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ON RELAXATION OSCILLATIONS GOVERNED BY A SECOND ORDER DIFFERENTIAL EQUATION FOR A LARGE PARAMETER AND WITH A PIECEWISE LINEAR FUNCTION[1]

BY

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ABSTRACT. This paper deals with the differential equation: $\ddot{x}+\mu F(\dot{x})+x=f(x,\dot{x},t/T\mu)$ for $\mu\gg 1$ where F is a piecewise linear function and f is a periodic function of period μT, where T is to be chosen. It is established that periodic forced vibrations exist in an annular domain $R(\mu)$ constructed for the free vibration ($f\equiv 0$), provided f is not of higher order than 0 ($\mu^{1/3}-r$), $0<r<\frac{1}{3}$. Subsequently with $f=A\cos(2\pi t/\mu T^*)$, an asymptotic treatment of the forced vibration problem is carried out, by finding the proper initial conditions and the proper period μT^* of f. Finally it is concluded that μT^* is close to the period of the free vibration.

1. **Introduction.** The mathematical formulation of the problem giving the current $x(t)$ in one branch of an electrical circuit containing a triode vacuum tube is:

$$\ddot{x}-\mu(\dot{x}-\tfrac{1}{3}\dot{x}^3)+x=f\left(x,\dot{x},\frac{t}{T\mu}\right),\qquad \mu>0, \tag{1}$$

where T is a parameter, whose usefulness will become apparent later in this paper. Doronitsyn [2] has dealt with this differential equation with $f\equiv 0$, that is the van der Pol equation, with μ, a large parameter. Haag [4] has considered differential equations of a very general type (including the van der Pol equation) in a series of important papers. In 1975, J. Grasman, E. J. M. Veling and G. M. Willems [3] published a joint paper called 'Relaxation Oscillations Governed by a van der Pol equation with periodic forcing term'. Of course J. J. Stoker [5] published 'Periodic Forced Vibrations of Systems of Relaxation Oscillators' more recently in the May 1980 issue of Communications in Pure and Applied Mathematics.

In this present paper, we mainly follow Stoker, use his methods with certain modifications and extend his work.

The equation dealt with in this paper is:

$$\ddot{x}+\mu F(\dot{x})+x=f\left(x,\dot{x}\frac{t}{T\mu}\right)\quad\text{for}\quad \mu\gg 1 \tag{2}$$

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where F is defined by,

$$\begin{aligned} -F(\dot{x}) &= \dot{x} && \text{for} \quad |\dot{x}| \le | \\ &= \pm 2 - \dot{x} && \text{for} \quad |\dot{x}| \ge | \end{aligned}$$

This equation for $f \equiv 0$, that is for the free vibration has a character similar to the van der Pol equation dealt with in Stoker's recent paper [5]. The equation (2) with $f \equiv 0$ describes a physical phenomenon, which may be viewed as a simplified version of the one described by the van der Pol equation.

Firstly following Stoker [6] and [5], we construct an annular domain $R(\mu)$ Fig. 1, in which, we show the existence of a unique periodic free vibration governed by the differential equation.

$$\begin{aligned} &\ddot{x} + \mu F(\dot{x}) + x = 0, && \mu \gg 1 \\ (3) \qquad & -F(\dot{x}) = \dot{x}, && |\dot{x}| \le 1 \\ &= \pm 2 - \dot{x}, && |\dot{x}| \ge 1 \end{aligned}$$

