ}(x; y)$$

and

$$(1.6) \qquad f'(x; y) \leqq f^{\circ}(x; y).$$

In the case of a convex function much more can be said.

PROPOSITION 1.2. ([**24**]). *If f is convex, then the function $f'(x; \cdot)$ is sublinear* (*possibly with $\pm\infty$ values*) *and for all y*

$$(1.7) \qquad f^{\uparrow}(x; y) = f_{+}(x; y) = \liminf_{y' \to y} f'(x; y') : = \operatorname{cl} f'(x; y),$$

and

$$(1.8) \qquad \partial^{\uparrow} f(x) = \partial_{+} f(x) = \partial f(x)$$

$$: = \{z \in E^* | f(x') - f(x) \geqq \langle x' - x, z \rangle \text{ for all } x' \in E\}.$$

The function f is said to be *subdifferentially regular* at x ([**23**]) if $f^{\uparrow}(x; \cdot)$ and $f_{+}(x; \cdot)$ coincide.

Thus it follows from Proposition 1.2 that a convex function finite at x is subdifferentially regular at x ([**23**]).

A considerable simplification in the analysis of the subderivatives can be obtained in the case of a directionally Lipschitzian function. This concept generalizes Lipschitz continuity in a neighbourhood of a point.

The function f is said to be *directionally Lipschitzian at x with respect to y* ([**23**]) if

$$(1.9) \quad f^*(x; y) := \limsup_{\substack{x' \to_f x \\ y' \to y \\ t \downarrow 0}} \frac{f(x' + ty') - f(x')}{t}$$

is less than $+\infty$.

It is assumed throughout this paper that f is finite and lower semicontinuous at x; however, the latter assumption can be relaxed in many cases ([**23**], [**24**]) and is kept here only for the purpose of simplifying notation.

The following assertions can be found in [**24**].

PROPOSITION 1.3. *f is Lipschitzian around x if and only if it is directionally Lipschitzian at x with respect to $y = 0$.*

PROPOSITION 1.4. *Suppose f is convex on E. Then f is directionally Lipschitzian at x with respect to y if and only if f is bounded above on a neighbourhood of $x + \lambda y$ for some $\lambda > 0$.*

The function f is said to be *directionally Lipschitzian at x* if $\operatorname{dom} f^*(x; \cdot) \neq \emptyset$.

The following theorem explains why it is desirable for the function f to be directionally Lipschitzian at x.

THEOREM 1.1 [**24**]. *Let f be directionally Lipschitzian at x. Then*

(a) $f^\uparrow(x; y) = \liminf_{y' \to y} f^\circ(x; y')$ *for all y,*

(b) $\operatorname{dom} f^*(x; \cdot) = \operatorname{int} \operatorname{dom} f^\uparrow(x; \cdot) = \operatorname{int} \operatorname{dom} f^\circ(x; \cdot)$,

(c) $f^\circ(x; \cdot)$ *is continuous on* $\operatorname{dom} f^*(x; \cdot)$ *and*

$$f^\uparrow(x; y) = f^\circ(x; y) = f^*(x; y) \text{ for all } y \in \operatorname{dom} f^*(x; \cdot),$$

(d) $\partial^\uparrow f(x) = \partial^\circ f(x)$.

The proof of (a), (b), (c) can be found in [**24**], (d) follows from (a) and the corresponding definitions.

As the simple consequence of this theorem and Proposition 1.3 we obtain the following result.

COROLLARY 1.1. *If f is Lipschitzian around x then the function $f^\circ(x; \cdot)$ is continuous* ([**6**], [7], [**16**]) *and*

$$f^\uparrow(x; \cdot) = f^\circ(x; \cdot),$$

(*Rockafellar* [**24**]).

The next two results are implicit in [**24**]. The explicit functional proofs may be found in [**2**].

PROPOSITION 1.5. *Let E be a locally convex vector space. Suppose that f and g are finite and lower semicontinuous at x. Then*

(1.10) $(f + g)^{\uparrow}(x; \cdot) \leqq f^{\uparrow}(x; \cdot) + g^{*}(x; \cdot).$

PROPOSITION 1.6. *Let E be a locally convex vector space. Suppose that f is finite and lower semicontinuous at x. Then for any* y_1, y_2, *and* $0 < \lambda \leqq 1$

(1.11) $f^{*}(x; \lambda y_1 + (1 - \lambda)y_2) \leqq \lambda f^{*}(x; y_1) + (1 - \lambda)f^{\uparrow}(x; y_2).$

Remark. It may be noticed that from Proposition 1.6 the proof of Theorem 1.1 can be easily obtained. If $y_1 \in \operatorname{dom} f^{*}(x; \cdot)$, letting $\lambda \to 0$ in (1.11) we get for $y_2 = :y$

(1.12) $f^{\uparrow}(x; y) = \operatorname{cl} f^{\circ}(x; y) = \operatorname{cl} f^{*}(x; y).$

The following theorem stated and proven in [**23**] follows from Propositions 1.5 and 1.6 ([**2**]).

THEOREM 1.2 (Rockafellar [**23**], Clarke [**8**] for locally Lipschitzian functions on normed spaces). *Let E be a locally convex vector space and let f and g be finite and lower semicontinuous at x. Suppose that*

$$\operatorname{dom} f^{\uparrow}(x; \cdot) \cap \operatorname{dom} g^{*}(x; \cdot) \neq \emptyset,$$

then

$$(f + g)^{\uparrow}(x; \cdot) \leqq f^{\uparrow}(x; \cdot) + g^{\uparrow}(x; \cdot)$$

and

$$\partial^{\uparrow}(f + g)(x) \subset \partial^{\uparrow}f(x) + \partial^{\uparrow}g(x).$$

The next theorem generalizes the Brøndsted-Rockafellar theorem about the density of subdifferentiability points of a convex function ([**4**]). It was observed and proven in [**18**] and independently in [**2**].

THEOREM 1.3 ([**18**]). *Let f be a lower semicontinuous proper function on a Banach space E. Then the set*

$$\{x \in E \mid f^{\uparrow}(x; 0) = 0\}$$

is dense in dom *f*.

Example 3 in Section 3 shows that in this theorem the assumption that *E* is a Banach space cannot be released.

It was observed ([**12**], [**24**], [**27**]) that the generalized directional derivatives which we consider in this paper are related to some types of tangent cones.

(a) For any set $C \subset E$ and any point $x \in C$, *the (Clarke) tangent cone to C at x* (Rockafellar [**23**], Clarke [**5**] for $E = \mathbf{R}^n$) is defined by

$$(1.13)\quad T_C(x) := \bigcap_{V \in \mathfrak{n}(0)} \bigcup_{\substack{X \in \mathfrak{n}(x)\\ \lambda > 0}} \bigcap_{\substack{x' \in C \cap X\\ t \in (0,\lambda)}} [t^{-1}(C - x') + V].$$

(b) The set

$$(1.14)\quad K_C(x) := \bigcap_{\substack{V \in \mathfrak{n}(0)\\ \lambda > 0}} \bigcup_{t \in (0,\lambda)} [t^{-1}(C - x) + V]$$

is called *the contingent cone to C at x* ([**3**]).

(c) *The hypertangent cone to C at x* ([**24**]) is defined as

$$(1.15)\quad H_C(x) := \{y \in E \mid \text{there exist } X \in \mathfrak{n}(x) \text{ and } \lambda > 0 \text{ with } x' + ty \in C \text{ for all } x' \in C \cap X, t \in (0, \lambda)\}.$$

The more general definitions are given in [**9**] by Dolecki, who presents a unified approach to various types of tangent cones and generalized derivatives based on convergence theory.

PROPOSITION 1.7 ([**24**]). *If C is convex then*

$$(1.16)\quad T_C(x) = K_C(x)$$

and

$$(1.17)\quad K_C(x) = \overline{\mathbf{P}(C - x)}.$$

A set which satisfies (1.16) is called *tangentially regular at x*. The fundamental relationship between tangent cones and generalized directional derivatives is the following.

PROPOSITION 1.8 ([**24**], [**27**]). (a) *The tangent cone $T_{\mathrm{epi}\,f}((x, f(x)))$ is the epigraph of the function $f^{\uparrow}(x; \cdot)$.*

(b) *The contingent cone $K_{\mathrm{epi}\,f}((x, f(x)))$ is the epigraph of the function $f_+(x; \cdot)$.*

(c) $f^{\circ}(x; y) = \inf \{\beta \in \mathbf{R} \mid (y, \beta) \in H_{\mathrm{epi}\,f}((x, f(x)))\}$.

The following geometric property is related to the concept of a directionally Lipschitzian function.

A set C is said to be *epi-Lipschitzian at x with respect to y* ([**23**]), if there exist $Y \in \mathfrak{n}(y)$, $X \in \mathfrak{n}(x)$, $\lambda > 0$ with $x' + ty' \in C$ for all $x' \in C \cap X$, $y' \in Y$ and $t \in (0, \lambda)$.

A set C is said to be *epi-Lipschitzian at x* if it is epi-Lipschitzian at x with respect to some y.

PROPOSITION 1.9. (a) *f is directionally Lipschitzian at x with respect to y if and only if* epi f *is epi-Lipschitzian at $(x, f(x))$ with respect to (y, β) for some $\beta \in \mathbf{R}$.*

(b) *C is epi-Lipschitzian at x with respect to y if and only if the indicator function of C, denoted Ψ_C, is directionally Lipschitzian at x with respect to y.*

(c) *For any set C one has*:

(1.18) $(\Psi_C)^{\uparrow}(x; y) = \Psi_{T_C(x)}(y)$ *for all y,*

and

(1.19) $(\Psi_C)^{\circ}(x; y) = \Psi_{H_C(x)}(y)$ *for all y.*

Parts (a), (b) and the first equality in (c) can be found in [**27**]. Equality (1.19) can be proved by simple calculations.

Using Theorem 1.1 and Proposition 1.8 we get:

THEOREM 1.4 ([**24**]). *If C is epi-Lipschitzian at x with respect to some y, then the vectors y with this property are those belonging to* int $H_C(x)$ *and*

$$\text{int } T_C(x) = \text{int } H_C(x).$$

For the case of a finite dimensional space the following result was established.

