Bull. Aust. Math. Soc. **98** (2018), 70–76
doi:10.1017/S0004972718000217

MAHLER MEASURE OF 'ALMOST' RECIPROCAL POLYNOMIALS

J. C. SAUNDERS

(Received 24 October 2017; accepted 22 January 2018; first published online 3 May 2018)

Abstract

We give a lower bound of the Mahler measure on a set of polynomials that are 'almost' reciprocal. Here 'almost' reciprocal means that the outermost coefficients of each polynomial mirror each other in proportion, while this pattern may break down for the innermost coefficients.

2010 *Mathematics subject classification*: primary 11R06; secondary 11R09.

Keywords and phrases: Lehmer's conjecture, Mahler measure, number theory, polynomials.

1. Introduction

The Mahler measure, $M(f)$, of a polynomial f with integer coefficients is defined to be the absolute value of the product of all of its roots having absolute value at least 1 and its leading coefficient. If no such roots exist, the Mahler measure is defined to be the absolute value of the leading coefficient. In other words, if

$$f(x) = a_n(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

then

$$M(f) = |a_n|\prod_{i=1}^{n}\max\{1,|\alpha_i|\}.$$

A major open problem is whether the Mahler measure can get arbitrarily close to 1 without actually being equal to 1. More specifically, for any $\epsilon > 0$, does there exist a polynomial f with integer coefficients such that $1 < M(f) < 1+\epsilon$? This problem was first posed in 1933 by Lehmer [2] and has since sparked many problems regarding the Mahler measure of polynomials. Lehmer was able to show that the polynomial

$$f(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$$

has Mahler measure $M(f) = 1.1762808\ldots$. This is the smallest Mahler measure greater than 1 that is currently known.

Research of J. C. Saunders was supported by NSERC and the Queen Elizabeth II Graduate Scholarship in Science and Technology program.

An important property of polynomials that has an impact on the value of the Mahler measure is whether or not they are reciprocal.

Definition 1.1. Let $f(x)$ be a polynomial of degree n. The reciprocal of $f(x)$ is $f^*(x) := x^n f(1/x)$. We say that $f(x)$ is a *reciprocal polynomial* if $f(x) = \pm f^*(x)$.

In 1971, Smyth [4] showed that if f is an irreducible polynomial with integer coefficients that does not have 0 or 1 as a root and is not reciprocal, then

$$M(f) \geq M(x^3 - x - 1) = 1.324717\ldots.$$

In 2004, Borwein *et al.* [1] modified Smyth's techniques to study polynomials with ± 1 coefficients which are not reciprocal ($f(x) \neq \pm f^*(x)$) but satisfy $f(x) \equiv \pm f^*(x) \pmod{m}$. Given these conditions, they prove that if $m \geq 2$, then

$$M(f) \geq \frac{m + \sqrt{m^2 + 16}}{4}$$

and this bound is sharp when m is even. The larger m is, the more impressive this bound becomes. Here we modify the approach of Borwein, Hare and Mossinghoff to study a new class of polynomials that we define to be ‘k-nonreciprocal’ for some integer $k > 0$.

Definition 1.2. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial in $\mathbb{Z}[x]$. For an integer $k \geq 1$, we say that $f(x)$ is k-nonreciprocal if $a_n a_i = a_0 a_{n-i}$ for $1 \leq i \leq k - 1$ and $a_n a_k \neq a_0 a_{n-k}$.

As with the result of Borwein, Hare and Mossinghoff, we prove a sharp lower bound for the Mahler measure of k-nonreciprocal polynomials which can get arbitrarily large, depending on the set of polynomials in question. More specifically, we prove the following result.

Theorem 1.3. *Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial in $\mathbb{Z}[x]$. Suppose that for some $k \in \mathbb{N}$ with $2k \leq n$, we have $a_n a_i = a_0 a_{n-i}$ for $1 \leq i \leq k - 1$. Let $\alpha = |a_k a_n - a_0 a_{n-k}|$. Then the Mahler measure $M(f)$ of f satisfies*

$$M(f) \geq \frac{\alpha + \sqrt{\alpha^2 + 4(|a_0| + |a_n|)^2 |a_0 a_n|}}{2(|a_0| + |a_n|)}.$$

We can see that if $f(x) \in \mathbb{Z}[x]$ is k-nonreciprocal for some $k > 0$, then $\pm(x - 1)f(x)$ is also k-nonreciprocal. Therefore, it is enough to consider polynomials where both the leading coefficient and the constant term are positive.