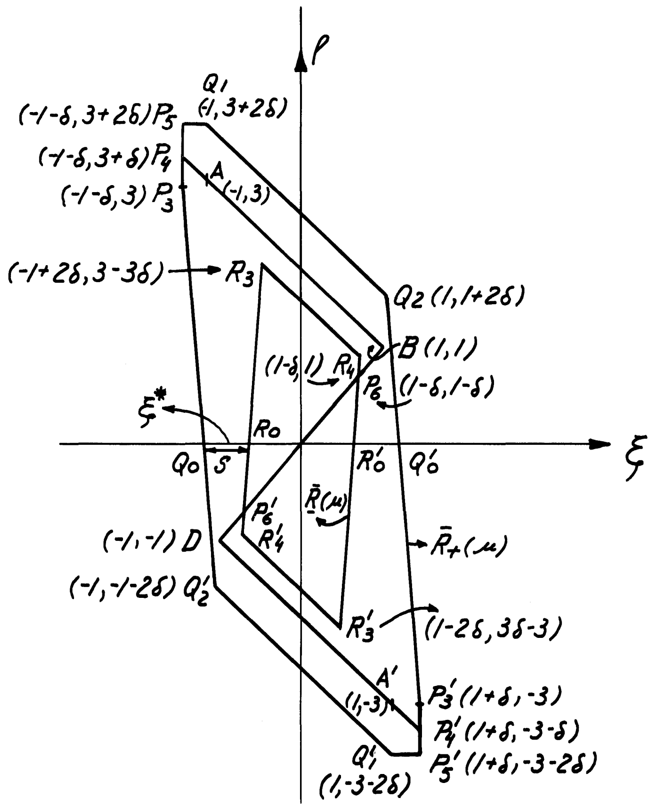

Figure 1. The annular region $R(\mu)$ with its outer boundary $\bar{R}_{+}(\mu)$ and its inner boundary $\bar{R}_{-}(\mu)$

Secondly we show the existence of a periodic forced vibration in the same annular domain, when the right-hand side of the above equation (3) is not zero but $f(x, \dot{x}, t/T\mu)$ and f is permitted to be of order, no higher than $0(\mu^{1/3-r})$, $0<r<\frac{1}{3}$. This restriction on f follows from Stoker [5] and requires modifications in the construction of $R(\mu)$.

Thirdly, we take f appearing in (2) to be a periodic function $A\cos(2\pi t/\mu T^*)$, where A is a constant. This function f certainly falls under the class of functions mentioned above. Consequently then, there does exist a periodic solution of the same equation with $f=A\cos(2\pi t/\mu T^*)$ given by a closed solution curve in the annular domain $R(\mu)$ constructed originally for the free vibration in the phase plane.

Lastly, an asymptotic treatment of the forced vibration problem,

$$\ddot{x}+\mu F(\dot{x})+x=A\cos\frac{2\pi t}{\mu T^*}, \qquad \mu\gg 1,$$

$$(4) \qquad \begin{aligned} -F(\dot{x})&=\dot{x} && \text{for} \quad |\dot{x}|\le 1\\ &=\pm 2-\dot{x} && \text{for} \quad |\dot{x}|\ge 1 \end{aligned}$$

comes about by finding the proper initial conditions and the proper period μT^* of the forcing term on the right-hand side of equation (4). This implies that the period of the forcing term is not prescribed in advance nor is the exact starting point at $t=0$. We aim at obtaining μT^* up to the leading term for the contribution of the forcing function. It turns out that the period μT^* is close to the period of the free vibration found in Stoker's 'Non Linear Vibrations' [6] to the order of accuracy required. The leading term for the period μT^* is $(2\ln 3)\mu$ as in the case of the free vibration.

2. **Treatment of the homogeneous case.** We deal with the homogeneous case first i.e. the equation (3). To start with, we change the variables by means of substitutions, $\xi=x/\mu$, $\tau=t/\mu$, $\rho=\xi'=d\xi/d\tau=\dot{x}$ and transform (3) into a system of two equations namely,

$$(5) \qquad \begin{aligned} \frac{d\xi}{d\tau}&=\rho\\ \frac{d\rho}{d\tau}&=\mu^2[-F(\rho)-\xi] \end{aligned}$$

where

$$\begin{aligned} -F(\rho)&=\rho, && |\rho|\le 1\\ &=\pm 2-\rho, && |\rho|\ge 1 \end{aligned}$$

Construction of $\bar{R}_+(\mu)$ and $\bar{R}_-(\mu)$. We construct $\bar{R}_+(\mu)$ and $\bar{R}_-(\mu)$, [Fig. 1], the outer and inner boundaries respectively of the annular domain $R(\mu)$ [Fig. 1] in the phase plane i.e. the (ξ, ρ) plane. We accomplish the drawing of $\bar{R}_+(\mu)$