THEOREM 1.5. ([**25**]). *Suppose E is finite dimensional and C is closed relative to some neighbourhood of x. Then C is epi-Lipschitzian at x with respect to y if and only if $y \in$* int $T_C(x)$.

From this theorem the following corollary can be obtained.

COROLLARY 1.2. *If E is finite dimensional and C is closed relative to some neighbourhood of $x \in C$ then*

$$\text{int } T_C(x) = \text{int } H_C(x).$$

In order to use Theorem 1.5 and Corollary 1.2 for a given function f and a point x at which f is finite, we assume that f is strictly lower semicontinuous at x.

A function f is said to be *strictly lower semicontinuous at* x if its epigraph is closed relative to some neighbourhood (in $E \times \mathbf{R}$) of the point $(x, f(x))$. This condition while stronger than lower semicontinuity at the point x is still weaker than the requirement for the function f to be lower semicontinuous on some neighbourhood of x (in E).

COROLLARY 1.3. *If E is finite dimensional and f is finite and strictly lower semicontinuous at x then*

$$(1.20) \quad \text{int dom } f^{\uparrow}(x; \cdot) = \text{int dom } f^{\circ}(x; \cdot) = \text{dom } f^{*}(x; \cdot).$$

THEOREM 1.6. ([**27**]). *Suppose E is finite dimensional and f is finite and strictly lower semicontinuous at x. Then the following are equivalent*:

(a) $y \in \text{int dom } f^{\uparrow}(x; \cdot)$,

(b) *f is directionally Lipschitzian at x with respect to y* ($y \in \text{dom } f^{*}(x; \cdot)$),

(c) $f^{\uparrow}(x; \cdot)$ *is continuous at y.*

Corollary 1.3 and Theorem 1.6 can be recovered from [**27**].

Now the question arises: could the same equivalencies be stated in the case of a general space? The answer is negative. The theorem is not true even for (infinite dimensional) Banach spaces. This is explained by Example 1 in Section 3.

It follows from Proposition 1.8 and Corollary 1.2 that in a finite dimensional space the upper subderivative of the function f being strictly lower semicontinuous at x can never be identically equal to $-\infty$. By Theorem 1.1 this statement is true in an infinite dimensional space under the assumption that the function f is directionally Lipschitzian at x. Without this assumption even an example of a convex function f in a Banach space with $f^{\uparrow}(x; \cdot) \equiv -\infty$ can be furnished. To observe this see Example 2 in Section 3.

While considering the conditions (a), (b), and (c) of Theorem 1.6 for an infinite dimensional space the question arises of substituting $f^{\circ}(x; \cdot)$ for $f^{\uparrow}(x; \cdot)$. This leads to the interesting results for the Baire metrizable spaces described in the next section. To observe that in an arbitrary space the condition (a), (b), (c) of Theorem 1.6 with $f^{\uparrow}(x; \cdot)$ replaced by $f^{\circ}(x; \cdot)$ are not equivalent see Examples 4, 5, 6, 7, 8, 9, 10, 11, and 12 in Section 3.

2. The Baire metrizable case. In addition to the assumptions made in the first paragraph in Section 1, let us assume that E is a metrizable space.

Suppose that $\{U_n | n \in \mathbf{N}\}$ is a base of neighbourhoods at zero in E and $U_{n+1} \subset U_n$ for $n \in \mathbf{N}$. Then clearly for any y

$$(2.1) \quad f^{\circ}(x; y) = \inf_{n \in \mathbf{N}} \sup_{\substack{0<t<2^{-n} \\ x' \in x+U_n \\ f(x') \leqq f(x)+2^{-n}}} \frac{f(x' + ty) - f(x')}{t}$$

and

$$(2.2) \quad f^{*}(x; y) = \inf_{n \in \mathbf{N}} \sup_{\substack{0<t<2^{-n} \\ x' \in x+U_n \\ f(x') \leqq f(x)+2^{-n} \\ y' \in y+U_n}} \frac{f(x' + ty') - f(x')}{t}.$$

In this section the theory presented in Section 1 is developed for Baire metrizable spaces (the Baire category theorem holds in these spaces). Let us recall that all Banach and Fréchet spaces are of this type. However the class of Baire metrizable spaces is wider. Indeed, it is easy to show that a linear space E is Baire if and only if E is second category in itself ([**28**]) and that incomplete dense subspaces of a Banach space exist with this property ([**28**], p. 29). It is an open question as to whether every finite codimension subspace of a Banach space is Baire.

THEOREM 2.1. *Let E be a Baire metrizable locally convex space. Suppose f is finite and strictly lower semicontinuous at x. Then the following are equivalent*:

(a) $y \in \operatorname{cor} \operatorname{dom} f^{\circ}(x; \cdot)$,

(b) *f is directionally Lipschitzian at x with respect to y*,

(c) *$f^{\circ}(x; \cdot)$ is continuous at y.*

Proof. Condition (b) always implies (regardless of the assumption about the Baire metrizable setting) that $f^{\circ}(x; \cdot)$ is bounded above on some neighbourhood of y as a comparison of (1.3) and (1.9) shows. Hence, being convex, it is continuous at y. Since (c) obviously implies (a), it is enough to show that (a) implies (b).

Let us pick a closed neighbourhood X of x and $\epsilon > 0$ such that epi f is closed relative to

$$X \times [f(x) - \epsilon, f(x) + \epsilon].$$

For n in $\mathbf{N}$, let

$$(2.3) \quad m_n(y') := \sup_{\substack{0<t<2^{-n} \\ x' \in x+U_n \\ f(x') \leqq f(x)+2^{-n}}} \frac{f(x' + ty') - f(x')}{t},$$

and let

(2.4) $M_n := \{y' \in E | x + U_n + ty' \subset X \text{ for } 0 < t < 2^{-n} \text{ and }$
$$m_n(y') \leqq 1\}.$$

Then by the definitions (2.3), (2.4), and (2.1) and by simple calculations we get:

(2.5) $$\operatorname{dom} f^{\circ}(x; \cdot) = \bigcup_{n \in \mathbf{N}} \bigcup_{k \in \mathbf{N}} kM_n = \bigcup_{n > \tilde{n}} \bigcup_{k \in \mathbf{N}} kM_n,$$

where $\tilde{n}$ is any natural number. Since f is lower semicontinuous at x we may assume that X was chosen such that the condition

(2.6) $f(x) - \epsilon < f(x')$ for all $x' \in X$

is satisfied. Let us pick $\tilde{n}$ such that $\epsilon > 2^{1-\tilde{n}}$. We claim that all sets M_n for $n > \tilde{n}$ are closed. Indeed let $y' \in \operatorname{cl} M_n$, where $n > \tilde{n}$. Take any x' and t such that:

(2.7) $x' \in x + U_n$,

(2.8) $0 < t < 2^{-n}$,

(2.9) $f(x') < f(x) + 2^{-n}$.

Let U be an arbitrary neighbourhood of zero. Since $y' \in \operatorname{cl} M_n$ there exists y'' such that

(2.10) $y'' \in (y' + (1/t)U) \cap M_n$.

Hence

(2.11) $x' + ty'' \in x' + ty' + U$, and $x' + ty'' \in X$.

By definitions (2.3) and (2.4) it follows from (2.7), (2.8), (2.9) that

(2.12) $f(x' + ty'') \leqq f(x') + t$.

Using now (2.6), (2.12), (2.9), and (2.8) we get:

$$f(x) - \epsilon < f(x' + ty'') \leqq f(x') + t$$
$$< f(x) + 2^{1-n} < f(x) + \epsilon.$$

Hence

$$(x' + ty'', f(x') + t) \in \operatorname{epi} f \cap (X \times [f(x) - \epsilon, f(x) + \epsilon]).$$

Since U is arbitrary, X is closed and epi f is closed relative to $X \times [f(x) - \epsilon, f(x) + \epsilon]$ we get that for all x' satisfying (2.7) and (2.9) and for all t satisfying (2.8)

(2.13) $\quad x' + ty' \in X$

and

(2.14) $\quad (x' + ty', f(x') + t) \in \text{epi}\, f.$

(2.13) together with (2.14) means that $y' \in M_n$. This establishes our claim above.

Let us assume that condition (a) is satisfied. Then

$$0 \in \text{cor}\,(\text{dom}\, f^{\circ}(x; \cdot) - y)$$

and hence

$$E = \bigcup_{p \in \mathbf{N}} p(\text{dom}\, f^{\circ}(x; \cdot) - y).$$

This together with 2.5 implies that E can be represented as the union of the sets

$$p(kM_n - y) \quad p, k, n \in \mathbf{N}, n > \tilde{n},$$

which by closedness of the corresponding sets M_n, are also closed. Since E is a Baire space, one of them has a nonempty interior. Hence also some $M_{\hat{n}}$ for $\hat{n} > \tilde{n}$ has a nonempty interior and clearly all M_n for $n \geqq \hat{n}$ have nonempty interiors. Take $n \geqq \hat{n}$ and $\hat{y}$ such that $\hat{y} + U_n \subset M_n$, then

(2.15) $$\sup_{\substack{0<t<2^{-n} \\ f(x') \leqq f(x)+2^{-n} \\ x' \in x+U_n \\ y' \in \hat{y}+U_n}} \frac{f(x' + ty') - f(x')}{t} \leqq 1.$$

Comparison of (2.15) with (2.2) gives

$$f^*(x; \hat{y}) \leqq 1.$$

Hence dom $f^*(x; \cdot)$ is not empty and it follows from Theorem 1.1 that:

$$\text{int dom}\, f^{\circ}(x; \cdot) = \text{dom}\, f^*(x; \cdot).$$

This implies that f is directionally Lipschitzian at x with respect to y, which completes the proof of the theorem.

Next some consequences of Theorem 2.1 are presented.

