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SOME RESULTS ON k-QUASI-HYPONORMAL OPERATORS

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k-Quasi-hyponormal operators are defined and some inclusion relations between these operators and k-paranormal operators are shown.

1. Preliminaries

An operator T defined on a Hilbert space H is said to be quasi-hyponormal if $||T^*Tx|| \le ||T^2x||$ for all x in H or equivalently $T^{*2}T^2 - (T^*T)^2 \ge 0$; T is k-paranormal $(k\ge2)$ if $||Tx||^k \le ||T^kx||$ for each unit vector x in H .

In the second section of this paper we define k-quasi-hyponormal ($k\ge2$, an integer) operators on a Hilbert space H and show the following inclusion relations which are proper.

(i) Quasi-hyponormal $\subseteq$ k-Quasi-hyponormal.

(ii) k-Quasi-hyponormal $\subseteq$ k-paranormal.

2. k-Quasi-hyponormal operators

An operator T defined on a Hilbert space H is k-quasi-hyponormal $(k\ge2)$ if $T^{*k}T^k - (T^*T)^k \ge 0$, or equivalently $||(T^*T)^{k/2}x|| \le ||T^kx||$,

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if k is even, and $\|T(T^*T)^{(k-1)/2}x\| \le \|T^kx\|$, if k is odd, for all x in H .

THEOREM 2.1. *Let T be a weighted shift with weights $\{\alpha_n\}$. Then T is k-quasi-hyponormal if and only if* $|\alpha_n|^{k-1} \le |\alpha_{n+1}||\alpha_{n+2}|\ldots|\alpha_{n+k-1}|$ *for all integers n .*

Proof. This follows immediately from the definition of k-quasi-hyponormal.

THEOREM 2.2. *If T is quasi-hyponormal and T^*T commutes with TT^* then T is k-quasi-hyponormal* $(k>2)$.

Proof. Let T be quasi-hyponormal. Then T is hyponormal on $T(H)$ and thus on $T(H)$, $(TT^*)^k \le (T^*T)^k \le T^{*^k}T^k$ [1]. If $k = 3$, then we must show that T is 3-quasi-hyponormal, that is $\|TT^*Tx\| \le \|T^3x\|$ for all x in H . Now

$$\|TT^*Tx\|^2 = (TT^*Tx, TT^*Tx) = ((TT^*)^2Tx, Tx)$$

or

$$\|TT^*Tx\|^2 \le (T^{*^2}T^2Tx, Tx) = (T^3x, T^3x) = \|T^3x\|^2 .$$

Hence T is 3-quasi-hyponormal. Suppose the result is true for $k = m$, m odd. To prove it for $k = m + 1$: since $m + 1$ is even, we must show that $\|(T^*T)^{(m+1)/2}x\| \le \|T^{m+1}x\|$ for all x in H .

$\|(T^*T)^{(m+1)/2}x\|^2 = ((T^*T)^{(m+1)/2}x, (T^*T)^{\frac{m+1}{2}}x) = ((TT^*)^mTx, Tx)$. Thus

$$\|(T^*T)^{(m+1)/2}x\|^2 \le (T^{*^m}T^mTx, Tx) = (T^{m+1}x, T^{m+1}x) = \|T^{m+1}x\|^2 .$$

The induction from even m to $m + 1$ works exactly as for the case going from 2 to 3. Hence a quasi-hyponormal operator is k-quasi-hyponormal.

THEOREM 2.3. *For $k > 2$, there exists a k-quasi-hyponormal operator which is not quasi-hyponormal.*

Proof. Let T be a bilateral weighted shift with weights $\{\alpha_n\}$, where

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$$\alpha_n = \begin{cases} \frac{1}{2} & \text{if } n \le -1 \\ \frac{1}{\sqrt{3}} & \text{if } n = 0 \\ \frac{n}{n+1} & \text{if } n \ge 1 . \end{cases}$$

Clearly $\alpha_n^{k-1} \le \alpha_{n+1}\alpha_{n+2}\cdots\alpha_{n+k-1}$ for $n \ne 0$. For $n = 0$, $k > 2$ we have $\alpha_o^{k-1} \le \alpha_1\alpha_2\cdots\alpha_{k-1}$. Therefore by Theorem 2.1, T is k-quasi-hyponormal $(k>2)$ but T is not quasi-hyponormal as $\alpha_o > \alpha_1$. Hence the result.