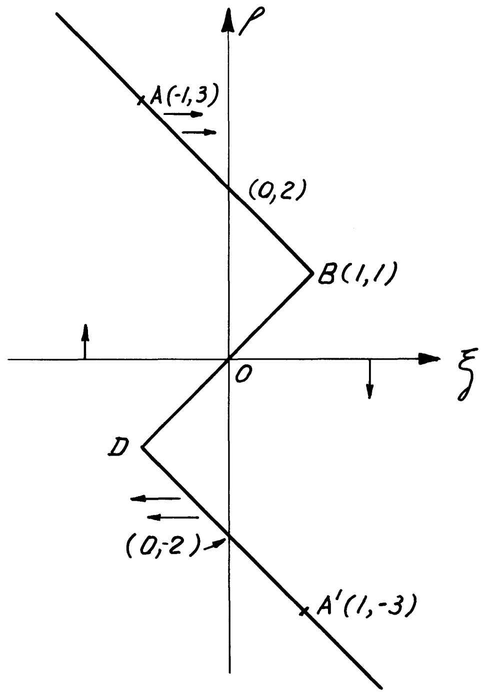

Figure 2. The piecewise linear characteristic $A'DBA$, $\xi=-F(\rho)$

by shifting the characteristic curve $\xi=-F(\rho)$ (Fig. 2) to the right by $2\,\delta$ units. At this point, we might mention that δ is small, while μ is arbitrarily large and positive [δ and μ are found to satisfy the relation $\delta=0(1/\mu^{2/3})$]. We let Q_1 and Q_2 be the points on the uppermost branch of the translated curve with ξ-co-ordinates -1 and 1 respectively. A parallel through Q_1 to the ξ-axis is drawn and a point P_5 is marked on it such that its ξ-co-ordinate is $-1-\delta$ and the ρ co-ordinate is the same as that for Q_1. We let $P_5P_4P_3$ be a vertical (parallel to the ρ-axis) drawn to cut the characteristic curve at P_4 so that P_5 is $(-1-\delta, 3+2\delta)$, P_4 is $(-1-\delta, 3+\delta)$ and P_3 is $(-1-\delta, 3)$. We join Q_2 with P_3', the point symmetrical to P_3. One half of the curve $\bar{R}_+(\mu)$ then consists of the vertical segment $P_3P_4P_5$, P_5Q_1, Q_1Q_2 and Q_2P_3'. In the lower half plane the construction is made symmetrically with respect to the origin. Next $\bar{R}_-(\mu)$ is constructed by translating $\xi=-F(\rho)$, a distance δ in the negative ξ direction. We let R_3 and R_4 be the points in the uppermost branch of the translated curve at which $\xi=-1+2\delta$ and $\xi=1-\delta$ respectively. Let P_6 be the point on $\xi=-F(\rho)$ with co-ordinates $\xi=1-\delta$, $\rho=1-\delta$. We join R_4 to P_6 and then P_6 with R_3', the point symmetrical to R_3 by a straight line segment cutting the

ξ-axis at R_0'. One half of $\bar{R}_-(\mu)$, then consists of the straight line segments R_3R_4, R_4P_6 and P_6R_3'. The other half is completed symmetrically with respect to the origin.

It is to be noted that everytime the characteristic curve is cut by $\bar{R}_+(\mu)$ or $\bar{R}_-(\mu)$, it is cut by means of a vertical line segment, that is a line segment parallel to the ρ axis. This happens to be essential to the argument that follows later for the corresponding non-homogeneous case.

By considering slopes of line segments bounding $\bar{R}_+(\mu)$ and $\bar{R}_-(\mu)$ and the directions of the field vectors $(d\xi/d\tau, d\rho/d\tau)$ at the boundary points, it is found that δ, the basic width of $R(\mu)$ cannot be too small in relation to μ. δ and μ are found to satisfy the relations $\delta > 2/\mu^{2/3}$. With such a δ, in relation to a given μ, the annular region is such that all field vectors point inwards of $R(\mu)$. Lipschutz condition on the system (5) is satisfied. Once an initial point for an integral curve of the system is chosen in $R(\mu)$ or on the boundary of $R(\mu)$ the solution curve stays inside of $R(\mu)$. Also from the first of equations in the system (5) namely $d\xi/d\tau = \rho$, we know that all solution curves make a complete clockwise circuit around $R(\mu)$. Thus a mapping of a segment S, (which is Q_0R_0, the intersection of the negative ξ axis with $\bar{R}(\mu)$ Fig. [1]), of $R(\mu)$ on itself is established. This mapping is well-defined and continuous because of continuous dependence of solutions on initial conditions. Hence from Brouwer's Fixed Point Theorem, this mapping has a fixed point. Hence there exists a point on S, such that an integral curve starting at such a point comes back to the same point after making a complete circuit of $R(\mu)$ in the clockwise sense. Hence we have the existence of a closed solution curve C_μ in every annular domain $R(\mu)$ in the phase plane implying a periodic solution of the system (5). This same result can be established by using Poincaré–Bendixson Theorem for the closed compact annular region $\bar{R}(\mu)$ containing no singular points of the system (5). Also all solution curves, which have once entered $\bar{R}(\mu)$ stay in there, as at all boundary points, the field vector points inwards of $\bar{R}(\mu)$. Thus a closed solution curve or a limit cycle C_μ results. The uniqueness of the limit cycle C_μ for a value of μ, is guaranteed because of Poincare's Orbital stability condition being satisfied, for