COROLLARY 2.1. *Let E be a Baire metrizable locally convex space. Suppose that f is lower semicontinuous at x and g is strictly lower semicontinuous at x. Let us suppose also that*

$$\operatorname{dom} f^{\uparrow}(x; \cdot) \cap \operatorname{cor} \operatorname{dom} g^{\circ}(x; \cdot) \neq \emptyset.$$

Then

(i) $(f + g)^{\uparrow}(x; \cdot) \leqq f^{\uparrow}(x; \cdot) + g^{\uparrow}(x; \cdot)$,
(ii) $\partial^{\uparrow}(f + g)(x) \subset \partial^{\uparrow}f(x) + \partial^{\uparrow}g(x)$.

Proof. The result follows from Theorem 1.2 via Theorem 2.1.

COROLLARY 2.2. *Let E be a Baire metrizable locally convex vector space and let a set C be closed relative to some neighbourhood of $x \in C$.*
Suppose that $y \in \operatorname{cor} H_C(x)$. Then C is epi-Lipschitzian at x with respect to y and

$$\operatorname{cor} H_C(x) = \operatorname{int} H_C(x) = \operatorname{int} T_C(x).$$

In particular $H_C(x) = E$ if and only if x lies interior to C.

Proof. The conclusion is obtained by applying Theorem 2.1 to the indicator function of C and using Proposition 1.8 and Proposition 1.9.

COROLLARY 2.3. *Let E be a Baire metrizable locally convex vector space. Let $x \in C_1 \cap C_2$. Suppose C_2 is closed relative to some neighbourhood of x and*

$$T_{C_1}(x) \cap \operatorname{cor} H_{C_2}(x) \neq \emptyset.$$

Then

$$T_{C_1 \cap C_2}(x) \supset T_{C_1}(x) \cap T_{C_2}(x).$$

Proof. The result follows by using indicators $f: = \Psi_{C_1}$ and $g: = \Psi_{C_2}$ in Corollary 2.1.

From Theorem 2.1 the conditions which characterize a function which is Lipschitzian around x can be obtained. First let us observe that in a finite dimensional space the following implications are valid.

PROPOSITION 2.1. ([**27**]). *Suppose E is finite dimensional and f is finite and strictly lower semicontinuous at x. Then the following are equivalent*:
(a) *f is Lipschitzian around x*;
(b) $f^{\circ}(x; \cdot) < +\infty$;
(c) *$f^{\uparrow}(x; \cdot)$ is finite*;
(d) *$\partial^{\uparrow}f(x)$ is bounded and nonempty.*

Examples in Section 3 show that in an infinite dimensional space the above conditions are not equivalent. However, in a general space the following assertions can be easily proved.

PROPOSITION 2.2. *Let E be a locally convex vector space and let f be finite and lower semicontinuous at x. Then*

(a) *If f is Lipschitzian around x then $f^{\circ}(x; \cdot)$ is continuous;*

(b) *If $f^{\circ}(x; \cdot)$ is continuous then $f^{\uparrow}(x; \cdot) \equiv -\infty$ or $f^{\uparrow}(x; \cdot)$ is also a continuous function;*

(c) *If $f^{\uparrow}(x; \cdot)$ is continuous on E then $\partial^{\uparrow} f(x)$ is nonempty and weak* compact;*

(d) *$\partial^{\uparrow} f(x)$ is nonempty and weak* bounded if and only if $f^{\uparrow}(x; \cdot)$ is finite everywhere;*

(e) *If E is a barreled space, then the following are equivalent:*

(i) *the set $\partial^{\uparrow} f(x)$ is nonempty and weak* compact,*

(ii) *$f^{\uparrow}(x; \cdot)$ is finite.*

Proof. (a) is the part of Corollary 1.1.

Let us assume that $f^{\circ}(x; \cdot)$ is continuous. Then

$$f^{\uparrow}(x; \cdot) \leqq f^{\circ}(x; \cdot) < +\infty. \tag{2.16}$$

From (2.16) and from convexity of $f^{\uparrow}(x; \cdot)$ follows that either it is identically equal to $-\infty$ or it must be everywhere finite. In the latter case by the inequality (2.16) the convex finite function $f^{\uparrow}(x; \cdot)$ is majorized by the continuous function $f^{\circ}(x; \cdot)$. Therefore $f^{\uparrow}(x; \cdot)$ is also continuous. This finishes the proof of (b).

To observe that (c) is true let us notice that if $f^{\uparrow}(x; \cdot)$ is continuous then by Proposition 1.1 $\partial^{\uparrow} f(x)$ is not empty and furthermore:

$$\partial^{\uparrow} f(x) \subset V^0 = \{z \in E^* | \sup_{v \in V} \langle v, z \rangle \mid \leqq 1\},$$

where

$$V = \{v \in E | f^{\uparrow}(x; v) | \leqq 1\}$$

is closed, convex neighbourhood of zero. Therefore the polar V^0 is weak* compact and so is $\partial^{\uparrow} f(x)$ being a weak* closed subset in V^0.

Part (d) follows from the formula (**[24]**)

$$f^{\uparrow}(x; y) = \sup_{z \in \partial^{\uparrow} f(x)} \langle y, z \rangle.$$

Equivalence in (e) is justified by (c) and the fact that in any barreled space every finite and lower semicontinuous convex function is continuous. This finishes the proof.

Remark. From Example 5(A) in Section 3 it follows that without the assumption that E is a barreled space part (e) in Proposition 2.2 does not hold. This shows that hypothesis that E is barreled must be added in the first paragraph on p. 275 and in Corollary 2 of [**24**] below it.

For a Baire metrizable space we are able to establish much stronger conditions than those from Proposition 2.2.

THEOREM 2.2. *Let E be a Baire metrizable locally convex space and suppose that f is finite and strictly lower semicontinuous at x. Then the following are equivalent*:

(a) $f^{\circ}(x; \cdot)$ *is finite*,

(b) f *is Lipschitzian around* x,

(c) $f^{\uparrow}(x; \cdot) = f^{\circ}(x; \cdot)$ *and* $f^{\uparrow}(x; \cdot)$ *is continuous*,

(d) $f^{\uparrow}(x; \cdot) = f^{\circ}(x; \cdot)$ *and* $\partial^{\uparrow}f(x)$ *is nonempty and weak* compact* (*weak* bounded*),

(e) $f^{\circ}(x; \cdot)$ *is lower semicontinuous and* $\partial^{\circ}f(x)$ *is nonempty and weak* compact* (*weak* bounded*),

(f) $f^{\circ}(x; \cdot)$ *is lower semicontinuous and finite*,

(g) $f^{\circ}(x; \cdot)$ *is continuous.*

Proof. It follows from (a) that $\operatorname{dom} f^{\circ}(x; \cdot) = E$, hence $0 \in \operatorname{int} \operatorname{dom} f^{\circ}(x; \cdot)$. Using Theorem 2.1 and Proposition 1.3 we get (b). By Corollary 1.1 (b) implies (c). From Proposition 2.2 (c) we obtain implication (c) $\Rightarrow$ (d). Since $f^{\uparrow}(x; \cdot)$ is lower semicontinuous we have (d) $\Rightarrow$ (e) $\Rightarrow$ (f). Implication (f) $\Rightarrow$ (g) is valid in any barreled space. (g) obviously implies (a), and the proof is complete.

Using Theorem 2.1 and the concept of polarity we can get the other formulation of conditions which characterize directionally Lipschitzian functions defined on a Baire metrizable space.

First let us recall that for any convex cone K its *polar cone* K^0 is defined as

$$K^0 := \{z \in E^* | \langle k, z\rangle \leqq 0 \text{ for all } k \in K\}.$$

Let us also state the known fact that

(2.17) $\quad (K^0)^0 = \operatorname{cl} K.$

https://doi.org/10.4153/CJM-1984-008-7 Published online by Cambridge University Press

COROLLARY 2.4. *Let E be a Baire metrizable locally convex vector space and let f be finite and strictly lower semicontinuous at x. Then the following are equivalent*:

(a) *f is directionally Lipschitzian at x with respect to y,*

(b) (i) *for all y' from some neighbourhood $Y \in \mathfrak{n}(y)$ one has*

$$\langle y', z\rangle \leqq 0 \text{ for all } z \in (\operatorname{dom} f^{\circ}(x; \cdot))^0$$

(ii) $\operatorname{int} \operatorname{cl} \operatorname{dom} f^{\circ}(x; \cdot) = \operatorname{int} \operatorname{dom} f^{\circ}(x; \cdot)$.

Proof. From (i) and (2.17) applied for $\operatorname{dom} f^{\circ}(x; \cdot)$ it follows that

$$Y \subset \operatorname{cl} \operatorname{dom} f^{\circ}(x; \cdot),$$

hence by (ii)

$$y \in \operatorname{int} \operatorname{dom} f^{\circ}(x; \cdot).$$

Now from Theorem 2.1 we get that f is directionally Lipschitzian at x with respect to y. On the other hand if f is directionally Lipschitzian at x with respect to y, then

$$y \in \operatorname{int} \operatorname{dom} f^{\circ}(x; \cdot)$$

and conditions (i) and (ii) are satisfied. This completes the proof.

COROLLARY 2.5. *Let E be a Baire metrizable locally convex space. Then the following are equivalent*:

(a) *f is Lipschitzian around x,*

(b) (i) *f is finite and lower semicontinuous at x,*

(ii) $\operatorname{int} \operatorname{cl} \operatorname{dom} f^{\circ}(x; \cdot) = \operatorname{int} \operatorname{dom} f^{\circ}(x; \cdot)$,

(iii) $(\operatorname{dom} f^{\circ}(x; \cdot))^0 = 0$.

Proof. Apply Corollary 2.4 and Proposition 1.3.

Remark. If E is a finite dimensional space $\operatorname{dom} f^{\circ}(x; \cdot)$ in Corollary 2.4 and Corollary 2.5 can be replaced by $\operatorname{dom} f^{\uparrow}(x; \cdot)$ ([**26**]). In this case condition (ii), being always satisfied, can be omitted. To observe that it cannot be omitted when E is infinite dimensional see Example 1 in Section 3.

3. Examples. Our goal is to find limiting examples for the theory developed in Sections 1 and 2. Whenever possible we require them to be convex. As Propositions 1.2 and 1.4 show this assumption makes our results stronger.

Example 1. We use the function given by Rockafellar in [**25**].