Remark 1.4. Borwein, Hare and Mossinghoff noted as a corollary to their result that if f is a nonreciprocal polynomial with all odd coefficients, then

$$M(f) \geq \frac{1 + \sqrt{5}}{2} = 1.618\ldots.$$

By Theorem 1.3, we may replace the condition that f has all odd coefficients with the condition that $|a_k a_n - a_0 a_{n-k}| \geq 2$ for the smallest k for which $a_k a_n \neq a_0 a_{n-k}$. Assuming that $|a_n| = |a_0| = 1$ (for otherwise $M(f) \geq \min\{|a_0|, |a_n|\} \geq 2$), this condition is substantially weaker than the condition that f is nonreciprocal and has all odd coefficients.

https://doi.org/10.1017/S0004972718000217 Published online by Cambridge University Press

2. Proof and example

Our proof follows that of Borwein, Hare and Mossinghoff in [1]. However, we now allow the innermost coefficients to depart from the reciprocal structure. We use the following result by Wiener (see [3, page 392]).

Lemma 2.1 (Wiener). *Suppose that* $\phi(z) = \sum_{i\geq 0} \gamma_i z^i$, *with* $\gamma_i \in \mathbb{C}$, *is analytic in an open disk containing* $|z| \leq 1$ *and satisfies* $|\phi(z)| \leq 1$ *on* $|z| = 1$. *Then* $|\gamma_i| \leq 1 - |\gamma_0|^2$ *for* $i \geq 1$.

We now prove Theorem 1.3.

Proof of Theorem 1.3. Suppose $f(z) = \sum_{i=0}^n a_i z^i = a_n(z-\alpha_1)\cdots(z-\alpha_n)$ satisfies the hypotheses in the theorem with a_0 and a_n both being positive. Write $f^*(z) = \sum_{i=0}^n d_i z^i$ so that $a_0 d_i = a_n a_i$ for $1 \leq i \leq k-1$. Let the power series expansion of $1/f^*(z)$ be $\sum_{i\geq 0} e_i z^i$ so that $e_0 = 1/a_n$. Let

$$G(z) = \frac{f(z)}{f^*(z)} = \sum_{i\geq 0} q_i z^i,$$

so that $q_0 = a_0/a_n$. From $f^*(z)G(z) = f(z)$, we obtain $\sum_{i=0}^j d_i q_{j-i} = a_j$. Thus, for $j \geq 1$,

$$a_n q_j = (a_j - q_0 d_j) - \sum_{i=1}^{j-1} d_i q_{j-i}.$$

From $a_0 d_i = a_n a_i$, we can see by induction that $q_i = 0$ for $1 \leq i \leq k-1$ and

$$q_k = \frac{a_k}{a_n} - \frac{a_o a_{n-k}}{a_n^2} \neq 0.$$

Let $\epsilon = -1$ if $f(z)$ has a zero of odd multiplicity at $z = 1$ and $\epsilon = 1$ otherwise. Since

$$\prod_{|\alpha_i|=1} \frac{z-\alpha_i}{1-\overline{\alpha_i}z} = \prod_{|\alpha_i|=1} \frac{-\alpha_i(1-z/\alpha_i)}{1-z/\alpha_i} = \prod_{|\alpha_i|=1}(-\alpha_i) = \epsilon,$$

we define

$$g(z) := \epsilon \prod_{|\alpha_i|<1} \frac{z-\alpha_i}{1-\overline{\alpha_i}z}$$

and

$$h(z) := \prod_{|\alpha_i|>1} \frac{1-\overline{\alpha_i}z}{z-\alpha_i}$$

so that

$$\frac{g(z)}{h(z)} = \frac{\prod_{i=1}^n (z-\alpha_i)}{\prod_{i=1}^n (1-\overline{\alpha_i}z)} = \frac{\prod_{i=1}^n (z-\alpha_i)}{\prod_{i=1}^n (1-\alpha_i z)} = \frac{f(z)}{f^*(z)} = G(z).$$

Since all poles of both $g(z)$ and $h(z)$ lie outside the unit disk, both functions are analytic in a region including $|z| \leq 1$. Also, if $|z| = 1$ and $\beta \in \mathbb{C}$, then

$$\left(\frac{z-\beta}{1-\overline{\beta}z}\right)\overline{\left(\frac{z-\beta}{1-\overline{\beta}z}\right)} = \left(\frac{z-\beta}{1-\overline{\beta}z}\right)\left(\frac{1/z-\overline{\beta}}{1-\beta/z}\right) = 1$$

so $|g(z)| = |h(z)| = 1$ on $|z| = 1$. Let

$$g(z) = \sum_{i\geq 0} b_i z^i$$

and

$$h(z) = \sum_{i\geq 0} c_i z^i.$$

Since $g(z) = h(z)G(z)$, we have $b_i = c_i q_0$ for $0 \leq i < k$ and $b_k = c_0 q_k + c_k q_0$. Thus

$$\left| c_0 \left(\frac{a_k}{a_n} - \frac{a_0 a_{n-k}}{a_n^2} \right) \right| = |c_0 q_k| = |b_k - c_k q_0| \leq |b_k| + |c_k| q_0.$$