$$P(\mu) = \oint_{C_\mu} \left(\frac{\partial \xi'}{\partial \xi} + \frac{\partial \rho'}{\partial \rho}\right) d\tau = -\oint_{C_\mu} \mu^2 F'(\rho)\, d\tau$$

The value of the above integral turns out to be $-(2 \ln 3)\mu^2 +$ terms which are of a lower order as $\mu \to \infty$. Hence $P(\mu)$ is negative as $\mu \to \infty$, and Poincare's stability condition is satisfied. Then proof by contradiction is utilized to demonstrate that a unique limit cycle C_μ exists in every annular domain $R(\mu)$. This $R(\mu)$ of course shrinks down on the curve called C_∞ in Fig. 3 because $\delta \to 0$ like $1/\mu^{2/3}$ when $\mu \to \infty$. Thus the limit of the limit cycles C_μ is C_∞.

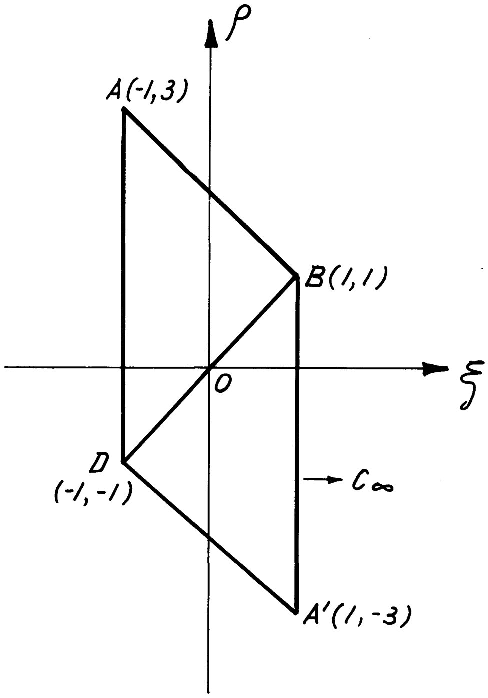

Figure 3. The limit of the limit cycles C_∞

3. **Forced vibrations in** $R(\mu)$. The equation (2) is transformed into the system

$$\text{(6)} \qquad \begin{aligned} \frac{d\xi}{d\tau} &= \rho \\ \frac{d\rho}{d\tau} &= \mu^2\left[-F(\rho) - \xi + \frac{1}{\mu^2} f\left(\mu\xi, \rho, \frac{\tau}{T}\right)\right] \end{aligned}$$

with

$$\begin{aligned} -F(\rho) &= \rho, \qquad & |\rho| \le 1 \\ &= \pm 2 - \rho, \qquad & |\rho| \ge 1 \end{aligned}$$