(A) Let $E := l^2$. For $x := (x_n) \in l^2$ let

$$f(x) := \sum_{n=1}^{\infty} n x_n^2.$$

For $x: = 0$ we get:

(a) E is a Banach space, f is convex, lower semicontinuous, finite at x,

(b) $f^{\uparrow}(x; \cdot)$ is continuous, $\partial^{\uparrow}f(x)$ consists of a single element, $\partial^{\circ}f(x) = \partial^{\uparrow}f(x)$,

(c) $(\operatorname{dom} f^{\circ}(x; \cdot))^{0} = 0$,

(d) f is not directionally Lipschitzian at x.

Compare. Corollary 1.3, Theorem 1.6, Proposition 2.1, Corollary 2.5.

Proof. Let us observe that for any y

$$(3.1) \quad f^{\circ}(0; y) = f'(0; y) = \begin{cases} 0 \text{ if } \sum_{n=1}^{\infty} n y_n^2 < +\infty, \\ +\infty \text{ otherwise.} \end{cases}$$

Using (3.1) one gets

$$f^{\uparrow}(0; \cdot) = \operatorname{cl} f'(0; \cdot) = \operatorname{cl} f^{\circ}(0; \cdot) \equiv 0.$$

Hence

$$\partial^{\circ}f(0) = \partial^{\uparrow}f(0) = \{0\}.$$

This proves (b). It can be easily noticed that

$$\operatorname{cl} \operatorname{dom} f^{\circ}(0; \cdot) = E,$$

which implies (c). Since f is not bounded on any neighbourhood, it follows from Proposition 1.4 that f is not directionally Lipschitzian at zero.

(B) Let $E: = l^2 \times \mathbf{R}$, $C: = \operatorname{epi} f$, $x: = (0, 0)$.

By Proposition 1.8(a) and 1.9(a) we get:

(a) E is a Banach space, C is closed, and convex,

(b) $\operatorname{int} T_C(x) \neq \emptyset$, $\operatorname{int} H_C(x) = \emptyset$

(c) C is not epi-Lipschitzian at x.

Compare. Theorem 1.5, Corollary 1.2.

Example 2. Let E be any infinite dimensional separable Banach space. Let K be a symmetric compact convex set whose core is empty but whose span (sp K) is dense ([**1**]).

(A) Let $\hat{x} \notin \operatorname{sp} K$. Consider the function

$$(3.2) \quad f(x): = \min \{\lambda \in \mathbf{R} | x + \lambda\hat{x} \in K\},$$

where the minimum is taken to be $+\infty$ when no such λ exists.

Let us notice that by the definition (3.2) one has

https://doi.org/10.4153/CJM-1984-008-7 Published online by Cambridge University Press

(3.3) $\quad f(s\hat{x}) = -s$ for any $s \in \mathbf{R}$,

(3.4) $\quad f(x) = 0$ for any $x \in K$,

(3.5) $$f'(0; y) = \begin{cases} -s \text{ if } y = \alpha k + s\hat{x} \text{ for some } k \in K, \alpha, s \in \mathbf{R}, \\ +\infty \text{ otherwise.} \end{cases}$$

For $x: = 0$ we get:

(a) E is an infinite dimensional Banach space, f is convex, lower semicontinuous, finite at x,

(b) $f^{\uparrow}(x; \cdot) \equiv -\infty$ on E,

(c) f is not directionally Lipschitzian at x.

Compare. Theorem 1.6.

Proof. Let $y \in K$. By (1.7) and (3.5) we obtain

$$f^{\uparrow}(0; y) \leqq f'(0; y) = 0$$

and since $f^{\uparrow}(0; \cdot)$ is sublinear and K is symmetric we have

$$f^{\uparrow}(0; y) \leqq 0 \text{ for all } y \in \text{sp } K.$$

Lower semicontinuity of the function $f^{\uparrow}(0; \cdot)$ implies now that

(3.6) $\quad f^{\uparrow}(0; y) \leqq 0$ for all $y \in \overline{\text{sp } K} = E$.

Furthermore: $\partial^{\uparrow}f(0) = \emptyset$. Indeed, suppose to the contrary that $z \in \partial^{\uparrow}f(0)$ for some $z \in E^*$. Using (1.8) we get

(3.7) $\quad \langle x, z \rangle \leqq f(x)$ for all $x \in E$.

Hence by (3.4)

$$\langle x, z \rangle \leqq 0 \text{ for all } x \in \overline{\text{sp } K} = E.$$

Thus $z = 0$. But this is impossible, because for $x: = s\hat{x}$ with $s > 0$ we have by (3.3) and (3.7)

$$\langle x, z \rangle \leqq -s.$$

Now Proposition 1.1 yields: $f^{\uparrow}(0; 0) = -\infty$ and since $f^{\uparrow}(0; \cdot)$ is a convex function which by (3.6) is nowhere $+\infty$, it must be identically equal to $-\infty$.

Let us also observe that it follows from (3.5) that

$$f^{\circ}(0; y) = +\infty \text{ for any } y \notin \text{sp }(K, \hat{x}).$$

Hence int dom $f^{\circ}(0; \cdot) = \emptyset$ and by Theorem 2.1 the function f is not directionally Lipschitzian. The proof is finished.

(B) Let $C: = \operatorname{epi} f$, $x: = (0, 0)$. Then in a similar way as in Example 1 we get:

(a) E is an infinite dimensional Banach space, C is convex, closed.

(b) $T_C(x) = E$, but x does not lie in int C and $H_C(x) \neq E$.

Compare. Theorem 1.5, Corollary 2.2.

Remark. If E is finite dimensional it follows from Theorem 1.5 that

$$T_C(x) = E \Leftrightarrow x \in \operatorname{int} C.$$

Hence the situation described in the example cannot happen in a finite dimensional space.

Before the next example, let us observe the following general equivalence.

PROPOSITION 3.1. *Let E be a locally convex topological vector space and let f be a convex function which is finite and lower semicontinuous at x. Then the following are equivalent*:

(a) $T_{\operatorname{dom} f}(x) = E$ *and* $\partial^{\uparrow} f(x) = \emptyset$,

(b) $T_{\operatorname{epi} f}(x, f(x)) = E \times \mathbf{R}$.

Proof. It follows from Proposition 1.7 that

$$(3.8) \qquad T_{\operatorname{dom} f}(x) = \overline{\mathbf{P}(\operatorname{dom} f - x)},$$

$$(3.9) \qquad T_{\operatorname{epi} f}(x, f(x)) = \overline{\mathbf{P}(\operatorname{epi} f - (x, f(x)))}.$$

Assume that (a) is satisfied but (b) does not hold. Then by the separation theorem there exists a convex functional $(z, \alpha) \in E^* \times \mathbf{R}$ such that

$$(3.10) \qquad \langle y\ z \rangle + \alpha\gamma \geqq 0 \text{ for all } (y, \gamma) \in T_{\operatorname{epi} f}(x, f(x)).$$

Combining (3.9) and (3.10) we get

$$(3.11) \qquad \langle x' - x, z \rangle + \alpha(f(x') - f(x)) \geqq 0 \text{ for all } x' \in \operatorname{dom} f.$$

Notice that α cannot be zero, otherwise

$$\langle x' - x, z \rangle \leqq 0 \text{ for all } x' \in \operatorname{dom} f,$$

which together with (3.8) and the first equality in (a) implies that $z = 0$. But this is impossible, therefore it follows from (3.11) that $-z/\alpha \in \partial^{\uparrow} f(x)$, which contradicts (a). This proves that (a) implies (b).

Let us assume that (b) holds. Then by Proposition 1.1 $\partial^{\uparrow} f(x) = \emptyset$. By (b), (3.8) and (3.9) we get the first part of (a) and the proof is finished.

Example 3. Let E be a Fréchet space which contains nonempty bounded closed convex set K with no support points ([**20**]). Following an example given in [**13**] we choose any nonzero $\hat{x} \in E$ and define a convex function by

$$f(x) := \min \{\lambda \in \mathbf{R} | x + \lambda\hat{x} \in K\}.$$

Similar functions were investigated in [**4**] for some other supportless closed convex sets: one of them constructed in some incomplete inner product space ([**15**]), the other one (not bounded) in some Fréchet space ([**14**]). Using any of the described functions we obtain:

(a) f is lower semicontinuous, proper, convex on some Fréchet space (or some incomplete inner product space) E,
(b) $f^{\uparrow}(x, \cdot) \equiv -\infty$ for all $x \in \operatorname{dom} f$.

Compare. Theorem 1.3.

Proof. (a) easily follows from the properties of the set K ([**13**], [**4**]). Furthermore, much as in [**4**] it may be proved that

(3.12) $\quad \partial^{\uparrow} f(x) = \emptyset$ for all $x \in \operatorname{dom} f$.

We give the proof of (b). First, observe that by the definition of f

$$f(x) \leqq 0 \text{ for all } x \in K;$$

$$x + f(x)\hat{x} \in K \text{ for all } x \in \operatorname{dom} f.$$

Hence

(3.13) $\quad \operatorname{dom} f = K + \mathbf{R}\hat{x}$

and

(3.14) $\quad f(x + s\hat{x}) = f(x) - s$ for all $x \in E$ and $s \in \mathbf{R}$.

Furthermore

$$T_K(x) = E \text{ for all } x \in K.$$

Indeed, if for some $x \in K$ $T_K(x) \neq E$, by a separation argument and Proposition 1.7 we would obtain the existence of some nonzero $z \in E^*$ such that

$$\langle x' - x, z \rangle \leqq 0 \text{ for all } x' \in K.$$

But this is impossible since K is supportless. Hence by (3.13) also

$$T_{\operatorname{dom} f}(x) = E.$$

Therefore, taking into account (3.12), we see that for all $x \in K$ condition (a) of Proposition 3.1 holds, which implies that

(3.15) $f^{\uparrow}(x; \cdot) \equiv -\infty$ if $x \in K$.

Take now any $x \in \operatorname{dom} f$. By (3.13)

(3.16) $x = \bar{x} + \alpha\hat{x}$ for some $\bar{x} \in K$ and some $\alpha \in \mathbf{R}$.