Notice that

$$c_0 = |h(0)| = \prod_{|\alpha_i|>1} 1/|\alpha_i| = |a_n|/M(f), \tag{2.1}$$

so that

$$\left| \frac{1}{M(f)} \left(a_k - \frac{a_0 a_{n-k}}{a_n} \right) \right| = |c_0 q_k| \leq |b_k| + |c_k| q_0. \tag{2.2}$$

By Lemma 2.1, we have $|c_k| \leq 1 - c_0^2$ and $|b_k| \leq 1 - b_0^2$. Notice that $b_0 = c_0 q_0$. Combining (2.1) and (2.2),

$$\begin{aligned}
\frac{1}{M(f)} \left| a_k - \frac{a_0 a_{n-k}}{a_n} \right| &\leq (1 - b_0^2) + (1 - c_0^2) q_0 \\
&= (1 - c_0^2 q_0^2) + (1 - c_0^2) q_0 \\
&= (1 + q_0)(1 - q_0 c_0^2) \\
&= (q_0 + 1) \left(1 - \frac{q_0 a_n^2}{M(f)^2} \right) \\
&= (q_0 + 1) \left(1 - \frac{a_0 a_n}{M(f)^2} \right).
\end{aligned}$$

Thus

$$M(f) \left| a_k - \frac{a_0 a_{n-k}}{a_n} \right| \leq (q_0 + 1)(M(f)^2 - a_0 a_n).$$

This gives

$$M(f) \geq \frac{1}{2(q_0 + 1)} \left(\left| a_k - \frac{a_0 a_{n-k}}{a_n} \right| + \sqrt{\left| a_k - \frac{a_0 a_{n-k}}{a_n} \right|^2 + 4(q_0 + 1)^2 a_0 a_n} \right)$$

and the result follows. □

REMARK 2.2. If $|a_na_k - a_0a_{n-k}| > |a_0^2 - a_n^2|$, then the above bound is nontrivial since then it will be greater than

$$\begin{aligned}\frac{|a_0^2 - a_n^2| + \sqrt{|a_0^2 - a_n^2|^2 + 4(a_0 + a_n)^2|a_0a_n|}}{2(a_0 + a_n)} &= \frac{|a_0 - a_n| + \sqrt{(a_0 - a_n)^2 + 4a_0a_n}}{2} \\ &= \frac{|a_0 - a_n| + \sqrt{(a_0 + a_n)^2}}{2} \\ &= \frac{|a_0 - a_n| + a_0 + a_n}{2} \\ &= \max\{a_n, a_0\},\end{aligned}$$

which is the trivial bound.

REMARK 2.3. If $f(x)$ is a reciprocal polynomial, the above bound is trivial. For we then have $a_n = a_0$ and $a_k = a_{n-k}$ so that

$$|a_na_k - a_0a_{n-k}| = 0$$

and Theorem 1.3 only gives

$$M(f) \geq \frac{\sqrt{4(a_0 + a_n)^2 a_na_0}}{2(a_0 + a_n)} = a_n,$$

which is trivial.

We now give some examples to show that the bound in Theorem 1.3 is sharp.

EXAMPLE 2.4. Let $k, n \in \mathbb{N}$ with $n > 2k$ and $n \neq 3k$, and $a, b, c \in \mathbb{Z}$ such that $a > 0 > c$ and $a - |b| \leq -c \leq a + |b|$. Consider the polynomial

$$f(x) = \sum_{i=0}^{n} a_ix^i = (ax^{2k} + bx^k + c)(x^{n-2k} - 1),$$

which satisfies $a_na_i = a_0a_{n-i}$ for $1 \leq i \leq k - 1$ and $a_na_k \neq a_0a_{n-k}$. As in Theorem 1.3, set $\alpha = |a_ka_n - a_0a_{n-k}|$. Then

$$M(f) = \frac{\alpha + \sqrt{\alpha^2 + 4(a_0 + a_n)^2 a_0a_n}}{2(a_0 + a_n)}.$$

Now

$$f(x) = \begin{cases} ax^n + bx^{n-k} + cx^{n-2k} - ax^{2k} - bx^k - c & \text{if } n > 4k, \\ ax^{4k} + bx^{3k} + (c - a)x^{2k} - bx^k - c & \text{if } n = 4k, \\ ax^n + bx^{n-k} - ax^{2k} + cx^{n-2k} - bx^k - c & \text{if } 4k > n > 3k, \\ ax^n - ax^{2k} + bx^{n-k} - bx^k + cx^{n-2k} - c & \text{if } 3k > n > 2k. \end{cases}$$