The proof for the existence of a forced vibration in $R(\mu)$ having the same period as the excitation f is taken up. Here f is assumed to be of no larger order than $0(\mu^{1/3-r})$, $0<r<\frac{1}{3}$ with $\mu \to \infty$. This restriction on f comes about because of the width δ of $R(\mu)$ satisfying the condition $\delta > 2/\mu^{2/3}$ uniformly over $R(\mu)$. The $R(\mu)$ we have constructed is a slight modification (and is a definite improvement) of the construction used by Stoker in his book 'Non Linear Vibrations' [6] or his recent paper [5] for the van der Pol equation. Our $\bar{R}_+(\mu)$, the outer boundary of $R(\mu)$ cuts the characteristic $\xi = -F(\rho)$ by a vertical segment, so that we experience no problem, when the existence of forced vibrations in the annular region $R(\mu)$, is shown. We are able to say, safely, that at all points of $\bar{R}(\mu)$, the field vector points inwards of $\bar{R}(\mu)$ as long as f is of no larger order than $0(\mu^{1/3-r})$, $0<r<\frac{1}{3}$. It is possible to say again that Brouwer's Fixed Point Theorem for the mapping of a segment on itself applies. Hence there exists a closed solution curve which begins at ξ^* on S (Fig. 1) and comes back to ξ^* after making a complete clockwise circuit of $R(\mu)$. It is imperative to point out that unlike the case of free vibrations, a periodic forced vibration could be seen to exist only if the time required to make a circuit around the ring starting from a point ξ^* on S and returning to it, happened to be the same as the period in τ of f, the forcing function. The parameter T in (1) and later in (2) was introduced to bring this about by a procedure that makes use of the Brouwer's Fixed Point Theorem for a mapping of the two dimensional closed rectangle (Fig. 4) given by $\xi_1 \leq \xi \leq \xi_2$ and $T_- \leq T \leq T_+$ on itself. This method has been used by Stoker [5] for the corresponding van der Pol case. Here $\xi_1\xi_2$ is the segment S already described.

A point (ξ, T) is chosen in the rectangle and a mapping (ξ^1, T^1) is defined by choosing a point ξ on S as initial point to define a trajectory of (6), when a specific value T is fixed in (6). The trajectory so defined cuts the segment S at a point ξ^1 after one circuit around the ring in a finite time $\tau = T^1$. Since T_- and T_+ are the bounds for all times for all integral curves (ξ^1, T^1) then lies in the rectangle $\xi_1 \leq \xi \leq \xi_2$, $T_- \leq T \leq T_+$. This continuous mapping of (ξ, T) to (ξ^1, T^1) has a fixed point (ξ^*, T^*). By inserting T^* for T in equation (2) and then in (6) and integrating (6) with ξ^* as initial point an integral curve results, which starts

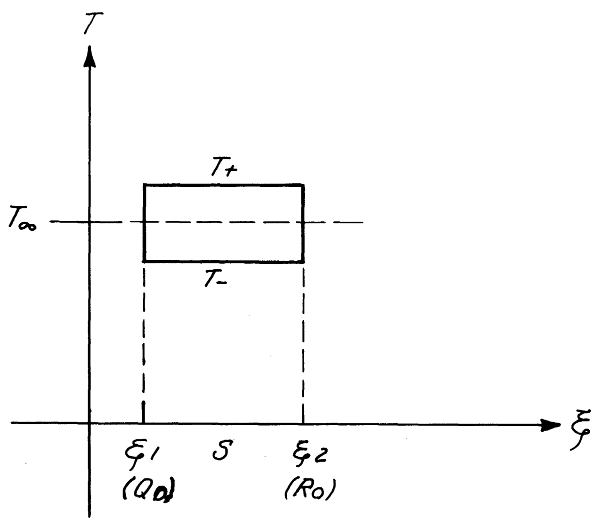

Figure 4. Mapping in a (ξ, T) plane

at $\xi=\xi^*$, $\tau=0$ and returns to that point in time T^*. But T^* is now the period of f in the differential equation (2) or the system (6). Thus existence of a periodic solution with the period of the forcing term is proved provided f is of order no larger than $0(\mu^{1/3-r})$, $0<r<\frac{1}{3}$.

4. **Asymptotic treatment of the non-homogeneous case.** We take f to be $A\cos(2\pi t/\mu T^*)$ in (2). This f is certainly of the class of functions f of no larger order than $0(\mu^{1/3-r})$, $0<r<\frac{1}{3}$. Thus we already know that a periodic solution in $R(\mu)$ exists for (2) with the period of the forcing term $f=A\cos(2\pi t/\mu T^*)$. We aim at finding the proper initial conditions, and the proper period μT^* of the forcing term.