Using (3.16) and (3.14) we get for any y'

$$(3.17)\quad f'(x; y') = \lim_{t\downarrow 0} \frac{f(\alpha\hat{x} + \bar{x} + ty') - f(\alpha\hat{x} + \bar{x})}{t}$$

$$= \lim_{t\downarrow 0} \frac{f(\bar{x} + ty') - f(\bar{x})}{t} = f'(\bar{x}; y').$$

Since f is convex it follows from (1.7), (3.17), and (3.15) that for any y

$$f^{\uparrow}(x; y) = \liminf_{y'\to y} f'(x; y') = \liminf_{y'\to y} f'(\bar{x}; y')$$

$$= f^{\uparrow}(\bar{x}; y) = -\infty.$$

Hence $f^{\uparrow}(x; \cdot) \equiv -\infty$ for all $x \in \operatorname{dom} f$, and the proof is complete.

Remark. Note that Example 3 shows that not only (3.12) holds, which was observed in [**4**], but also what is more, that $f^{\uparrow}(x; \cdot) \equiv -\infty$ for all $x \in \operatorname{dom} f$.

Example 4. (A) Let

$$E: = \{x \in l^{\infty} | \text{ support of } x \text{ is finite}\}$$

and for $x: = (x_n) \in E$ let

$$f(x): = \max \{2^n x_n | n \in \mathbf{N}\}.$$

Taking $x: = e^0: = (1, 0, 0 \dots)$ we get:

(a) E is a non-Baire normed space, f is convex, lower semicontinuous, finite,

(b) $f^{\circ}(x; \cdot)$ is continuous,

(c) $f^{\uparrow}(x; \cdot)$ is continuous,

(d) $f^{\circ}(x; \cdot) \neq f^{\uparrow}(x; \cdot)$; $\emptyset \neq \partial^{\uparrow} f(x) \subsetneq \partial^{\circ} f(x)$,

(e) f is not directionally Lipschitzian at x.

Compare. Corollary 1.3, Theorem 1.6, Theorem 2.1, Theorem 2.2, Proposition 2.1.

Proof. Let $y = (y_n) \in E$ and suppose

$$\operatorname{supp} y \subset \{0, 1, \dots m\}.$$

Pick $\epsilon > 0$ such that for all $x' = (x'_n)$ and t satisfying:

$$\|x' - e^0\| < \epsilon,\ 0 < t < \epsilon,$$

the following condition holds

$$(3.18)\quad 2^n(x'_n + ty_n) \leqq (x'_0 + ty_0) \quad \text{for all } n \leqq m.$$

Then the ratio

$$(3.19)\quad \frac{f(x' + ty) - f(x')}{t}$$

is not greater than zero, if

$$\max \{2^n(x'_n + ty_n) | n \in \mathbf{N}\}$$

is attained for $n > m$, or it is not greater than

$$\frac{\max \{2^n(x'_n + ty_n) | n \leqq m\} - \max \{2^n x'_n | n \in \mathbf{N}\}}{t},$$

otherwise. Applying now (3.18) we get that the ratio (3.19) is not greater than $\max \{0, y_0\}$. This implies that

$$(3.20)\quad f^{\circ}(e^0; y) \leqq \max \{0, y_0\}.$$

For $k \in \mathbf{N}$ let

$$e^k: = (0, 0, \dots 0, \overset{k}{1}, 0, 0, \dots) \quad \text{and}$$

$$x^k: = e^0 + 2^{-k}e^k.$$

Then for $k > m$ and $t > 0$ sufficiently small

$$\frac{f(x^k + ty) - f(x^k)}{t} = \frac{\max \{1 + ty_0, 1\} - 1}{t}$$

$$= \max \{0, y_0\}.$$

Since $x^k \to 0, f(e^0) = 1$ and $f(x^k) = 1$ for all k, we get

$$f^{\circ}(e^0; y) \geqq \max \{0, y_0\}.$$

This and (3.20) imply

$$f^{\circ}(e^0; y) = \max \{0, y_0\}.$$

Hence $f^{\circ}(e^0; \cdot)$ is a continuous function. However f is convex and it is not bounded on any neighbourhood. By Proposition 1.4, f is not directionally Lipschitzian at e^0.

To prove (c) let us notice that for any $y \in E$ and $t > 0$ sufficiently small

$$f(e^0 + ty) = 1 + ty_0.$$

Furthermore

$$f^{\uparrow}(e^0; y) = \liminf_{y' \to y} f'(e^0; y') = \liminf_{y' \to y} y'_0 = y_0.$$

Obviously $f^{\uparrow}(e^0; \cdot)$ is a continuous (linear) function and

$$f^{\uparrow}(e^0; \cdot) \neq f^{\circ}(e^0; \cdot).$$

This finishes the proof.

(B) Let us consider $C: = \text{epi} f$ and $x: = (e^0, 1)$. We get:
(a) E is a non-Baire normed space, C is convex, closed,
(b) $\text{int } H_C(x) \neq \emptyset$,
(c) $\text{int } T_C(x) \neq \emptyset$,
(d) $\text{int } H_C(x) \neq \text{int } T_C(x)$,
(e) C is not epi-Lipschitzian at x.
Compare. Theorem 1.5, Corollary 1.2, Corollary 2.2.

Example 5. (A) Let E be any non-barreled space (hence a non-Baire space). We can find a set $C \subset E$ such that C is convex and symmetric, weak*-closed and bounded, but not equicontinuous ([**28**]). Consider the function f defined on E as

$$f(x): = \sup_{z \in C} \langle x, z \rangle.$$

Let $x: = 0$, we get:
(a) E is any non-barreled space, f is convex, lower semicontinuous and finite,
(b) $f^{\uparrow}(x; \cdot) = f_{+}(x; \cdot) = f'(x; \cdot) = f^{\circ}(x; \cdot) = f$,
(hence also $f^{\circ}(x; \cdot)$ is lower semicontinuous and finite),
(c) $f^{\uparrow}(x; \cdot)$ is finite, $\partial^{\uparrow} f(x)$ is non-empty but not weak* compact,
(d) f is not directionally Lipschitzian at x.

Compare. Proposition 2.2(e), Theorem 2.2.

Proof. Obviously f is sublinear and as the "sup" of continuous functions it is lower semicontinuous. Since $f(0) = 0$, f must be a proper function. But C is weak* bounded, hence f is also nowhere $+\infty$. Now by Proposition 1.2 and lower semicontinuity of f we have for any y

$$(3.21)\quad f^{\uparrow}(0; y) = f_{+}(0; y) = \liminf_{y' \to y} f'(0; y')$$
$$= \liminf_{y' \to y} f(y') = f(y).$$

From sublinearity of f it follows that

$$(3.22)\quad f^{\circ}(0; y) \leqq f(y) \quad \text{for any } y.$$

(3.21) and (3.22) imply (b). Since $f^{\uparrow}(0; \cdot)$ is finite, $\partial^{\uparrow} f(0)$ is not empty and

$$(3.23)\quad f^{\uparrow}(0; y) \sup_{z \in \partial^{\uparrow} f(0)} \langle y, z \rangle.$$

(3.21) and (3.23) and definition of f imply

$$\partial^{\uparrow} f(0) = C.$$

But C is not equicontinuous, hence $\partial^{\uparrow} f(0)$ is not weak* compact and f is nowhere continuous. By Proposition 1.4 it is not directionally Lipschitzian at x.

(B) Let us consider $C: = \text{epi} f$ and $x: = (0, 0)$. We get:

(a) E is any non-barreled space, C is a closed convex subset of $E \times \mathbf{R}$,

(b) $K_C(x) = T_C(x) = H_C(x) = C$,

(c) C is not epi-Lipschitzian at x.

A large variety of interesting examples can be obtained by considering an infinite dimensional Banach space and exploring properties of a function such as the norm considered in the weak (or weak*) topology.

Let $\widetilde{E}$ be an infinite dimensional Banach space and let E denote $\widetilde{E}$ considered in its weak topology. Let f be defined on E as

$$f(x): = \|x\|.$$

Then for any x and any y we have

$$(3.24)\quad f^{\uparrow}(x; y) = f'(x; y) \leqq f^{\circ}(x; y) \leqq \|y\|.$$

The first equality is due to Proposition of 1.2 and to the fact that $f^{\uparrow}(x; \cdot)$ is convex and continuous on $\widetilde{E}$, hence it is weakly lower semicontinuous on

$\tilde{E}$. The final inequality is a consequence of a sublinearity of f. (3.24) also shows that f and $f^{\circ}(x; \cdot)$ are finite and lower semicontinuous on E (weakly lower semicontinuous on $\tilde{E}$). However f is nowhere continuous on E and therefore it is not directionally Lipschitzian at any x.

One interesting thing about this function is that while $f^{\uparrow}(x; \cdot)$ and $f^{\circ}(x; \cdot)$ may be not continuous, the sets $\partial^{\uparrow}f(x)$ and $\partial^{\circ}f(x)$ are always non-empty and weak* compact. To see this let us observe that since $f^{\circ}(x; \cdot)$ is a lower semicontinuous, proper, sublinear function we have that $\partial^{\circ}f(x) \neq \emptyset$ and

$$f^{\circ}(x; y) = \sup_{z \in \partial^{\circ}f(x)} \langle y, z \rangle.$$

Clearly the similar formula for $f^{\uparrow}(x; \cdot)$ also holds, because by Proposition 1.1 and the first equality in (3.24) $\partial^{\uparrow}f(x) \neq \emptyset$. By (3.24) and "sup" formulas for $f^{\circ}(x; \cdot)$ and $f^{\uparrow}(x; \cdot)$, $\partial^{\uparrow}f(x)$ and $\partial^{\circ}f(x)$ are bounded in norm topology of E^* (which is the same as $\tilde{E}^*$). Since $\partial^{\uparrow}f(x)$ and $\partial^{\circ}f(x)$ are weak* closed and norm bounded they are also weak* compact.