In all cases, we can easily see that $a_n = a$, $a_0 = -c$, $a_k = -b$, $a_{n-k} = b$, $a_na_i = a_0a_{n-i}$ for $1 \leq i \leq k - 1$ and $a_na_k \neq a_0a_{n-k}$. Therefore $\alpha = |a_na_k - a_0a_{n-k}| = |b(a - c)|$.

https://doi.org/10.1017/S0004972718000217 Published online by Cambridge University Press

Since all the roots of $x^{n-2k} - 1$ have absolute value 1, it follows that $M(f) = M(ax^{2k} + bx^k + c)$. By the quadratic formula, the roots of $ax^{2k} + bx^k + c$ are the kth roots of the numbers

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Since $c < 0 < a$, the absolute values of these numbers are

$$\frac{\pm|b| + \sqrt{b^2 - 4ac}}{2a}.$$

First, consider

$$\frac{|b| + \sqrt{b^2 - 4ac}}{2a}. \tag{2.3}$$

If $|b| > a$, then clearly this number is greater than 1 and we may assume that $|b| \leq a$. By assumption $-c \geq a - |b|$, so

$$\begin{aligned} \frac{|b| + \sqrt{b^2 - 4ac}}{2a} &\geq \frac{|b| + \sqrt{b^2 + 4a(a - |b|)}}{2a} \\ &= \frac{|b| + \sqrt{4a^2 - 4a|b| + b^2}}{2|a|} \\ &= \frac{|b| + 2a - |b|}{2a} \\ &= 1. \end{aligned}$$

Now consider

$$\frac{\sqrt{b^2 - 4ac} - |b|}{2a}. \tag{2.4}$$

By assumption $-c \leq a + |b|$, so

$$\begin{aligned} \frac{\sqrt{b^2 - 4ac} - |b|}{2a} &\leq \frac{\sqrt{b^2 + 4a(a + |b|)} - |b|}{2a} \\ &= \frac{\sqrt{4a^2 + 4a|b| + b^2} - |b|}{2a} \\ &= \frac{2a + |b| - |b|}{2a} \\ &= 1. \end{aligned}$$

All of the nth roots of (2.3) have absolute value at least 1, while all of the nth roots of (2.4) have absolute value at most 1. Hence

$$M(f) = \frac{|b| + \sqrt{b^2 - 4ac}}{2}.$$

Note that

$$\frac{|b| + \sqrt{b^2 - 4ac}}{2} = \frac{|ba - cb| + \sqrt{(ba - cb)^2 - 4(a - c)^2 ca}}{2(a - c)}$$

since a and c have opposite signs. Thus we attain our bound.

https://doi.org/10.1017/S0004972718000217 Published online by Cambridge University Press

REMARK 2.5. If we impose the restriction $a_0, a_n = \pm 1$ in the above example, then α will be even. It is unknown whether the inequality in Theorem 1.3 is still sharp if we impose $a_0, a_n = \pm 1$ with α being odd.

3. Future work

The results obtained in this paper and those of Borwein, Hare and Mossinghoff in [1] suggest some further interesting questions about the Mahler measure of reciprocal or 'almost reciprocal' polynomials. For example:

(1) If a polynomial $f(x)$ is 'almost reciprocal' as defined in this paper and 'almost reciprocal' as defined by Borwein, Hare and Mossinghoff in [1], can we get better bounds than the bounds we have shown here and the bounds proved by Borwein, Hare and Mossinghoff?

(2) Can we use the ideas presented here to give bounds for the Mahler measure of sparse polynomials?

Acknowledgements

The author would like to thank Dr Kevin Hare and Dr Yu-Ru Liu for their support and suggestions for this paper.

References

[1] P. Borwein, K. G. Hare and M. J. Mossinghoff, 'The Mahler measure of polynomials with odd coefficients', *Bull. Lond. Math. Soc.* **36**(3) (2004), 332–338.

[2] D. H. Lehmer, 'Factorization of certain cyclotomic functions', *Ann. of Math. (2)* **34**(3) (1933), 461–479.

[3] A. Schinzel, *Polynomials With Special Regard to Reducibility*, Encyclopedia of Mathematics and its Applications, 77 (Cambridge University Press, Cambridge, 2000).

[4] C. J. Smyth, 'On the product of the conjugates outside the unit circle of an algebraic integer', *Bull. Lond. Math. Soc.* **3**(2) (1971), 169–175.

J. C. SAUNDERS, Department of Pure Mathematics,
University of Waterloo, Waterloo, Ontario N2L 3G1, Canada
e-mail: j8saunde@uwaterloo.ca