A change of variables with $x=\mu\xi$, $t=\mu\tau$, $\dot{x}=\xi'=\rho$ leads us to solving,

$$\frac{d^2\xi}{d\tau^2}+\mu^2[F(\rho)]+\mu^2\xi=A\mu\cos\frac{2\pi\tau}{T^*} \tag{7}$$

with

$$\begin{aligned} -F(\rho)&=\rho, && |\rho|\le 1\\ &=\pm 2-\rho, && |\rho|\ge 1 \end{aligned}$$

in the regions $|\rho|\le 1$ and $|\rho|\ge 1$ for a periodic solution such that $\tau=0$, $\xi=1+\xi_0$, $\rho=1$ at P_1 and $\tau=T^*$, $\xi=1+\xi_0$, $\rho=1$ at P_1 after one circuit (Fig. 5). The general solutions of (7) in the four different regions are:

$$\xi^{(1)}=c_1e^{\lambda_1\tau}+c_2e^{\lambda_2\tau}+\frac{A}{D\mu}\cos\omega(\mu\tau+\alpha) \tag{8}$$

for $-1\le\xi'\le 1$ from P_1 to P_2 (Fig. 5)

$$\xi^{(2)}=-2+c_3e^{-\lambda_1\tau}+c_4e^{-\lambda_2\tau}+\frac{A}{D\mu}\cos\omega(\mu\tau-\alpha) \tag{9}$$

for $\xi'\le -1$ from P_2 to P_3 (Fig. 5)

$$\xi^{(3)}=c_1'e^{\lambda_1\tau}+c_2'e^{\lambda_2\tau}+\frac{A}{D\mu}\cos\omega(\mu\tau+\alpha) \tag{10}$$

for $-1\le\xi'\le 1$ from P_3 to P_4 (Fig. 5)

$$\xi^{(4)}=2+c_5e^{-\lambda_1\tau}+c_6e^{-\lambda_2\tau}+\frac{A}{D\mu}\cos\omega(\mu\tau-\alpha) \tag{11}$$

for $\xi'\ge|$ from P_4 to P_1 (Fig. 5).

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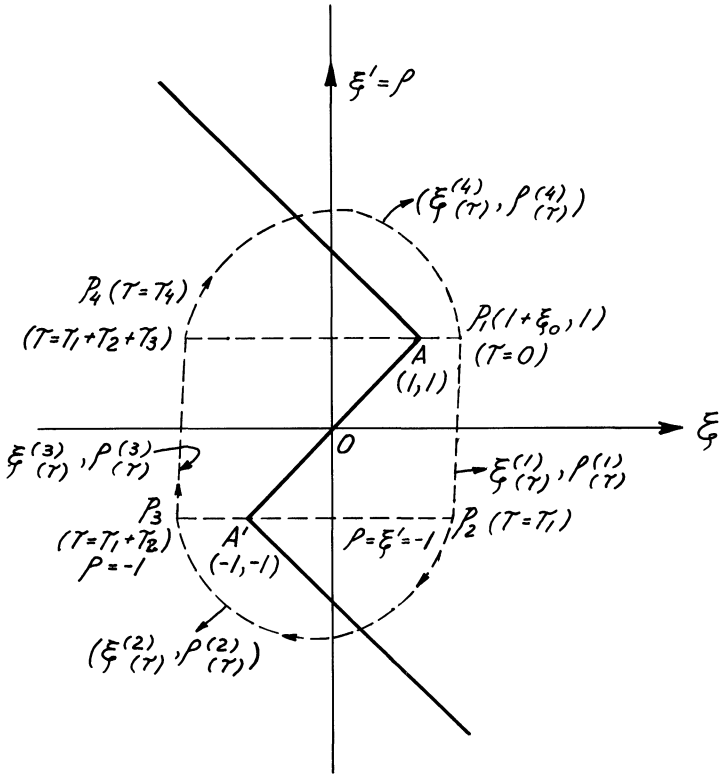

Figure 5. The patched up solution curve $(\xi_{(\tau)}\rho_{(\tau)})$

where

(12) $$\omega\mu = \frac{2\pi}{T^*}$$

(13) $$D^2 = (1-\omega^2)^2 + \omega^2\mu^2 = 1 + \frac{4\pi^2}{T^{*2}} + O\left(\frac{1}{\mu^2}\right)$$
$$= D_0^2 + O\left(\frac{1}{\mu^2}\right)$$