Example 6. Let $\tilde{E}$, E and f be as described above. Then

$$f^{\uparrow}(0; \cdot) = f'(0; \cdot) = f^{\circ}(0; \cdot) = f$$

and

$$\partial^{\circ}f(0) = \partial^{\uparrow}f(0) = \{z \in E | \, \|z\| \leqq 1\}.$$

Thus for $x := 0$ we get:

(a) E is any infinite dimensional Banach space viewed in its weak topology, f is convex lower semicontinuous, finite,

(b) $f^{\uparrow}(x; \cdot) = f^{\circ}(x; \cdot)$ and $f^{\circ}(x; \cdot)$ is finite, $\emptyset \neq \partial^{\circ}f(x) = \partial^{\uparrow}f(x)$ and $\partial^{\uparrow}f(x)$ is weak* compact,

(c) $f^{\uparrow}(x; \cdot) = f'(x; \cdot) = f^{\circ}(x; \cdot)$ and $f^{\circ}(x; \cdot)$ is lower semicontinuous but nowhere continuous,

(d) f is not directionally Lipschitzian at x.

Compare. Theorem 2.2.

Example 7. (A) Consider $\tilde{E}$, E and f as above. Suppose that $\tilde{E}$ has a locally uniformly convex (LUC) norm (as any L^P space for $1 < p < \infty$ ([**10**]), ([**11**])) or more generally Kadec-norm as in l. Then for any net $(x_\gamma) \subset \tilde{E}$ the following Kadec condition is satisfied ([**10**], [**11**]).

(3.25) If $x_\gamma \to x$ weakly and $\|x_\gamma\| \to \|x\|$ then $\|x_\gamma - x\| \to 0$.

With this assumption we have for all $x \in E$

(3.26) $f^{\circ}(x; \cdot) = f'(x; \cdot)$.

This follows from (3.25) and the fact that f is norm-Lipschitzian ([**23**], Corollary p. 339).

Combining (3.24) and (3.26) we get

$$f^{\circ}(x; \cdot) = f'(x; \cdot) = f^{\uparrow}(x; \cdot) \quad \text{for all } x \in E.$$

Let x be any smooth point in $\widetilde{E}$. Then $f'(x; \cdot)$ is linear and weakly continuous on $\widetilde{E}$ because it is equal to the gradient of f at x (considered in $\widetilde{E}$). The gradient being continuous is weakly continuous. Therefore $f^{\uparrow}(x; \cdot)$, $f'(x; \cdot)$ and $f^{\circ}(x; \cdot)$ are all equal and continuous on E.

Recall that a Banach space is *weakly compactly generated* (WCG) if it contains a weakly compact set K whose closed span is the space ([**10**], [**11**]). Each reflexive or separable space is WCG. It is a consequence of Trojanski's renorming Theorem and the Asplund averaging theorem ([**10**] or [**11**], Corollary 2, p. 167) that every WCG space has an equivalent norm which is both smooth and LUC.

Hence we get:

(a) E is any WCG Banach space viewed in its weak topology, f is convex, lower semicontinuous, finite,

(b) $f^{\uparrow}(x; \cdot) = f'(x; \cdot) = f^{\circ}(x; \cdot)$ and $f^{\circ}(x; \cdot)$ is continuous and linear,

(c) f is not directionally Lipschitzian at x.

Compare. Corollary 1.3, Theorem 1.6, Theorem 2.1, Theorem 2.2.

(B) Let $C: = \operatorname{epi} f$, $x: = (0, 0)$. We get:

(a) E is any WCG Banach space viewed in its weak topology, C is closed, convex subset of $E \times \mathbf{R}$,

(b) $\operatorname{int} T_C(x) = \operatorname{int} H_C(x) \neq \emptyset$,

(c) C is not epi-Lipschitzian at x.

Compare. Theorem 1.5, Corollary 2.2.

Example 8. Let $\widetilde{E}: = c$ (the space of all convergent sequences in the supremum norm) and let E and f be defined as previously. Consider $x: = (x_n)$ such that

(3.27) $\quad x_n \geqq 0$ for $n \in \mathbf{N}$ and $\lim_{n\to\infty} x_n = 1 = \|x\|$.

Then for any $y: = (y_n)$

(3.28) $\quad f^{\circ}(x; y) \geqq \lim_{n\to\infty} |y_n| \geqq 0.$

https://doi.org/10.4153/CJM-1984-008-7 Published online by Cambridge University Press

To see (3.28) let us observe that the sequence $(x^n)_{n \in \mathbf{N}}$ defined as

$$x^n: = x - 2x_n e^n$$

tends weakly to x. Furthermore

$$\|x^n\| = 1 \text{ for } n \in \mathbf{N}.$$

Let

$$\lim_{n \to \infty} |y_n| = : \alpha.$$

Then

$$\lim_{n \to \infty} y_n = -\alpha \quad \text{or} \quad \lim_{n \to \infty} y_n = \alpha. \tag{3.29}$$

Assume that first equality in (3.29) holds. Then

$$f^{\circ}(x; y) \geqq \lim_{t \downarrow 0} \lim_{n \to \infty} \frac{\|x - 2x_n e^n + ty\| - 1}{t}. \tag{3.30}$$

Let $\epsilon > 0$ be arbitrary. Then for n sufficiently large and $t > 0$ sufficiently small

$$\|x - 2x_n e^n + ty\| \geqq x_n + t(\alpha - \epsilon).$$

Using this inequality in (3.30) we get

$$f^{\circ}(x; y) \geqq \alpha - \epsilon \quad \text{if} \quad \lim_{n \to \infty} y_n = -\alpha. \tag{3.31}$$

Suppose now that the second equality in (3.29) holds. Then

$$f^{\circ}(x; y) \geqq \lim_{t \downarrow 0} \frac{\|x + ty\| - \|x\|}{t} \tag{3.32}$$

and for sufficiently small $t > 0$ we have by (3.27)

$$\sup_{n \in \mathbf{N}} |x_n + ty_n| \geqq \sup_{n \in \mathbf{N}} (x_n + t(\alpha - \epsilon)) = 1 + t(\alpha - \epsilon).$$

This together with (3.32) implies

$$f^{\circ}(x; y) \geqq \alpha - \epsilon \quad \text{if} \quad \lim_{n \to \infty} y_n = \alpha \tag{3.33}$$

and since ϵ can be arbitrarily small (3.31) and (3.33) justify (3.28).

Let us observe that (3.28) says that $0 \in \partial^{\circ} f(x)$, while $0 \notin \partial^{\uparrow} f(x)$ because $\|x\| = 1$.

Suppose that x is such that in addition to (3.27) also

(3.34) $\quad x_n < 1 \quad$ for all $n \in \mathbf{N}$.

Then $f^{\circ}(x; \cdot)$ is continuous and

(3.35) $\quad f^{\circ}(x; y) = \lim_{n\to\infty} |y_n| \quad$ for all y.

Indeed, take any $k \in \mathbf{N}$. Pick $\eta > 0$ satisfying

(3.36) $\quad x_n < 1 - 6\eta \quad$ for all $n \leqq k$.

Let $\tilde{U}$ be a neighbourhood of zero such that for all $u = (u_n) \in \tilde{U}$

(3.37) $\quad |u_n| < \eta \quad$ for $n \leqq k$, $\quad$ and

(3.38) $\quad |\lim u_n| < \eta$.

Choose $n_k > k$ such that

(3.39) $\quad 1 - \eta < x_{n_k} \quad$ and $\quad \lim_{n\to\infty} u_n - \eta < u_{n_k}$.

Furthermore, pick $\tilde{\lambda} < 0$ satisfying

(3.40) $\quad |ty_n| < \eta$ for all $0 < t < \tilde{\lambda}$ and all $n \in \mathbf{N}$.

Then using (3.35)-(3.40) we obtain

(3.41)
$$|x_n + u_n + ty_n| \leqq |x_n| + |u_n| + |ty_n| < 1 - 6\eta + \eta + \eta$$
$$= 1 - 4\eta < 1 + \lim_{n\to\infty} u_n - 3\eta < x_{n_k} + u_{n_k} + ty_{n_k}$$

for all $n \leqq k$, $u \in \tilde{U}$, $0 < t < \tilde{\lambda}$. It follows from (3.41) that

$$\|x + u + ty\| = \sup_{n>k} |x_n + u_n + ty_n| \text{ for } u \in \tilde{U},\ 0 < t < \tilde{\lambda}.$$

Hence

$$f^{\circ}(x; y) = \inf_{\substack{U\in n(0)\\ \delta>0\\ \lambda>0}} \sup_{\substack{u\in U\\ \|x+u\|\leqq 1+\delta\\ 0<t<\lambda}} \frac{\|x + u + ty\| - \|x + u\|}{t}$$
$$\leqq \sup_{\substack{u\in\tilde{U}\\ 0<t<\tilde{\lambda}}} \frac{\sup_{n>k} |x_n + u_n + ty_n| - \sup_{n>k} |x_n + u_n|}{t}$$
$$\leqq \sup_{n>k} |y_n|.$$

https://doi.org/10.4153/CJM-1984-008-7 Published online by Cambridge University Press

Since k was arbitrary and $y = (y_n)$ converges we get

$$f^{\circ}(x; y) \leqq \lim_{n\to\infty} |y_n|$$

which together with (3.28) implies (3.35) and that $f^{\circ}(x; \cdot)$ is continuous (weakly continuous on c).

One can check that any point x satisfying (3.27) and (3.34) is a smooth point in $\widetilde{E}$ at which

$$f^{\uparrow}(x; y) = f'(x; y) = \lim_{n\to\infty} y_n \quad \text{for any } y.$$

Summing up our considerations for x satisfying (3.27) and (3.34) we get:

(a) E is some infinite dimensional Banach space viewed in its weak topology, f is convex, lower semicontinuous, finite,

(b) $f^{\uparrow}(x; \cdot)$ and $f^{\circ}(x; \cdot)$ are both continuous but different,

(c) $\partial^{\uparrow}f(x)$ and $\partial^{\circ}f(x)$ are both nonempty and weak* compact,

(d) $\partial^{\uparrow}f(x) \subsetneq \partial^{\circ}f(x)$,

(e) f is not directionally Lipschitzian at x.

Compare. Corollary 1.3, Theorem 1.6, Theorem 2.1, Theorem 2.2, Proposition 2.1.