THEOREM 2.4. *If T is k-quasi-hyponormal, then T is k-paranormal* $(k\ge2)$.

Proof. To prove the result, it is enough to show the following inequalities:

$$\|Tx\|^k \le \|(T^*T)^{k/2}x\| \qquad \text{if } k \text{ is even}$$

and

$$\|Tx\|^k \le \|T(T^*T)^{(k-1)/2}x\| \quad \text{if } k \text{ is odd}$$

for each unit vector x in H. We prove these inequalities in induction.

Let x be a unit vector in H. Then

$$\|Tx\|^2 = (Tx,Tx) = (T^*Tx,x) \le \|T^*Tx\| ,$$

and

$$\|Tx\|^4 \le \|(T^*T)x\|^2 = (T^*Tx,T^*Tx) = (TT^*Tx,Tx) ,$$

or

$$\|Tx\|^4 \le \|TT^*Tx\| \quad \|Tx\|$$

that is

$$\|Tx\|^3 \le \|TT^*Tx\| .$$

Let the result be true for all $k < m$. To prove for $k = m$. If m is

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odd then $m + 1 = 2n$ for some $n \in N$, clearly $n < m$. Now

$$\|Tx\|^{m+1} = \|Tx\|^{2n} = (\|Tx\|^{n})^2 \le \|(T^*T)^{n/2}x\|^2 \text{ if } n \text{ is even,}$$

or

$$\|Tx\|^{m+1} \le \left((T^*T)^{n/2}x, (T^*T)^{n/2}x\right) = \left(T(T^*T)^{\frac{n}{2}+\frac{n}{2}-1}, Tx\right)$$

or

$$\|Tx\|^{m+1} \le \|T(T^*T)^{n-1}x\| \, \|Tx\|$$

or

$$\|Tx\|^{m} \le \|T(T^*T)^{(m-1)/2}x\| .$$

If n is odd then, we have

$$\|Tx\|^{m+1} \le \|T(T^*T)^{(n-1)/2}x\|^2 = \left(T(T^*T)^{(n-1)/2}, T(T^*T)^{(n-1)/2}x\right)$$

that is

$$\|Tx\|^{m+1} \le \left(T(T^*T)^{\frac{n-1}{2}+1+\frac{n-1}{2}-1}, Tx\right)$$

that is

$$\|Tx\|^{m+1} \quad \|T(T^*T)^{n-1}x\| \, \|Tx\|$$

that is

$$\|Tx\|^{m} \le \|T(T^*T)^{(m-1)/2}x\| .$$

Similarly it can be proved if m is even. Hence the result.

In the following theorem we show that the inclusion given in Theorem 2.4 is proper.

THEOREM 2.5. *For $k \ge 2$, there exists a k-paranormal operator which is not k-quasi-hyponormal.*

Proof. Let M be the direct sum of a countable number of copies of H. For given positive operators A and B on H, define the operator $T_{A,B}$ on M as follows:

$$T_{A,B}(x_1, x_2, \ldots) = (0, Ax_n, \ldots, Ax_n, Bx_{n+1}, \ldots) .$$

A simple computation shows that the operator $T = T_{A,B}$ is k-quasi-hyponormal if and only if $AB^{2k-2}A - A^{2k} \ge 0$.

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Now if H is a two dimensional Hilbert space and $A = C^{\frac{1}{2}}$, $B = \left(C^{-\frac{1}{2}}DC^{-\frac{1}{2}}\right)^{\frac{1}{2}}$ where

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix},$$

then $T = T_{A,B}$ is paranormal and therefore k-paranormal $(k \geq 2)$ [2], but it is not k-quasi-hyponormal.

References

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