(14) $$D_0^2 = 1 + \frac{4\pi 2}{T^{*2}}$$

Defining,

(15) $$D = D_0 + O\left(\frac{1}{\mu^2}\right)$$

(16) $$\cos\alpha\omega = \frac{1-\omega^2}{D} = \frac{1}{D_0} + O\left(\frac{1}{\mu^2}\right)$$

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$$\sin\alpha\omega=\frac{\omega\mu}{D}=\frac{2\pi}{T^*D_0}+O\left(\frac{1}{\mu^2}\right) \tag{17}$$

$$\lambda_1=\mu^2-1-\frac{1}{\mu^2}-\frac{2}{\mu^4}-\frac{5}{\mu^6}-\frac{14}{\mu^8}+O\left(\frac{1}{\mu^{10}}\right) \tag{18}$$

$$\lambda_2=1+\frac{1}{\mu^2}+\frac{2}{\mu^4}+\frac{5}{\mu^6}+\frac{14}{\mu^8}+O\left(\frac{1}{\mu^{10}}\right) \tag{19}$$

and $c_1, c_2, c_1', c_2', c_3, c_4, c_5, c_6$, are arbitrary constants.

Considering the continuity of the solution curve at P_1, P_2, P_3, P_4, we get the following seventeen equations:

At P_1, $\tau=0$ (Fig. 5)

$$1+\xi_0=c_1+c_2+\frac{A}{D\mu}\cos\omega\alpha \tag{20}$$

$$1=c_1\lambda_1+c_2\lambda_2-\frac{A\omega}{D}\sin\omega\alpha \tag{21}$$

$$1+\xi_0=2+c_5+c_6+\frac{A}{D\mu}\cos\omega\alpha \tag{22}$$

$$1=-c_5\lambda_1-c_6\lambda_2+\frac{A\omega}{D}\sin\omega\alpha \tag{23}$$

At P_2, (Fig. 5) $\tau=\tau_1$,

$$\xi^{(1)}(\tau_1)=c_1e^{\lambda_1\tau_1}+c_2e^{\lambda_2\tau_1}+\frac{A}{D\mu}\cos\omega(\mu\tau_1+\alpha) \tag{24}$$

$$\xi'^{(1)}(\tau_1)=-1=c_1\lambda_1e^{\lambda_1\tau_1}+c_2\lambda_2e^{\lambda_2\tau_1}-\frac{A\omega}{D}\sin\omega(\mu\tau_1+\alpha) \tag{25}$$

$$\begin{aligned}\xi^{(2)}(\tau_1)&=-2+c_3e^{-\lambda_1\tau_1}+c_4e^{-\lambda_2\tau_1}+\frac{A}{D\mu}\cos\omega(\mu\tau_1-\alpha)\\&=\xi^{(1)}(\tau_1)\end{aligned} \tag{26}$$

$$\xi'^{(2)}(\tau_1)=-1=-c_3\lambda_1e^{-\lambda_1\tau_1}-c_4\lambda_2e^{-\lambda_2\tau_1}-\frac{A\omega}{D}\sin\omega(\mu\tau_1-\alpha) \tag{27}$$

At P_3, (Fig. 5), $\tau=\tau_1+\tau_2$

$$\xi^{(2)}(\tau_1+\tau_2)=-2+c_3e^{-\lambda_1(\tau_1+\tau_2)}+c_4e^{-\lambda_2(\tau_1+\tau_2)}+\frac{A}{D\mu}\cos\omega(\mu(\tau_1+\tau_2)-\alpha) \tag{28}$$

$$\begin{aligned}\xi'^{(2)}(\tau_1+\tau_2)=-1=-c_3\lambda_1e^{-\lambda_1(\tau_1+\tau_2)}-c_4\lambda_2e^{-\lambda_2(\tau_1+\tau_2)}\\-\frac{A\omega}{D}\sin\omega[\mu(\tau_1+\tau_2)-\alpha]\end{aligned} \tag{29}$$