If the same example is considered for $\widetilde{E} := l^{\infty}$ then (3.28) holds with "lim sup" replacing "lim". Note that $e^n \to 0$ weakly in l^{∞}.

Example 9. Let $\widetilde{E} := C([0, 1])$ (the space of real continuous functions on $[0, 1]$) and let E and f be defined as previously. Denote by e a function constantly equal to 1 on $[0, 1]$.

For $x := e$ we obtain:

(a) E is a Banach space viewed in its weak topology, f is convex, lower semicontinuous, finite,

(b) $f^{\uparrow}(x; \cdot)$ is finite,

(c) $f^{\circ}(x; \cdot) = f$ (hence $f^{\circ}(x; \cdot)$ is lower semicontinuous and finite but nowhere continuous),

(d) $f^{\uparrow}(x; \cdot) \neq f^{\circ}(x; \cdot)$,

(e) $\partial^{\uparrow}f(x)$ and $\partial^{\circ}f(x)$ are both nonempty and weak* compact but different,

(f) f is not directionally Lipschitzian at x.

Compare. Corollary 1.3, Theorem 1.6, Theorem 2.1, Theorem 2.2.

Proof. Let us observe that for any y

$$f'(e; y) = \max_{s\in[0,1]} y(s).$$

Hence function $f'(e; \cdot)$ is lower semicontinuous on E (since it is continuous and convex on $\widetilde{E}$). Therefore

$$f^{\uparrow}(e; y) = f'(e; y) = \max_{s \in [0,1]} y(s).$$

This proves (b). By the formula obtained for $f'(e; \cdot)$ to prove (c) it is enough to show that for any y

$$f^{\circ}(e; y) \geqq \max_{s \in [0,1]} - y(s).$$

So let y be arbitrary and choose s satisfying

$$y(s) = \min_{s \in [0,1]} y(s).$$

For each $n \in \mathbf{N}$ pick $x^n \in C([0, 1])$ such that $-2 \leqq x_n(t) \leqq 0$ for $t \in [0, 1]$, $x^n(s) = -2$ and $x^n(t) = 0$ if $|t - s| \geqq 1/2^n$. Then $x^n \to 0$ weakly in $C([0, 1])$, hence $x^n \to 0$ in E. Furthermore

$$\|e + x^n\| = 1 \quad \text{for } n \in \mathbf{N}$$

and for small t

$$\frac{\|e + x^n + ty\| - \|e + x^n\|}{t} \geqq \frac{|-1 + ty(s)| - 1}{t}$$

$$= -y(s) = \max_{y \in [0,1]} - y(s).$$

Thus we proved that for any y

$$f^{\circ}(e; y) = \|y\|.$$

Since $E^* = \widetilde{E}^* = \mathrm{NBV}([0, 1])$ (the space of real normalized functions of bounded variation on $[0, 1]$) we get that

$$\partial^{\circ} f(e) = \{\Psi \in \mathrm{NBV}([0, 1]) \mid \|\Psi\| \leqq 1\}$$

and hence $\partial^{\circ} f(e)$ is nonempty and weak* compact. To complete the proof of (e) let us observe that

$$\partial^{\uparrow} f(e) = \{\Psi \in \mathrm{NBV}([0, 1]) \mid \int_0^1 y(s) d\Psi(s) \leqq \max_{s \in [0,1]} y(s), y \in C([0,1])\}$$

$$= \{\Psi \in \text{NBV}\,([0, 1]) \mid \int_0^1 d\Psi(s)$$
$$= 1 \text{ and } \Psi \text{ is nondecreasing}\}.$$

Obviously f is not directionally Lipschitzian since it is nowhere continuous.

Example 10. Let $\widetilde{E}$ be a non-reflexive Banach space and let E be the continuous dual of $\widetilde{E}$ considered in weak* topology. We define a function f on E as

(3.42) $f(x) := \|x\|$

where the norm above is the dual norm on $\widetilde{E}^*$. Notice that f is lower semicontinuous on E (because it is weak* lower semicontinuous on $\widetilde{E}^*$). However it is not continuous at any point, and hence also not directionally Lipschitzian at any point. As in our previous analysis it can be easily proved that $f^{\circ}(x; \cdot)$ is everywhere finite for any x. Since $\widetilde{E}$ is non-reflexive, by the theorem of James there exists $x \in E$ which does not attain its norm on $\widetilde{E}$. For this x

(3.43) $\partial^{\uparrow} f(x) = \emptyset.$

Indeed, otherwise since $\partial^{\uparrow} f(x) \subset \widetilde{E}$, there would exist some $z \in \widetilde{E}$ such that:

$$\|x'\| - \|x\| \geqq \langle x' - x, z \rangle \text{ for all } x' \in E.$$

But it follows from the above inequality that

$$\|x\| = \langle x, z \rangle \text{ and } \|z\| \leqq 1,$$

which is impossible. Hence (3.43) holds.

Since (3.43) is equivalent to

$$f^{\uparrow}(x; 0) = -\infty,$$

and $f^{\uparrow}(x; \cdot)$ is convex and nowhere $+\infty$ we obtain that

$$f^{\uparrow}(x; \cdot) \equiv -\infty.$$

So in this case we get:

(a) E is dual to any non-reflexive Banach space considered in weak* topology, f is convex, lower semicontinuous, finite,

(b) $f^{\circ}(x; \cdot)$ is finite (but not continuous, see Proposition 3.2 below),

(c) $f^{\uparrow}(x; \cdot) \equiv -\infty$,

(d) f is not directionally Lipschitzian at x.

Compare. Theorem 2.1, Proposition 2.3, Theorem 2.2.

Let us observe that in the case of a Banach space considered in its weak topology the Clarke derivative of the norm was lower semicontinuous. For the function defined by (3.42) on the dual space considered in its weak* topology this assertion is not always true. This follows from the next example.

Example 11. Let $\widetilde{E}: = c_0$ (the space of zero-convergent sequences in supremum norm) and let $E: = l^1$ considered in the weak* topology induced by c_0. Now l^1 is "weak* Kadec" ([**19**]). This is to say that for any net (x_γ)

(3.44) if $x_\gamma \to x$ weak* and $\|x_\gamma\| \to \|x\|$ then

$$\|x_\gamma - x\| \to 0.$$

It follows much as in Example 7 that

(3.45) $f^\circ(x; \cdot) = f'(x; \cdot)$ for any x.

Let $x: = (2^{-n})$, then

(3.46) $\partial f(x) = \emptyset$.

Indeed, otherwise since $\partial f(x) \subset c_0$, we would get for some $z \in c_0$

$$\|x'\| - \|x\| \geqq \langle x' - x, z\rangle \quad \text{for all } x' \in l^1$$

which is only possible when $z = (1, 1, 1 \dots)$, but then $z \notin c_0$. Hence (3.46) holds, and arguing as in the previous example shows that $f^\uparrow(x; \cdot) \equiv -\infty$. It follows now by Proposition 1.4 that $f'(x; \cdot)$ is not lower semicontinuous, hence by (3.45) $f^\circ(x; \cdot)$ is also not lower semicontinuous.

By (3.45) we also obtain that

$$f^\uparrow(x; \cdot) = \operatorname{cl} f'(x; \cdot) = \operatorname{cl} f^\circ(x; \cdot)$$

and this together with (3.46) gives

$$\partial^\uparrow f(x) = \partial^\circ f(x) = \emptyset.$$

Thus we get:

(a) E is some dual to Banach space considered in weak* topology, f is convex, lower semicontinuous, finite,

(b) $f^\circ(x; \cdot) = f'(x; \cdot)$ and $f^\circ(x; \cdot)$ is finite but not lower semicontinuous,

(c) $f^\uparrow(x; \cdot) \equiv -\infty$, $\partial^\uparrow f(x) = \partial^\circ f(x) = \emptyset$,

(d) f is not directionally Lipschitzian at x.

Compare. Theorem 2.1.

Example 12. Much as in Example 10 consider $\tilde{E}: = l^1$, $E: = l^{\infty}$ in its weak* topology and f defined by (3.42). Let $x: = (1 - 2^{-n})$. Then the analogous argument to that in Example 8 shows that for any y

$$\|y\| \geqq f^{\circ}(x; y) \geqq \limsup_{n\to\infty} |y_n| \geqq 0,$$

while considerations as in Example 10 show that

$$f^{\uparrow}(x; \cdot) \equiv -\infty.$$

Hence $\partial^{\uparrow}f(x; \cdot) = \emptyset$ but $\partial^{\circ}f(x; \cdot) \neq \emptyset$, since $0 \in \partial^{\circ}f(x; \cdot)$. Furthermore $f^{\circ}(x; \cdot)$ is finite, but by Proposition 3.2 given below not continuous.

Thus we get:

(a) E is dual to some Banach space considered in its weak* topology, f is convex, lower semicontinuous, finite,

(b) $f^{\uparrow}(x; \cdot) \equiv -\infty$, $\partial^{\uparrow}f(x) = \emptyset$,

(c) $f^{\circ}(x; \cdot)$ is finite but not continuous,

(d) f is not directionally Lipschitzian at x.

Compare. Corollary 1.3, Theorem 1.6, Theorem 2.1, Theorem 2.2.

Remark. Note that much of the theory developed in Examples 6-12 works also for more general norm continuous homogeneous functions. Notice also that in our convex examples $f^{\uparrow}(x; \cdot)$ is dependent only on the dual pair while $f^{\circ}(x; \cdot)$ is topology dependent.

In Proposition 2.1 we discovered that if $f^{\circ}(x; \cdot)$ is continuous then one of the following alternatives must hold: $f^{\uparrow}(x; \cdot) \equiv -\infty$ or $f^{\uparrow}(x; \cdot)$ is also a continuous function. However if f is a convex function, the first situation can never happen. This is explained below.

PROPOSITION 3.2. *Let E be a real locally convex vector space and let f be a convex function on E, finite and lower semicontinuous at x. Suppose $f^{\circ}(x; \cdot)$ is continuous on E. Then $f^{\uparrow}(x; \cdot)$ is finite and continuous and $f^{\uparrow}(x; \cdot) = f'(x; \cdot)$.*

Proof. Since $f'(x; \cdot) \leqq f^{\circ}(x; \cdot) < +\infty$ it follows that

$$f'(x; y) > -\infty \quad \text{for all } y,$$

otherwise we would get that

$$f'(x; \cdot) \equiv -\infty.$$

https://doi.org/10.4153/CJM-1984-008-7 Published online by Cambridge University Press

But this is impossible because $f'(x; 0) = 0$. Since $f'(x; \cdot)$ is finite, convex and majorized by continuous function it also must be continuous. Therefore for any y

$$f^{\uparrow}(x; y) = \liminf_{y' \to y} f'(x; y') = f'(x; y) > -\infty.$$

This completes the proof.

There is still an open question of deriving an example of a function for which, at some point x, the Clarke derivative is continuous on E but the upper subderivative is equal to $-\infty$ everywhere. As the above proposition and our previous consideration show, the function with this feature cannot be convex or directionally Lipschitzian at x. Can it be subdifferentially regular?

4. Conclusions. We have shown that in a Baire metrizable setting relations between various generalized derivatives (and tangent cones) simplify considerably. We have also shown that outside this setting, even for convex functions all these simplifications vanish. In particular in the absence of directional Lipschitzness $f^{\circ}(x; \cdot)$ and $f^{\uparrow}(x; \cdot)$ may be continuous and agree or disagree. Similarly $f^{\circ}(x;\cdot)$ can be finite but not lower semicontinuous or lower semicontinuous but not continuous. Even in the case of sublinear functions (norms) it may range between $f'(x; \cdot)$ and f.

We were also able in certain cases to give explicit formulae for $f^{\uparrow}(x; \cdot)$ and $f^{\circ}(x; \cdot)$ even when they diverged. As was apparent such estimates are quite delicate.

Our limiting examples describing various relations between the generalized derivatives and subgradients are presented in the following table (P, T, C mean Proposition, Theorem and Corollary respectively).

TABLE

No.	Space	f is convex, lower semicontinuous, not directionally Lipschitzian at x: $f^{\circ}(x;\cdot)$	$f^{\uparrow}(x;\cdot)$	$\partial^{\uparrow}f(x)$	$\partial^{\circ}f(x)$	Compare
1	Banach space	nowhere continuous $(\operatorname{dom} f^{\circ}(x;\cdot))^{0} = 0$	continuous $f^{\uparrow}(x;\cdot) \equiv 0$	$\partial^{\uparrow}f(x) = \{0\}$ $\partial^{\uparrow}f(x) =$	$\partial^{\circ}f(x)$	C 1.3, T 1.6, P 2.1, C 2.5
2	Banach space	nowhere continuous	$f^{\uparrow}(x;\cdot) \equiv -\infty$	$\partial f(x) = \emptyset$		T 1.6
3	Frechet or incomplete inner product space	not continuous	$f^{\uparrow}(x;\cdot) \equiv -\infty$ for all $x \in \operatorname{dom} f$	$\partial^{\uparrow}f(x) = \emptyset$ for all $x \in \operatorname{dom} f$		T 1.3
4	non-Baire normed space	continuous $f^{\circ}(x;\cdot) \neq$	continuous $f^{\uparrow}(x;\cdot)$	weak* compact $\emptyset \neq \partial f(x) \subsetneq$	weak* compact $\partial^{\circ}f(x)$	C 1.3, T 1.6 T 2.1, T 2.2 P 2.1
5	any nonbarreled space	lower semi-continuous, finite, nowhere continuous $f^{\circ}(x;\cdot) =$	$f^{\uparrow}(x;\cdot) = f'(x;\cdot)$	weak* bounded, not weak* compact $\partial^{\uparrow}f(x) =$	$\partial^{\circ}f(x)$	P 2.2 T. 2.2
6	any Banach space in weak topology	lower semi-continuous, finite, nowhere continuous $f^{\circ}(x;\cdot) = f^{\uparrow}(x;\cdot) = f'(x;\cdot)$		weak* compact $\emptyset \neq \partial^{\uparrow}f(x) =$	$\partial^{\circ}f(x)$	T 2.2

No.	Space	f is convex, lower semicontinuous, not directionally Lipschitzian at x				Compare
		$f^{\circ}(x;\cdot)$	$f^{\uparrow}(x;\cdot)$	$\partial^{\uparrow}f(x)$	$\partial^{\circ}f(x)$	
7	any WCG Banach space in weak topology	continuous, linear $f^{\circ}(x;\cdot) = f^{\uparrow}(x;\cdot) = f'(x;\cdot)$		weak* compact $\partial^{\uparrow}f(x) = \partial^{\circ}f(x)$		C 1.3, T 1.6, T 2.1, T 2.2
8	Banach space in weak topology	continuous $f^{\circ}(x;\cdot) \neq f^{\uparrow}(x;\cdot)$	continuous	weak* compact $\emptyset \neq \partial f(x) \subsetneq \partial^{\circ}f(x)$	weak* compact	C 1.3, T 1.6, T 2.1, T 2.2, P 2.1
9	Banach space in weak topology	lower semi-continuous, finite, nowhere continuous $f = f^{\circ}(x;\cdot) \neq f^{\uparrow}(x;\cdot)$	finite	weak* compact $\emptyset \neq \partial f(x) \subsetneq \partial^{\circ}f(x)$	weak* compact	C 1.3, T 1.6, T 2.1, T 2.2
10	dual to any non-reflexive Banach space in weak* topology	finite, not continuous	$f^{\uparrow}(x;\cdot) \equiv -\infty$	$\partial^{\uparrow}f(x) = \emptyset$		T 2.1, P 2.1, T 2.2
11	dual to Banach space in weak* topology	finite, not lower semicontinuous $f'(x;\cdot) = f^{\circ}(x;\cdot)$	$f^{\uparrow}(x;\cdot) \equiv -\infty$	$\partial^{\uparrow}f(x) = \partial^{\circ}f(x) = \emptyset$		T 2.1
12	dual to Banach space in weak* topology	finite, not continuous	$f^{\uparrow}(x;\cdot) \equiv -\infty$	$\partial^{\uparrow}f(x) = \emptyset$	$\partial^{\circ}f(x) \neq \emptyset$	C 1.3, T 1.6, T 2.1, T 2.2

References

1. J. M. Borwein, *Stability and regular points of inequality systems*, Research Report 82-2, Department of Mathematics, Mellon Institute of Science, Carnegie-Mellon University. To appear in Numerical Functional Analysis and Applications.

2. J. M. Borwein and H. M. Stròjwas, *Directionally Lipschitzian mappings on Baire spaces*, Research Report 83-3, Department of Mathematics, Mellon Institute of Science, Carnegie-Melon University.

3. G. Bouligand, *Introduction à la gèometrie infinite-simale directe* (Vuibert, Paris, 1932).

4. A. Brøndsted and R. T. Rockafellar, *On the subdifferentiability of convex functions*, Proc. Amer. Math. Soc. *16* (1965), 605-11.

5. F. H. Clarke, *Necessary conditions for nonsmooth problems in optimal control and the calculus of variation*, Thesis, University of Washington, Seattle (1973).

6. ——— *Generalized gradients and applications*, Trans. Amer. Math. Soc. *205* (1975), 247-62.

7. ——— *Generalized gradients of Lipschitz functions*, Adv. in Math. *40* (1981), 52-67.

8. ——— *A new approach to Lagrange multipliers*, Math of Operations Research *1* (1976), 165-74.

9. S. Dolecki, *Tangency and differentiation: some applications of convergence theory*, Annali di Matematica pura ed applicata *130* (1982), 223-255.

10. M. M. Day, *Normed linear spaces*, (Springer-Verlag, New York, 1973).

11. J. Diestel, *Geometry of Banach spaces – selected topics* (Springer-Verlag, New York, 1975).

12. J. B. Hiriart-Urruty, *Tangent cones, generalized gradients and mathematical programming in Banach spaces*, Math. of Op. Res. *4* (1975), 79-97.

13. R. B. Holmes, *Geometric functional analysis* (Springer-Verlag, New York, 1975).

14. V. Klee, *On a question of Bishop and Phelps*, Amer. J. Math. *85* (1963), 95-98.

15. ——— *Extremal structure of convex sets II*, Math. A. *69* (1958), 90-104.

16. G. Lebourg, *Valeur moyenne pour gradient généralisé*, C. R. Acad. Aci. Paris, Ser. A. *281* (1975), 795-797.

17. ——— *Generic differentiability of Lipschitzian functions*, Trans. Am. Math. Soc. *256* (1979), 125-144.

18. L. McLinden, *An application of Ekeland's theorem to minimax problems*, Nonlin. Anal. Th. Meth. Appl. *6* (1982), 189-196.

19. I. Namioka, *Separate continuity and joint continuity*, Pacific J. Math *51* (1974), 515-531.

20. N. Peck, *Support points in locally convex spaces*, Duke Math. J. *38*, 271-278.

21. J. P. Penot, *The use of generalized subdifferential calculus in optimization theory*, Proc. Third Symp. on Operations Research, Mannheim (1978), Methods of Operations Research *31*, 495-511, Athenäum, Berlin.

22. ——— *A characterization of tangential regularity*, Nonlinear Anal., Theory, Math. Appl. *5* (1981), 625-663.

23. R. T. Rockafellar, *Directionally Lipschitzian functions and subdifferential calculus*, Bull. London, Math. Soc. *39* (1979), 331-355.

24. ——— *Generalized directional derivatives and subgradients of nonconvex functions*, Can. J. Math. *32* (1980), 257-280.

25. ——— *Clarke's tangent cones and the boundaries of closed sets in* $\mathbf{R}^n$, Nolin. Analysis. Th. Meth. Appl. *3* (1979), 145-154.

26. ——— *Lagrange multipliers and subderivatives of optimal value functions in nonlinear programming*, to appear in Math. Prog. Study 15.

27. ——— *The theory of subgradients and its applications to problems of optimization. Convex and nonconvex functions* (Heldermann-Verlag, Berlin, 1981).

28. A. Wilansky, *Modern methods in topological vector spaces* (McGraw-Hill, New York, 1978).

Dalhousie University, Halifax, Nova Scotia;
Carnegie-Mellon University, Pittsburgh, Pennsylvania