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$$\xi^{(2)}(\tau_1+\tau_2)=\xi^{(3)}(\tau_1+\tau_2)=c_1'e^{\lambda_1(\tau_1+\tau_2)} + c_2'e^{\lambda_2(\tau_1+\tau_2)}+\frac{A}{D\mu}\cos\omega[\mu(\tau_1+\tau_2)+\alpha] \tag{30}$$

$$\begin{aligned}-1=\xi'^{(2)}(\tau_1+\tau_2)&=\xi'^{(3)}(\tau_1+\tau_2)\\ &=c_1'\lambda_1e^{\lambda_1(\tau_1+\tau_2)}+c_2'\lambda_2e^{\lambda_2(\tau_1+\tau_2)}-\frac{A\omega}{D}\sin\omega[\mu(\tau_1+\tau_2)+\alpha]\end{aligned} \tag{31}$$

At P_4, $\tau=\tau_1+\tau_2+\tau_3$ referring to Fig. 5 again,

$$\begin{aligned}\xi^{(3)}(\tau_1+\tau_2+\tau_3)&=c_1'e^{\lambda_1(\tau_1+\tau_2+\tau_3)}+c_2'e^{\lambda_2(\tau_1+\tau_2+\tau_3)}\\ &\quad+\frac{A}{D\mu}\cos\omega[\mu(\tau_1+\tau_2+\tau_3)+\alpha]\\ &=\xi^{(4)}(\tau_4)\end{aligned} \tag{32}$$

$$\begin{aligned}1=\xi'^{(3)}(\tau_1+\tau_2+\tau_3)&=c_1'\lambda_1e^{\lambda_1(\tau_1+\tau_2+\tau_3)}+c_2'\lambda_2e^{\lambda_2(\tau_1+\tau_2+\tau_3)}\\ &\quad-\frac{A\omega}{D}\sin\omega[\mu(\tau_1+\tau_2+\tau_3)+\alpha]\\ &=\xi'^{(4)}(\tau_4)\end{aligned} \tag{33}$$

Also, at P_4 (Fig. 5), $\tau=\tau_4$ where τ_4 is the time required to pass in the region $\rho\geq 1$ from the point $P_1(1+\xi_0, 1)$ to $P_4(\xi^{(4)}(\tau_4), 1)$. Thus,

$$\xi^{(4)}(\tau_4)=2+c_5e^{-\lambda_1\tau_4}+c_6e^{-\lambda_2\tau_4}+\frac{A}{D\mu}\cos\omega(\mu\tau_4-\alpha) \tag{34}$$

$$\begin{aligned}\xi'^{(4)}(\tau_4)=1=-c_5\lambda_1e^{-\lambda_1\tau_4}-c_6\lambda_2e^{-\lambda_2\tau_4}\\ -\frac{A\omega}{D}\sin\omega(\mu\tau_4-\alpha)\end{aligned} \tag{35}$$

Lastly,

$$T^*=\tau_1+\tau_2+\tau_3-\tau_4 \tag{36}$$

We solve the above seventeen equations (out of which only fourteen are independent), with the aim of obtaining T^*, the period of the solution in the form

$$T^*=[\cdots]+BA\phi(\mu),$$

where we wish the square bracket to contain all of terms, (which arise from the free vibration) that dominate the leading term involving A, that is one of highest order involving μ. The quantity B is to be a numerical constant. We also wish to approximate ξ_0 to fix the initial conditions for the solution curve. The asymptotic treatment of the corresponding homogeneous case (3) in Dr. Stoker's Non-Linear Vibrations [6] provided a guide as to what must be kept while making approximations. At each step, in the very involved computations in solving what are mostly transcendental equations, a term due to the free vibration, plus the order of the next term due to the free vibration was noted, besides keeping the leading term involving A and the order of the next term involving A. The details of the calculations can be found in the author's [1].

We find that the starting point for the solution curve turns out to be

$$\tau = 0, \qquad \xi = 1 + \xi_0 = \left\{1 + O\left(\frac{1}{\mu^2}\right)\right\} + A\left[\frac{1}{\mu} + O\left(\frac{1}{\mu^3}\right)\right] \quad \text{and} \quad \rho = 1.$$

This is evidently within the annular region $R(\mu)$ of basic width δ of $O(1/\mu^{2/3})$.

Lastly we find,

$$T^* = 2\ln 3 + \frac{8\pi^2}{3\mu} \cdot \frac{A}{\pi^2 + \ln^2 3}$$

Thus the period μT^* in t for the forced oscillation has the same leading term $(2\ln 3)\mu$ as in the case of the free vibration found in Stoker's [6].

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